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VERY INTUITIONISTIC THEORIES AND QUANTIFIER ELIMINATION

WIM RUITENBURG

ABSTRACT. We show methods to construct, and give examples of, consistent intuitionistic theories that admit quantifier elimination. The examples are very intuitionistic in the sense that they prove the negation of the schema of linearity.

1. INTRODUCTION

We present a simple method by which one can construct intuitionistic theories that admit quantifier elimination, and that are in some sense very intuitionistic. Our task splits into two main components.

First, we offer a pragmatic criterion for an intuitionistic theory to be very intuitionistic. We do not claim that ours is the only right definition of very intuitionistic theory.

Second, we present a simple method by which one can construct intuitionistic theories over which each formula is equivalent to a positive existential formula, that is, equivalent to a formula built from the atoms using existential quantification, disjunction, and conjunction only.

Additionally, we give some examples of very intuitionistic theories that admit quantifier elimination. This part is presented so that the reader should be able to verify the main facts about these examples, without any need to seriously read the earlier sections.

Intuitionistic theories that admit quantifier elimination, have been and are studied elsewhere. For examples, see Craig Smoryński's paper [3], or more recent work by Seyed Mohammad Bagheri.

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2. Very intuitionistic theories

In this paper, first-order languages use $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$ as basic symbols. Symbols $\top$ and $\bot$ are both atoms and nullary connectives. Negation $\neg\varphi$ is short for $\varphi \rightarrow \bot$. Bi-implication $\varphi \leftrightarrow \psi$ is short for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We write $\mathbf{x}$ as short for finite lists of variables $x_1, \ldots, x_n$. Notations like $\mathbf{a}$ are defined the same way, in terms of finite lists of constant symbols; the expression '$x^\frown\mathbf{y}$' to denote the sequence resulting from prepending x to the sequence $\mathbf{y}$. We write $\varphi(\mathbf{x})$ to indicate that $\mathbf{x}$ includes all free variables of formula $\varphi(\mathbf{x})$. Formally, intuitionistic predicate logic IQC is a proper subsystem of classical predicate logic CQC.

We should not expect one unique definition of very intuitionistic theory. Our choice is a pragmatic one. Let Γ be a theory over IQC. One way to classify Γ as very intuitionistic is, by finding a sentence φ such that CQC $\vdash \varphi$ (so φ is a classical tautology), and $\Gamma \cup \{\neg\varphi\}$ is consistent (such φ can always be assumed to be conjunctions of sentences of the form $\forall\mathbf{x}(\psi(\mathbf{x}) \vee \neg\psi(\mathbf{x}))$). However, we can still find theories that look and feel classical, but satisfy this criterion. Therefore we prefer the stronger criterion given below.

Define the theory AQC of almost-classical logic as the extension of IQC, axiomatizable by the schemas

A1 $\forall x(\varphi(\mathbf{y}) \vee \psi(x\mathbf{y})) \rightarrow (\varphi(\mathbf{y}) \vee \forall x\psi(x\mathbf{y}))$ (Constant Domains)
A2 $(\varphi(\mathbf{x}) \rightarrow (\psi(\mathbf{x}) \vee \sigma(\mathbf{x}))) \rightarrow ((\varphi(\mathbf{x}) \rightarrow \psi(\mathbf{x})) \vee (\varphi(\mathbf{x}) \rightarrow \sigma(\mathbf{x})))$ (Linearity)
A3 $(\varphi(\mathbf{y}) \rightarrow \exists x\psi(x\mathbf{y})) \rightarrow \exists x(\varphi(\mathbf{y}) \rightarrow \psi(x\mathbf{y}))$

Next, we motivate the use of the expression *almost-classical*.

A consistent theory Γ is *first-order prime* if it satisfies

$\Gamma \vdash \varphi \vee \psi$ implies $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$, for all sentences $\varphi \vee \psi$
$\Gamma \vdash \exists x\varphi(x)$ implies $\Gamma \vdash \varphi(c)$ for some c, for all sentences $\exists x\varphi(x)$

A theory Γ is *universally first-order prime* if, additionally, it satisfies

For all sentences $\forall x\varphi(x)$, if $\Gamma \vdash \varphi(c)$ for all constant symbols c, then $\Gamma \vdash \forall x\varphi(x)$

It is well-known that if Γ is a theory such that $\Gamma \nvdash \varphi$, then we can extend Γ to a first-order prime theory $\Gamma' \nvdash \varphi$ over a language $\mathcal{L}[C]$, obtained from the original language $\mathcal{L}$ by adding a set C of new constant symbols. It is slightly less well-known that if Γ satisfies condition A1 of Constant Domains, then Γ' can be chosen universally first-order prime; see [2] or [1].

