Mapping Characterization Theorems via Model Theory

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Introduction. The basic theme of this talk is the extrinsic description of objects by means of morphisms. One way to do this is to say that all monomorphisms from the object are “special” in some way; the dual of this is to single out epimorphisms to the object.

Injectivity and projectivity are the most familiar to algebraists:

- An abelian group is injective (resp., projective): every monomorphism from it (resp., every epimorphism to it) is a coretraction (resp., retraction).

- A normal topological space is an absolute retract: it is a retract of every normal space in which it is embedded as a closed subset (a kind of injectivity).
And in model theory, we have:

- A model of a universal theory is existentially closed: whenever it is embedded in another model of the theory, existential statements (with parameters in the small model) which are true in the larger model are true in the smaller model as well.

- A weak version of existential closedness for abelian groups, where the existential statements have the form $\exists x \ (nx = a)$, is called absolute purity.
In this talk our objects are continua (= connected compact Hausdorff spaces), and our morphisms are epimorphisms (a.k.a., continuous surjections).

By a *mapping characterization theorem* we mean a proposition that takes the form:

*A continuum $Y$ is in class $\mathcal{K}$ iff whenever $X$ is a continuum and $f : X \to Y$ is an epimorphism in mapping class $\mathcal{F}$, then $f$ is necessarily in mapping class $\mathcal{G}$.***
Continuum theory is rich in its capacity to describe interesting mapping classes—see [J. J. Charatonik and W. J. Charatonik, *Continua determined by mappings*, Pub. de L’Institut Math. 67 (2000), 133-144]. The ones we take up today are defined by what the pre-images of subcontinua look like.

Definition. Let $f : X \to Y$ be an epimorphism between continua, with $K$ an arbitrary subcontinuum of $Y$. Then $f$ is:

- **monotone**: if $f^{-1}[K]$ is a subcontinuum of $X$;

- **semi-monotone** if there is a component $C$ of $f^{-1}[K]$ such that $f[C] = K$ and $f^{-1}[\text{Int}(K)] \subseteq C$;

- **confluent** if $f[C] = K$ for each component $C$ of $f^{-1}[K]$; and

- **weakly confluent** if $f[C] = K$ for some component $C$ of $f^{-1}[K]$. 
(Note: The coinage *semi-monotone* first appears in the topological literature in [P. B., *Defining topological properties via interactive mapping classes*, Top. Proc. 34 (2009), 39-45].)