First-order prime theories play a key role in the construction of Kripke models: Let $\mathbf{M} = \mathbf{M}(\mathcal{L})$ be the category of (classical) models and morphisms over language $\mathcal{L}$. Then a *Kripke model* is a functor $\mathcal{K} : \mathbf{C} \to \mathbf{M}$, where $\mathbf{C}$ is a small category. So for each arrow $f : k \to m$ of $\mathbf{C}$, there is a morphism $\mathcal{K}f : \mathcal{K}_k \to \mathcal{K}_m$. Node structures can be constructed from first-order prime theories in the usual way. We write K_k for the domain of $\mathcal{K}_k$. Forcing is inductively defined in the usual way by

$k \Vdash P$ exactly when $\mathcal{K}_k \models P$, $P \in \mathcal{L}[K_k]$ atomic
$k \Vdash \varphi \wedge \psi$ exactly when $k \Vdash \varphi$ and $k \Vdash \psi$
$k \Vdash \varphi \vee \psi$ exactly when $k \Vdash \varphi$ or $k \Vdash \psi$
$k \Vdash \exists x \varphi(x)$ exactly when $k \Vdash \varphi(a^k)$ for some $a \in K_k$
$k \Vdash \varphi \to \psi$ exactly when $m \Vdash \varphi^f$ implies $m \Vdash \psi^f$, for all $f : k \to m$
$k \Vdash \forall x \varphi(x)$ exactly when $m \Vdash \varphi^f(a^m)$, for all $f : k \to m$ and $a \in K_m$

where $\varphi^f \in \mathcal{L}[K_m]$ is constructed from $\varphi \in \mathcal{L}[K_k]$ by replacing all constant symbols a^k with $a \in K_k$, by $(\mathcal{K}fa)^m$ with $\mathcal{K}fa \in K_m$. We may drop the superscripts when the meaning is clear from the context.

By the strong completeness theorem, if $\Gamma \cup \{\varphi\}$ is a set of sentences, then the following are equivalent:

- $\Gamma \vdash \varphi$
- For all $\mathcal{K}$ and nodes k, if $k \Vdash \gamma$ for all $\gamma \in \Gamma$, then $k \Vdash \varphi$

Usually we are allowed to restrict the choice of Kripke models. For example, if $\Gamma \supseteq$ CQC, then we can restrict ourselves to one-node Kripke models with only the identity arrow: Such Kripke models essentially are classical models, for if $\mathbf{C}$ has one node k, and just the identity arrow, then $k \Vdash \varphi$, if and only if $\mathcal{K}_k \models \varphi$. We are particularly interested in the following almost-classical Kripke models. Let $\Gamma \supseteq$ AQC be a universally first-order prime theory. Construct a Kripke model $\mathcal{K}(\Gamma)$ as follows. As objects of the underlying category $\mathbf{C}$ we have all equivalence classes $[\varphi]$ of sentences such that $\Gamma \nvdash \neg\varphi$, taken modulo the equivalence relation $\Gamma \vdash \varphi \leftrightarrow \psi$. The category is a linear order with $[\varphi] \preceq [\psi]$ if and only if $\Gamma \vdash \psi \to \varphi$. As structure above each node $[\varphi]$, choose the model constructed in the usual way from the universally first-order prime theory axiomatized by $\Gamma \cup \{\varphi\}$. The implied morphisms $\mathcal{K}(\Gamma)_{[\varphi]} \to \mathcal{K}(\Gamma)_{[\psi]}$ are onto. Then $[\varphi] \Vdash \psi$ if and only if $\Gamma \vdash \varphi \to \psi$, for all appropriate φ, and all ψ. Almost-classical models share with classical models that they are essentially definable inside their universally first-order prime theories. If $\Gamma \cup \{\varphi\}$ is a set of sentences such that $\Gamma \supseteq$ AQC, then the following are equivalent:

- $\Gamma \vdash \varphi$
- For all almost-classical $\mathcal{K}$ and nodes k, if $k \Vdash \gamma$ for all $\gamma \in \Gamma$, then $k \Vdash \varphi$

Definition: A theory Γ is *very intuitionistic* if there exists a sentence φ such that $\mathrm{AQC} \vdash \varphi$, and $\Gamma \cup \{\neg\varphi\}$ is consistent.

3. Simple quantifier elimination

Next we show that it is not too difficult to construct very intuitionistic theories that admit quantifier elimination.

A formula is called *positive existential* if it is built from atoms using at most disjunctions, conjunctions, and existential quantifiers. One easily verifies:

Proposition 3.1. *Let $\varphi(\mathbf{x})$ be a positive existential formula. Let k be a node of a Kripke model, and let $\mathbf{a} \in K_k$. Then the following are equivalent.*

- $k \Vdash \varphi(\mathbf{a})$
- $\mathcal{K}_k \models \varphi(\mathbf{a})$

A formula is called *geometric* if it is of the form $\forall\mathbf{x}(\varphi(\mathbf{xy}) \to \psi(\mathbf{xy}))$, with $\varphi(\mathbf{xy})$ and $\psi(\mathbf{xy})$ positive existential. One easily verifies:

Proposition 3.2. *Let $\varphi(\mathbf{x})$ be a geometric formula. Let k be a node of a Kripke model $\mathcal{K}$, and let $\mathbf{a} \in K_k$. Then the following are equivalent.*

- $k \Vdash \varphi(\mathbf{a})$
- *For all arrows $f : k \to m$ of the underlying small category of $\mathcal{K}$, we have $\mathcal{K}_m \models \varphi(\mathbf{a})^f$*

Let Γ be a set of sentences. We write $G(\Gamma)$ for the theory axiomatizable by the geometric sentences that follow from Γ. A Kripke model is *locally* Γ if all its classical node structures are models of Γ. Proposition 3.2 implies that a Kripke model satisfies $G(\Gamma)$ if and only if it is locally $G(\Gamma)$. A Kripke model $\mathcal{K}$ is called *fully* Γ, if for all geometric formulas $\varphi(\mathbf{x})$, all nodes k, and all $\mathbf{a} \in K_k$, if there is a classical model $\mathcal{B} \models \Gamma$ and morphism $g : \mathcal{K}_k \to \mathcal{B}$ such that $\mathcal{B} \not\models \varphi(\mathbf{a})^g$, then there is an arrow $f : k \to m$ of the underlying small category of $\mathcal{K}$ such that $\mathcal{K}_m \not\models \varphi(\mathbf{a})^f$. A model $\mathcal{K}$ is *fully locally* Γ if it is both locally Γ and fully Γ.