Here is a schematic of how these properties are implicationally related:

```
(M)monotone
   \simeq
   \simeq
(S)semi-monotone (C)confluent
   \simeq
   \simeq
(W)weakly confluent
   ↓
(E)epimorphic
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For \( \mathcal{F}, \mathcal{G} \in \{M,S,C,W,E\} \), we denote by \( \mathcal{F}\mathcal{G} \) the class of continua \( Y \) such that:
if \( f : X \to Y \) is in class \( \mathcal{F} \), then \( f \) is also in class \( \mathcal{G} \).
Results I. Of the 25 possibilities for $\mathcal{G}$, exactly eleven yield potentially new classes of continua. Five of these we can deal with quickly:

1. CM, WM and EM all comprise the degenerate continua. (Exercise.)

2. WS and CS both comprise the indecomposable continua. (This is easy, given the characterization of indecomposability as the condition that every proper subcontinuum has empty interior. So if $Y$ is decomposable, with $K$ a proper subcontinuum, $U \subseteq K$ a nonempty open set, and $y \in Y \setminus K$, let $X$ consist of two copies of $Y$, “spot-welded” at $y$. Then the obvious projection map $f : X \to Y$ is confluent without being semi-monotone.)
Of the remaining six possible instances of $\mathcal{C}$, EW has received the most attention in the literature; but the known results concern metrizable continua only. Lelek’s designation Class(W) refers to the metrizable members of EW, and Grispolakis-Tymchatyn (1978) have provided interesting characterizations in terms of hyperspace notions.

Once EW is known, ES simply consists of the members of EW that are indecomposable (easy, given (2) above). The two classes are distinct because all arc-like continua are well known to be in Class(W). In particular, arcs are in EW but not in ES.

This leaves EC, WC, SM and SC, which we focus on for the remainder of the talk.
Results II.

Theorem (H. Cook, A. Lelek and D. R. Read). \textit{EC and WC consist of the hereditarily indecomposable continua.}

Remark. In view of this, we see that EC is contained within ES; however the two classes are distinct because the Knaster bucket-handle is arc-like and indecomposable, and so is in ES. It is not hereditarily indecomposable, hence it is not in EC.
Proof. Note first that a continuum is hereditarily indecomposable iff no two of its incomparable subcontinua can intersect.

Fix continuum $Y$, and assume there is an epimorphism $f : X \to Y$ that is not confluent. Then there is a subcontinuum $K$ of $Y$ and a component $C$ of $f^{-1}[K]$ such that $f[C]$ is properly contained in $K$.

Let $y \in K \setminus f[C]$ be fixed. Then $C$ is disjoint from $f^{-1}[y] := f^{-1}[\{y\}]$; and, by “boundary bumping,” there is a subcontinuum $M$ of $X$ such that $C \subseteq M$, $C \neq M$, and $M \cap f^{-1}[y] = \emptyset$. 

Thus \( f[M] \) is a subcontinuum of \( Y \) that intersects \( K \) because it contains \( C \). We have \( y \in K \setminus f[M] \) because \( M \cap f^{-1}[y] = \emptyset \), and we have \( f[M] \setminus K \neq \emptyset \) because \( C \) is maximally connected in \( f^{-1}[K] \). This says that \( Y \) is not hereditarily indecomposable.

If continuum \( Y \) is not hereditarily indecomposable, then there are subcontinua \( M \) and \( N \) with \( M \cap N, M \setminus N, \) and \( N \setminus M \) all nonempty. Fix \( y \in M \setminus N \), and let

\[
X = ((Y \times \{0\}) \cup (M \times \{1\})) / \sim,
\]

where \( \sim \) identifies \( \langle y, 0 \rangle \) and \( \langle y, 1 \rangle \) only. With \( f : X \to Y \) induced by the standard projection, we see that \( f \) is weakly confluent. However, \( (M \cap N) \times \{1\} \) contains components of \( f^{-1}[N] \) that \( f \) does not send onto \( N \); hence \( f \) is not confluent. \( \square \)
Results III.

Theorem (PB). \textit{SM consists of the locally connected continua.}

\textbf{Proof} (Locally connected $\implies$ SM). Suppose $Y$ is a locally connected continuum, with $f : X \to Y$ a semi-monotone map. To show $f$ to be monotone, it suffices to prove that $f^{-1}[y]$ is connected for each $y \in Y$.

Indeed, let $\mathcal{B}$ be a base at $y$, consisting of connected open sets. For each $U \in \mathcal{B}$, use semi-monotonicity to find subcontinuum $C_U$ of $X$ such that $f[C_U] = \overline{U}$ and $f^{-1}[U] \subseteq C_U$.

Then, because $\bigcap \mathcal{B} = \{y\}$, $\mathcal{C} = \{C_U : U \in \mathcal{B}\}$ is a family of subcontinua of $X$ whose intersection is $f^{-1}[y]$. Moreover, it is easy to show that $\mathcal{C}$ is directed downwards: for if $U, V \in \mathcal{B}$, there is some $W \in \mathcal{B}$ with $\overline{W} \subseteq U \cap V$. Then $C_W \subseteq C_U \cap C_V$. By elementary continuum theory, $f^{-1}[y] = \bigcap \mathcal{C}$ is connected. \hfill \Box
Proof Sketch (SM $\implies$ locally connected).

Suppose $Y$ is a continuum that is not locally connected. Our plan is to create an ultracopower $Y_D$, dually analogous with the ultrapowers from model theory, whose canonical codiagonal epimorphism $p_D : Y_D \to Y$ (dually analogous with the canonical ultrapower monomorphism) is not monotone. Codiagonal maps are always semi-monotone (and much more), so this will give us our result.