Let $\Gamma \subseteq \Gamma'$ be sets of sentences, and $\mathcal{K}$ be a Kripke model. If $\mathcal{K}$ is locally Γ', then $\mathcal{K}$ is locally Γ. If $\mathcal{K}$ is fully Γ, then $\mathcal{K}$ is fully Γ'.

Proposition 3.3. *Let Δ be a classical theory, and let $\mathcal{K}$ be a Kripke model. Then the following are equivalent.*

- *$\mathcal{K}$ is fully Δ*
- *For all geometric formulas $\varphi(\mathbf{x})$, all nodes k, and all $\mathbf{a} \in K_k$, if $k \Vdash \varphi(\mathbf{a})$, then there is a positive existential formula $\sigma(\mathbf{x})$ such that $k \Vdash \sigma(\mathbf{a})$ and $G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})$*

Proof. Assume $\mathcal{K}$ is fully Δ. Suppose that $k \Vdash \varphi(\mathbf{a})$, with $\varphi(\mathbf{a})$ geometric. So $\mathcal{K}_m \models \varphi(\mathbf{a})^f$, for all arrows $f : k \to m$. So, by assumption, $\mathcal{B} \models \varphi(\mathbf{a})^g$, for all morphisms $g : \mathcal{K}_k \to \mathcal{B} \models \Delta$. Let D_k^+ be the positive atomic sentence diagram of $\mathcal{K}_k$ over the language $\mathcal{L}[K_k]$. Then $\Delta \cup D_k^+ \vdash \varphi(\mathbf{a})$. There is a finite conjunction $\gamma(\mathbf{ab})$ of atoms of D_k^+ such that $\Delta \vdash \gamma(\mathbf{ab}) \to \varphi(\mathbf{a})$. So $\Delta \vdash \exists \mathbf{y} \gamma(\mathbf{xy}) \to \varphi(\mathbf{x})$. Set $\sigma(\mathbf{x})$ equal to $\exists \mathbf{y} \gamma(\mathbf{xy})$. With the intuitionistically valid principle

$$\forall x(\alpha(\mathbf{y}) \to \beta(x\mathbf{y})) \leftrightarrow (\alpha(\mathbf{y}) \to \forall x \beta(x\mathbf{y}))$$

and intuitionistic propositional logic, formula $\sigma(\mathbf{x}) \to \varphi(\mathbf{x})$ is equivalent to a geometric formula. So $G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})$.

Conversely, assume the second item. Suppose that $k \Vdash \varphi(\mathbf{a})$, with $\varphi(\mathbf{a})$ geometric. It suffices to show that $\Delta \cup D_k^+ \vdash \varphi(\mathbf{a})$. By assumption, there is a positive existential formula $\sigma(\mathbf{x})$ such that $D_k^+ \vdash \sigma(\mathbf{a})$ and $G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})$. Thus $\Delta \cup D_k^+ \vdash \varphi(\mathbf{a})$. □

Proposition 3.3 implies that if Δ is a classical theory, and $\mathcal{K}$ is a fully Δ Kripke model, then $\mathcal{K}$ is also fully $G(\Delta)$.

There is a natural way to extend the inductive definition of forcing $k \Vdash \varphi$ to sentences in the construction of which we allow infinite disjunctions and conjunctions, by:

$k \Vdash \bigvee_i \varphi_i$ if and only if $k \Vdash \varphi_i$, for some i
$k \Vdash \bigwedge_i \varphi_i$ if and only if $k \Vdash \varphi_i$, for all i

Let Δ be a classical theory, and $\varphi(\mathbf{x})$ be a geometric formula. Define

$$\Sigma(\mathbf{x}) = \Sigma(\Delta)_{\varphi(\mathbf{x})}(\mathbf{x}) = \bigvee\{\sigma(\mathbf{x}) \mid \sigma(\mathbf{x}) \text{ is positive existential and } G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})\}$$

Modulo provable equivalence, the set of positive existential formulas that make up the definition of $\Sigma(\mathbf{x})$ forms an ideal on the lattice of all positive existential formulas $\sigma(\mathbf{x})$. Clearly, $\Sigma(\mathbf{x})$ is equal to

$$\bigvee\{\sigma(\mathbf{x}) \mid \sigma(\mathbf{x}) \text{ is positive existential and } \Delta \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})\}$$

Proposition 3.4. *Let $\varphi(\mathbf{x})$ be a geometric formula, Δ be a classical theory, and $\mathcal{K}$ be a Kripke model. Set $\Sigma(\mathbf{x}) = \Sigma(\Delta)_{\varphi(\mathbf{x})}(\mathbf{x})$. Then*

If $\mathcal{K}$ is locally Δ, then $\mathcal{K} \models \Sigma(\mathbf{x}) \to \varphi(\mathbf{x})$
If $\mathcal{K}$ is fully Δ, then $\mathcal{K} \models \varphi(\mathbf{x}) \to \Sigma(\mathbf{x})$

Proof. Let k be a node of $\mathcal{K}$, and $\mathbf{a} \in K_k$. If $\mathcal{K}$ is locally Δ, then

$$k \Vdash \bigwedge\{\sigma(\mathbf{a}) \to \varphi(\mathbf{a}) \mid \sigma(\mathbf{x}) \text{ is positive existential and } G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})\}$$

So $k \Vdash \Sigma(\mathbf{a}) \to \varphi(\mathbf{a})$.

Assume $\mathcal{K}$ is fully Δ. Suppose $k \Vdash \varphi(\mathbf{a})$. By Proposition 3.3 there is a positive existential formula $\sigma(\mathbf{x})$ such that $k \Vdash \sigma(\mathbf{a})$ and $G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})$. So $k \Vdash \Sigma(\mathbf{a})$. □

Let Δ be a classical theory. A fully locally Δ Kripke model $\mathcal{K}$ is called *positive existentially compact* if for all geometric formulas $\varphi(\mathbf{x})$ there exist positive existential $\sigma(\mathbf{x})$ such that

$$\mathcal{K} \models \Sigma(\Delta)_{\varphi(\mathbf{x})}(\mathbf{x}) \leftrightarrow \sigma(\mathbf{x})$$

Proposition 3.5. *Let Δ be a classical theory. Let $\mathcal{K}$ be a fully locally Δ Kripke model which is positive existentially compact. Then for all formulas $\varphi(\mathbf{x})$ there exist positive existential $\sigma(\mathbf{x})$ such that*

$$\mathcal{K} \models \varphi(\mathbf{x}) \leftrightarrow \sigma(\mathbf{x})$$

Proof. We complete the proof by induction on the complexity of $\varphi(\mathbf{x})$. The case holds for atoms. The cases for which it holds are closed under conjunction, disjunction, and existential quantification. Suppose $\varphi(\mathbf{x})$ is of the form $\alpha(\mathbf{x}) \to \beta(\mathbf{x})$. By induction there are positive existential $\sigma(\mathbf{x})$ and $\tau(\mathbf{x})$ such that $\mathcal{K} \models \alpha(\mathbf{x}) \leftrightarrow \sigma(\mathbf{x})$ and $\mathcal{K} \models \beta(\mathbf{x}) \leftrightarrow \tau(\mathbf{x})$. Now $\sigma(\mathbf{x}) \to \tau(\mathbf{x})$ is geometric. So there is $\Sigma(\mathbf{x}) = \Sigma(\Delta)_{\sigma(\mathbf{x})\to\tau(\mathbf{x})}(\mathbf{x})$ such that

$$\mathcal{K} \models \varphi(\mathbf{x}) \leftrightarrow \Sigma(\mathbf{x})$$

Apply positive existential compactness. Suppose $\varphi(\mathbf{x})$ is of the form $\forall y\, \psi(y\mathbf{x})$. By induction there is positive existential $\sigma(y\mathbf{x})$ such that $\mathcal{K} \models \psi(y\mathbf{x}) \leftrightarrow \sigma(y\mathbf{x})$. Now $\forall y\, \sigma(y\mathbf{x})$ is geometric, up to a trivial translation. So there is $\Sigma(\mathbf{x}) = \Sigma(\Delta)_{\forall y(\top\to\sigma(y\mathbf{x}))}(\mathbf{x})$ such that

$$\mathcal{K} \models \varphi(\mathbf{x}) \leftrightarrow \Sigma(\mathbf{x})$$

Apply positive existential compactness. □

There are several situations where positive existential compactness easily follows. Below is an example.

Lemma 3.6. *Suppose that language $\mathcal{L}$ has finitely many predicates and constant symbols, and no function symbols. Let $\mathbf{x}$ be a finite list of variables. Then there are, up to (intuitionistic) provable equivalence, only finitely many quantifier free positive existential formulas with free variables all among $\mathbf{x}$.*

Proof. There are only finitely many atomic formulas in the variables $\mathbf{x}$. These finitely many atomic formulas can only generate a finite lattice modulo provable equivalence. □

Lemma 3.6 is false when we remove the quantifier free restriction.

Theorem 3.7. *Suppose that language $\mathcal{L}$ has finitely many predicates and constant symbols, and no function symbols. Let Δ be a classical theory such that for all positive existential quantifier free $\alpha(x\mathbf{y})$ there exist positive existential quantifier free $\beta(\mathbf{y})$ such that*

$$\Delta \vdash \exists x \alpha(x\mathbf{y}) \leftrightarrow \beta(\mathbf{y})$$

Let $\mathcal{K}$ be a fully locally Δ Kripke model. Then for all formulas $\varphi(\mathbf{x})$ there exist positive existential quantifier free $\sigma(\mathbf{x})$ such that

$$\mathcal{K} \models \varphi(\mathbf{x}) \leftrightarrow \sigma(\mathbf{x})$$

Proof. With Proposition 3.5, it suffices to show that $\mathcal{K}$ is positive existentially compact. Let $\varphi(\mathbf{x})$ be a geometric formula. Set

$$\Xi(\mathbf{x}) = \bigvee\{\sigma(\mathbf{x}) \mid \sigma(\mathbf{x}) \text{ is positive existential quantifier free and } G(\Delta) \vdash \sigma(\mathbf{x}) \to \varphi(\mathbf{x})\}$$