Since $Y$ is not locally connected, there is a point $x \in Y$ at which $Y$ is not connected im kleinen; i.e., there is an open neighborhood $U$ of $x$ such that for any open neighborhood $V$ of $x$ contained in $U$, there is some $y \in V$ such that no subcontinuum of $U$ contains both $x$ and $y$. 
Fix neighborhood $W$ of $x$ such that $\overline{W} \subseteq U$, and let $\{V_i : i \in I\}$ be an indexed open neighborhood base for $x$, consisting of sets in $W$. By the failure of connectedness *im kleinen* at $x$, we may pick $y_i \in V_i$ such that $y_i$ and $x$ are not in the same component of $W$. Since $\overline{W}$ is a compactum, there is a set $H_i$ that is clopen in $\overline{W}$, contains $y_i$, and doesn’t contain $x$.

For each $i \in I$, let $i^+ := \{j \in I : V_j \subseteq V_i\}$. Then the collection $\{i^+ : i \in I\}$ satisfies the finite intersection property and is hence contained in an ultrafilter $\mathcal{D}$ on $I$. 
The next step is to form the topological ultracopower \( p_D : Y_D \to Y \), and show that \( p_D \) is a semi-monotone mapping that is not monotone.

The ultracopower, along with its canonical ("co-elementary") codiagonal epimorphism, is exactly dual to the model-theoretic ultrapower, along with its canonical elementary monomorphism. Ultracopowers of \( Y \) may be obtained as Stone spaces of ultrapowers of lattice bases for \( Y \); however a purely topological version of \( Y_D \) arises as follows:

Given the diagram

\[
Y \times I \xrightarrow{q} I \\
\downarrow p \\
Y
\]

where \( p \) and \( q \) are the standard projection maps, we apply the Stone-Čech functor to obtain the diagram
Now the ultrafilter $\mathcal{D}$ is a point in $\beta(I)$, and it turns out that $Y_\mathcal{D}$ is canonically homeomorphic to the pre-image of $\mathcal{D}$ under $q^\beta$.

What’s more, the restriction $p_\mathcal{D} := p^\beta|Y_\mathcal{D}$ is a semi-monotone map from $Y_\mathcal{D}$ onto $Y$. (Indeed, for any subcontinuum $K$ of $Y$, the signal component of $p_{\mathcal{D}}^{-1}[K]$ is a canonical copy of $K_\mathcal{D}$ in $Y_\mathcal{D}$.)
The idea at this juncture—details omitted—is to use the sets \( \{H_i : i \in I\} \) and the points \( \{y_i : i \in I\} \) to form a subcompactum \( \sum_D H_i \) of \( \overline{W_D} \) and a point \( \sum_D y_i \in \sum_D H_i \) such that:

- \( \sum_D H_i \) is clopen in \( \overline{W_D} \);

- \( p_D^{-1}[x] \subseteq \overline{W_D} \);

- \( x_D \in p_D^{-1}[x] \setminus \sum_D H_i \); and

- \( \sum_D y_i \in p_D^{-1}[x] \cap \sum_D H_i \).

These four assertions immediately imply that \( p_D^{-1}[x] \) is disconnected; witnessing the fact that \( p_D \) is a non-monotone, semi-monotone mapping onto \( Y \). \( \square \)
Remark 1. The question naturally arises whether a non-locally connected metrizable continuum is the image of a metrizable continuum under a semi-monotone mapping that is not monotone. The answer is yes: more generally, if $Y$ is not locally connected and $p_D : Y_D \to Y$ is constructed as above, then—by means of the Löwenheim-Skolem theorem from model theory—one may obtain a commutative diagram

$$
\begin{array}{c}
Y_D \\ \downarrow p_D \\
Y
\end{array} \xrightarrow{g} 
\begin{array}{c}
X \\ \downarrow f \\
Y
\end{array}
$$

where $f$ is “enough like” $p_D$ to be semi-monotone but not monotone, and $X$ has the same weight as $Y$. 
Remark 2. In the diagram above we start with $Y$ and $\mathcal{D}$, and construct $X$, $f$, and $g$. If, on the other hand, we’re given $f : X \to Y$ and are able to find $\mathcal{D}$ and $g$ making the diagram commute, this situation is exactly dual to the ultrapower characterization of existential embeddings in model theory. In this situation, we call $f$ a co-existential map. Co-existential maps are semi-monotone, but not necessarily confluent [P. B., *Not every co-existentiel map is confluent*, Houston J. Math. 36 (4) (2010), 1233-1242]. And in view of the characterization of SM above, it is clear that every co-existentiel map onto $Y$ is monotone iff $Y$ is locally connected.
So we end with the following two questions:

Question 1. With \( \mathcal{C} \) denoting the class of co-existential maps between continua, what is a suitable (“intrinsically defined”) \( \mathcal{K} \) to characterize \( \mathcal{E}\mathcal{C} \), the co-existentially closed continua?

So far, all we know is that:

- \( \mathcal{K} \) is contained within the class of hereditarily indecomposable continua of covering dimension 1; and

- If \( X \) is any continuum, then \( X \) is a continuous image of a co-existentially closed continuum of the same weight as \( X \).
Question 2. What about SC?

SC clearly contains both SM and WC, so a continuum is in SC if it is either locally connected or hereditarily indecomposable. We don't know whether the containment is proper.
DIOLCH AM WRANDO!