If $\alpha(x\mathbf{y})$ and $\beta(\mathbf{y})$ are positive existential, then both $\exists x\, \alpha(x\mathbf{y}) \to \beta(\mathbf{y})$ and $\beta(\mathbf{y}) \to \exists x\, \alpha(x\mathbf{y})$ are geometric. So, by Proposition 3.2, positive existential quantifier elimination applies to the theory of $\mathcal{K}$. By Proposition 3.4, we have

$$\mathcal{K} \models \varphi(\mathbf{x}) \leftrightarrow \Sigma_{\varphi(\mathbf{x})}(\mathbf{x}) \leftrightarrow \Xi(\mathbf{x})$$

Modulo provable equivalence, the set of positive existential quantifier free formulas that make $\Xi(\mathbf{x})$ forms an ideal on the lattice of positive existential quantifier free formulas with all variables among $\mathbf{x}$. By Lemma 3.6, this lattice is finite. So $\Xi(\mathbf{x})$ equals its maximal element. Thus $\mathcal{K}$ is positive existentially compact. □

We are almost ready for some simple examples of very intuitionistic theories that admit quantifier elimination.

Consider the following principle. A Kripke model is *locally atomically prime* if for all nodes k, all positive existential quantifier free $\delta(\mathbf{x})$, $\sigma(\mathbf{x})$, and $\tau(\mathbf{x})$, with $\delta(\mathbf{x})$ atomic, and all $\mathbf{a} \in K_k$, we have $k \Vdash [\delta(\mathbf{a}) \to (\sigma(\mathbf{a}) \vee \tau(\mathbf{a}))] \to [(\delta(\mathbf{a}) \to \sigma(\mathbf{a})) \vee (\delta(\mathbf{a}) \to \tau(\mathbf{a}))]$.

Proposition 3.8. *Let Δ be a classical theory such that*

> *for all models $\mathcal{A}$ of Δ, all atomic $\delta(\mathbf{x})$, and all $\mathbf{a} \in A$, there is a morphism $e : \mathcal{A} \to \mathcal{B} \models \Delta$ such that for all morphisms $f : \mathcal{A} \to \mathcal{C} \models \Delta$, if $\mathcal{C} \models \delta(\mathbf{a})^f$, then $\mathcal{B} \models \delta(\mathbf{a})^e$, and there is a morphism g such that $f = ge$.*

Then locally fully Δ Kripke models are locally atomically prime.

Proof. Let $\mathcal{K}$ be locally fully Δ, and let k be a node such that $k \Vdash \delta(\mathbf{a}) \to (\sigma(\mathbf{a}) \vee \tau(\mathbf{a}))$, where $\delta(\mathbf{x})$, $\sigma(\mathbf{x})$, and $\tau(\mathbf{x})$ are positive existential quantifier free with $\delta(\mathbf{x})$ atomic, and $\mathbf{a} \in K_k$. To show: $k \Vdash (\delta(\mathbf{a}) \to \sigma(\mathbf{a})) \vee (\delta(\mathbf{a}) \to \tau(\mathbf{a}))$. Let $e : \mathcal{K}_k \to \mathcal{B} \models \Delta$ be the morphism given by the assumption about Δ and $\delta(\mathbf{a})$. We may assume that $k \not\Vdash \neg\delta(\mathbf{a})$. There is $f : k \to m$ such that $\mathcal{K}_m \models \delta(\mathbf{a})^f$. So, by the assumption, $\mathcal{B} \models \delta(\mathbf{a})^e$. Since $\mathcal{K}$ is fully Δ, also $\mathcal{B} \models \sigma(\mathbf{a}) \vee \tau(\mathbf{a})$, say $\mathcal{B} \models \sigma(\mathbf{a})$. Now whenever $h : k \to n$ is such that $\mathcal{K}_n \models \delta(\mathbf{a})^h$, there is a morphism $g : \mathcal{B} \to \mathcal{K}_n$ such that $h = ge$. So $\mathcal{K}_n \models \sigma(\mathbf{a})^h$. Thus $k \Vdash \delta(\mathbf{a}) \to \sigma(\mathbf{a})$. □

4. Simple examples of quantifier elimination

In the examples of this section, both Theorem 3.7 and Proposition 3.8 apply. The earlier theory helped in discovery and development of the examples. Next, we re-arranged the material so that the reader need not seriously look at the earlier sections to follow most details of this section. Standard knowledge of intuitionistic predicate logic IQC and Kripke models suffices.

The following proposition-logical intuitionistically valid equivalences are useful in quantifier elimination:

$$[\varphi \to (\psi \wedge \theta)] \leftrightarrow [(\varphi \to \psi) \wedge (\varphi \to \theta)]$$
$$[(\psi \vee \theta) \to \varphi] \leftrightarrow [(\psi \to \varphi) \wedge (\theta \to \varphi)]$$
$$[(\psi \wedge \theta) \to \varphi] \leftrightarrow [\psi \to (\theta \to \varphi)]$$

As with theories over classical logic, quantifier elimination requires us to find, for each quantifier free formula $\varphi(x\mathbf{y})$, a quantifier free formula $\psi(\mathbf{y})$, such that the intuitionistic theory satisfies

$$\exists x\, \varphi(x\mathbf{y}) \leftrightarrow \psi(\mathbf{y})$$

We also must find quantifier free $\theta(\mathbf{y})$ such that the intuitionistic theory satisfies

$$\forall x \varphi(x\mathbf{y}) \leftrightarrow \theta(\mathbf{y})$$

The following intuitionistically valid equivalences may help with quantifier elimination involving $\exists$ and $\forall$:

$$\forall x(\varphi(x\mathbf{y}) \wedge \psi(x\mathbf{y})) \leftrightarrow (\forall x\varphi(x\mathbf{y}) \wedge \forall x\psi(x\mathbf{y}))$$
$$\exists x(\varphi(x\mathbf{y}) \vee \psi(x\mathbf{y})) \leftrightarrow (\exists x\, \varphi(x\mathbf{y}) \vee \exists x\, \psi(x\mathbf{y}))$$
$$\exists x(\varphi(\mathbf{y}) \wedge \psi(x\mathbf{y})) \leftrightarrow (\varphi(\mathbf{y}) \wedge \exists x\, \psi(x\mathbf{y}))$$
$$\forall x(\varphi(x\mathbf{y}) \to \psi(\mathbf{y})) \leftrightarrow (\exists x\, \varphi(x\mathbf{y}) \to \psi(\mathbf{y}))$$
$$\forall x(\varphi(\mathbf{y}) \to \psi(x\mathbf{y})) \leftrightarrow (\varphi(\mathbf{y}) \to \forall x\psi(x\mathbf{y}))$$

4.1. **Equality.** Let $\mathcal{L}$ be the minimal language with just the equality predicate $x = y$. We already have positive existential quantifier elimination, since we have the most difficult cases

$$\vdash (\exists x \textstyle\bigwedge_{i<n} x = x_i) \leftrightarrow \bigwedge_{i<j<n} x_i = x_j$$

The general case then follows with the intuitionistic equivalences from the beginning of Section 4.

The Kripke model below is fully locally CQC. Its underlying small category **C** needs only one node. As this node we choose an infinite set, say ω. As arrows we have all functions which are almost identities, that is, functions which are equal to the identity map except for at most finitely many elements. (As will be clear later, this is a choice. To get the theory Γ_E below, it suffices if for all arrows f, all a and c, and all finite sets B with $a \notin B$, there is an arrow g satisfying $ga = c$, and g is identical to f on B. So we could choose all arrows onto.) Our Kripke model is the identity functor from **C** to $\mathbf{M}(\mathcal{L})$. We leave it as easy exercises to show that this Kripke model satisfies the following axioms:

E1 $(x = y \to \bigvee_{i<n} u_i = v_i) \to \bigvee_{i<n}(x = y \to u_i = v_i)$
E2 $(x = y \to u = v) \to (u = v \vee (x = u \wedge y = v) \vee (x = v \wedge y = u))$
E3 $\forall x(\bigvee_{i<m} x = x_i \vee \bigvee_{j<n} u_j = v_j) \to \bigvee_{j<n} u_j = v_j$

We allow the indexed disjunctions to be empty. Empty disjunctions equal $\perp$. So E1 includes statement $\neg\neg x = y$ as a special case; and E3 includes statement $\neg\forall x(\bigvee_{i<m} x = x_i)$ as a special case. The reverses of the principal implications of E1 through E3 are tautologies over IQC. Let Γ_E be the theory axiomatized by E1 through E3.

Proposition 4.1. *Over Γ_E, each quantifier free formula is equivalent to a disjunction of conjunctions of atoms.*

Proof. By the intuitionistic proposition-logically valid equivalences from the beginning of Section 4, we may assume that innermost implications are of the form

$$(\bigwedge_{i<m} x_i = y_i) \to \bigvee_{j<n} u_j = v_j$$

Again using the proposition-logical equivalences, we can rewrite this as a nested implication

$$(x_0 = y_0 \to \ldots (x_{m-2} = y_{m-2} \to (x_{m-1} = y_{m-1} \to \bigvee_{j<n} u_j = v_j)) \ldots)$$

Repeated application of E1 and E2 shows that this is equivalent to a disjunction of conjunctions of atoms. An induction argument on the depth of nested implications shows that each quantifier free formula is equivalent, over Γ_E, to a disjunction of conjunctions of atoms. □

In fact, over Γ_E all formulas are equivalent to disjunctions of conjunctions of atoms, because:

Proposition 4.2. *The theory Γ_E admits quantifier elimination.*

Proof. By induction on the complexity of formulas. In the case for $\exists$ we may assume a starting formula of the form

$$\exists x \bigwedge_{i<n} x = x_i$$

This is equivalent to

$$\bigwedge_{i<j<n} x_i = x_j$$

Finally, the case for $\forall$. We may assume a starting formula of the form

$$\forall x(\bigvee_{i<m} x = x_i \vee \bigvee_{j<n} u_j = v_j)$$

Apply E3. □

Applications:

Proposition 4.3. *$\Gamma_E \vdash \varphi$ or $\Gamma_E \vdash \neg\varphi$, for all sentences φ.*
$\Gamma_E \vdash \neg\forall xyuv((x = y \to u = v) \vee (u = v \to x = y))$.

Proof. We only check the second half. Over Γ_E, the expression $(x = y \to u = v) \vee (u = v \to x = y)$ is equivalent to

$$x = y \vee u = v \vee (x = u \wedge y = v) \vee (x = v \wedge y = u)$$

So $\forall v((x = y \to u = v) \vee (u = v \to x = y))$ implies

$$\forall v(x = y \vee u = v \vee y = v \vee x = v)$$

Apply E3. This implies $x = y$. So

$$\Gamma_E \vdash [\forall v((x = y \to u = v) \vee (u = v \to x = y))] \leftrightarrow x = y$$

□

So Γ_E is a very intuitionistic complete theory because inconsistent with AQC. Γ_E is complete, so it is the intuitionistic theory of the one-node Kripke model at the top of this Subsection 4.1.

Proposition 4.4. *$\Gamma_E \vdash \neg\varphi$ or $\Gamma_E \vdash \neg\neg\varphi$, for all formulas φ.*

Proof. By quantifier elimination over Γ_E, each formula φ is equivalent to a disjunction of conjunctions of atoms. Now $\Gamma_E \vdash \neg\neg x = y$. So if at least one conjunction is non-trivial, then $\Gamma_E \vdash \neg\neg\varphi$. Otherwise, trivially $\Gamma_E \vdash \neg\varphi$ or $\Gamma_E \vdash \neg\neg\varphi$. □

Note that Γ_E is the theory of a directed Kripke model.

Here is another Kripke model of Γ_E. As nodes of the underlying small category $\mathbf{C}$, take all finite sets $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$. As arrows between $\mathbf{m}$ and $\mathbf{n}$, take all functions. As Kripke model, take the identity functor to $\mathbf{M}(\mathcal{L})$. We easily verify that this Kripke model satisfies axioms E1 through E3. Completeness of Γ_E implies that it is the theory of this Kripke model.

Obviously, Γ_E satisfies the Constant Domain schema A1. But Γ_E satisfies a much stronger property:

Proposition 4.5. *Γ_E satisfies the schema*

$$\forall x(\varphi(x\mathbf{y}) \vee \psi(x\mathbf{y})) \leftrightarrow (\forall x\,\varphi(x\mathbf{y}) \vee \forall x\,\psi(x\mathbf{y}))$$

Proof. By Proposition 4.1 and quantifier elimination, we may assume that $\varphi(x\mathbf{y})$ and $\psi(x\mathbf{y})$ are disjunctions of conjunctions of atoms. Since IQC satisfies the schema

$$\forall x((\sigma_1(x\mathbf{y}) \wedge \sigma_2(x\mathbf{y})) \vee \tau(x\mathbf{y})) \leftrightarrow [\forall x(\sigma_1(x\mathbf{y}) \vee \tau(x\mathbf{y})) \wedge \forall x(\sigma_2(x\mathbf{y}) \vee \tau(x\mathbf{y}))]$$

we may assume $\varphi(x\mathbf{y})$ and $\psi(x\mathbf{y})$ to be disjunctions of atoms. Apply schema E3. □

4.2. **Linear order.** Let $\mathcal{L}$ be the language with equality $x = y$ and a binary predicate $x \leq y$. The theory of linear order is axiomatizable by

L1 $x \leq x$
L2 $x \leq y \wedge y \leq x \rightarrow x = y$
L3 $x \leq y \wedge y \leq z \rightarrow x \leq z$
L4 $x \leq y \vee y \leq x$

We write Γ_4 for the theory axiomatized by L1 through L4. It satisfies

$$\Gamma_4 \vdash [\exists x(\bigwedge_{i<m} x_i \leq x \wedge \bigwedge_{j<n} x \leq y_j)] \leftrightarrow \bigwedge_{i<m,j<n} x_i \leq y_j$$

By axioms L1 and L2, equality $x = y$ is equivalent to a conjunction $x \leq y \wedge y \leq x$. So over Γ_4 we have positive existential quantifier elimination.

The Kripke model below is fully locally $\mathrm{CQC} \cup \Gamma_4$. Its underlying small category $\mathbf{C}$ has as only node the dense linear order without endpoints, $\mathbb{Q}$. As arrows we have all order preserving maps which are almost identities in the following sense: There are $a < b$ such that the morphism is the identity on all $x \leq a$ and all $x \geq b$. (This is a choice. To get the theory Γ_L below, it suffices if for all arrows f, all a and c, and all finite sets B such that for all $b \in B$, if $a \leq b$ then $c \leq fb$, and if $b \leq a$, then $fb \leq c$, we have a map g such that $ga = c$, and g equals f on B. So we could choose all arrows onto.) Our Kripke model is the identity functor from $\mathbf{C}$ to $\mathbf{M}(\mathcal{L})$. We leave it as easy exercises to show that this Kripke model satisfies Γ_4, plus the axioms:

L5 $(x \leq y \rightarrow \bigvee_{i<n} u_i \leq v_i) \rightarrow \bigvee_{i<n}(x \leq y \rightarrow u_i \leq v_i)$
L6 $(x \leq y \rightarrow u \leq v) \rightarrow ((u \leq v) \vee (u \leq x \wedge y \leq v))$
L7 $\forall x(y \leq x \vee x \leq z \vee \bigvee_{k<p} u_k \leq v_k) \rightarrow (y \leq z \vee \bigvee_{k<p} u_k \leq v_k)$
L8 $\forall x(y \leq x \vee \bigvee_{k<p} u_k \leq v_k) \rightarrow \bigvee_{k<p} u_k \leq v_k$
L9 $\forall x(x \leq z \vee \bigvee_{k<p} u_k \leq v_k) \rightarrow \bigvee_{k<p} u_k \leq v_k$

Let Γ_L be the theory axiomatized by L1 through L9. The reverses of the principal implications of L5 through L9 are tautologies over IQC, or trivially follow from Γ_4.

Proposition 4.6. Γ_L *satisfies the schema*

$$\forall x(\bigvee_{i<m} x_i \leq x \vee \bigvee_{j<n} x \leq y_j \vee \bigvee_{k<p} u_k \leq v_k) \rightarrow (\bigvee_{i<m,j<n} x_i \leq y_j \vee \bigvee_{k<p} u_k \leq v_k)$$

where indexed disjunctions are allowed to be empty, hence to be $\perp$.

Proof. First assume $m > 0$ and $n > 0$. By Γ_4 we have

$$\bigvee_{\sigma,\tau}(\bigwedge_{i<m-1} x_{\sigma(i)} \leq x_{\sigma(i+1)} \wedge \bigwedge_{j<n-1} y_{\tau(j)} \leq y_{\tau(j+1)})$$

where σ and τ range over all permutations on the appropriate index sets. Take this disjunction in conjunction with the left hand side of the implication of this Proposition, and repeatedly apply axiom L7:

$$\bigvee_{\sigma,\tau}(x_{\sigma(0)} \leq y_{\tau(n-1)} \vee \bigvee_{k<p} u_k \leq v_k)$$

If $m > 0$ and $n = 0$ we repeatedly apply L8, and get

$$\bigvee_{\sigma} \bigvee_{k<p} u_k \leq v_k$$

The other cases are similar or easier. □

The reverse of the main implication in the formula of Proposition 4.6, clearly holds over Γ_4.

Proposition 4.7. *Over* Γ_L*, each quantifier free formula is equivalent to a disjunction of conjunctions of atoms.*

Proof. The quantifier free equivalences from the beginning of Section 4, plus L1 and L2, imply that we may assume that innermost implications are of the form

$$(\bigwedge_{i<m} x_i \leq y_i) \rightarrow \bigvee_{j<n} u_j \leq v_j$$

Again using the quantifier free equivalences, we can rewrite this as a nested implication

$$(x_0 \leq y_0 \rightarrow \ldots (x_{m-2} \leq y_{m-2} \rightarrow (x_{m-1} \leq y_{m-1} \rightarrow \bigvee_{j<n} u_j \leq v_j)) \ldots)$$

Repeated application of L5 and L6, shows that this is equivalent to a disjunction of conjunctions of atoms. An induction argument on the depth of nested implications, shows that each quantifier free formula is equivalent, over Γ_L, to a disjunction of conjunctions of atoms. □

Over Γ_L all formulas are equivalent to disjunctions of conjunctions of atoms:

Proposition 4.8. *The theory* Γ_L *admits quantifier elimination.*

Proof. By induction on the complexity of formulas. In the case for $\exists$ we may assume a starting formula of the form

$$\exists x(\bigwedge_{i<m} x_i \leq x \wedge \bigwedge_{j<n} x \leq y_j)$$

This is equivalent to

$$\bigwedge_{i<m,j<n} x_i \leq y_j$$

Finally, the case for $\forall$. We may assume a starting formula of the form

$$\forall x(\bigvee_{i<m} x_i \leq x \vee \bigvee_{j<n} x \leq y_j \vee \bigvee_{k<p} u_k \leq v_k)$$

Apply Proposition 4.6. □

Just as for theory Γ_E of Subsection 4.1, we have

Proposition 4.9. *$\Gamma_L \vdash \varphi$ or $\Gamma_L \vdash \neg\varphi$, for all sentences φ.*
$\Gamma_L \vdash \neg\forall xyuv((x \leq y \to u \leq v) \vee (u \leq v \to x \leq y))$.
$\Gamma_L \vdash \neg\varphi$ or $\Gamma_L \vdash \neg\neg\varphi$, for all formulas φ.

Proof. All quantifier free sentences are equivalent to $\top$ or $\bot$. So, by quantifier elimination over Γ_L, all sentences are equivalent to $\top$ or $\bot$.

Over Γ_L, the expression $(x \leq y \to u \leq v) \vee (u \leq v \to x \leq y)$ is equivalent to

$$(u \leq v) \vee (u \leq x \wedge y \leq v) \vee (x \leq y) \vee (x \leq u \wedge v \leq y)$$

So $\forall v((x \leq y \to u \leq v) \vee (u \leq v \to x \leq y))$ is equivalent to

$$((x \leq y) \vee (x \leq u)) \wedge ((u \leq x) \vee (x \leq y) \vee (u \leq y))$$

So

$$\Gamma_L \vdash \forall uv((x \leq y \to u \leq v) \vee (u \leq v \to x \leq y)) \leftrightarrow x \leq y$$

By quantifier elimination over Γ_L, each formula φ is equivalent to a disjunction of conjunctions of atoms. Now $\Gamma_L \vdash \neg\neg x \leq y$. So if at least one conjunction is non-trivial, then $\Gamma_L \vdash \neg\neg\varphi$. Otherwise, trivially $\Gamma_L \vdash \neg\varphi$ or $\Gamma_L \vdash \neg\neg\varphi$. □

So Γ_L is a very intuitionistic complete theory, and is the theory of the Kripke model at the beginning of this Subsection 4.2.

Here is another Kripke model of Γ_L. As nodes of the underlying small category $\mathbf{C}$, take all finite sets $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$ with their standard ordering. As morphisms between $\mathbf{m}$ and $\mathbf{n}$, take all order preserving maps. As Kripke model, take the identity functor to $\mathbf{M}(\mathcal{L})$. We easily verify that this Kripke model satisfies axioms L1 through L9. Completeness of Γ_L implies that it is the theory of this Kripke model.

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Department of Mathematics, Statistics and Computer Science, Marquette University, P.O. Box 1881, Milwaukee, WI 53201, U.S.A

E-mail address: wimr@mscs.mu.edu