# Mind the Gaps

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## 0. Introduction.

If $[\ ,\ ,\ ]$ is a ternary relation on a set $X$ interpreting a notion of betweenness, we say the structure $\langle X, [\ ,\ ,\ ]\rangle$ is *gap free* if each two elements of $X$ always have a third element between them. This is the first-order sentence

- Gap Freeness:
  $\forall ab(a \neq b \rightarrow \exists x([a,x,b] \wedge x \neq a \wedge x \neq b))$

For example, if we start with a totally ordered set $\langle X, \leq\rangle$ and define $[a,c,b]$ to mean $(a \leq c \leq b) \vee (b \leq c \leq a)$, then gap freeness in this interpretation means order-density.

We take an "inclusive" view of betweenness; meaning that $[a,c,b]$ automatically holds if $c \in \{a,b\}$.

In this talk we are interested in gap free betweenness relations naturally arising in the context of (Hausdorff) continua.

## 1. Three Topological Interpretations.

There are (at least) three interpretations of betweenness in continua deserving mention; they're all closely related.

If $X$ is a continuum, $a, b, c \in X$, and $c \notin \{a, b\}$, we have:

- $[a, c, b]_Q$ iff there's a disconnection $\langle A, B\rangle$ of $X \setminus \{c\}$ such that $a \in A$ and $b \in B$; i.e., $a$ and $b$ lie in different quasicomponents of $X \setminus \{c\}$.

- $[a, c, b]_C$ iff there's no connected $A \subseteq X \setminus \{c\}$ with $a, b \in A$; i.e., $a$ and $b$ lie in different components of $X \setminus \{c\}$; and

- $[a, c, b]_K$ iff there's no continuum $A \subseteq X \setminus \{c\}$ with $a, b \in A$; i.e., $a$ and $b$ lie in different continuum components of $X \setminus \{c\}$.

## 2. Q-gap Freeness.

Clearly $[\,,\,,\,]_Q \subseteq [\,,\,,\,]_C \subseteq [\,,\,,\,]_K$; hence Q-gap free $\Longrightarrow$ C-gap free $\Longrightarrow$ K-gap free.

2.1 Proposition. *If $X$ is an aposyndetic continuum, then $[\,,\,,\,]_K = [\,,\,,\,]_C$. If $X$ is also locally connected, then $[\,,\,,\,]_K = [\,,\,,\,]_Q$.* $\square$

Q-gap freeness is a very strong property.

2.2 Theorem (L. E. Ward). *Q-gap freeness in a continuum implies local connectedness and hereditary decomposability. It is equivalent to the* connected intersection property—*the intersection of any two connected subsets is connected.* $\square$

Ward uses what we call *Q-gap free* as the defining condition for a continuum to be a *tree*. Less overloaded terminology is *dendron*; indeed the metrizable dendrons are the *dendrites*—locally connected and containing no simple closed curves.

Currently we do not know of any literature on the C-interpretation of betweenness, so here is an opportunity to ask some questions:

- Does C-gap freeness imply Q-gap freeness?

- Failing this, are C-gap free continua locally connected? Aposyndetic?

- Or, is there some weakened form of the connected intersection property that characterizes C-gap freeness?

**3. K-gap Freeness.**

Given a continuum $X$ and $a, b \in X$, let $\mathcal{K}(a, b)$ constitute the subcontinua of $X$ that contain both $a$ and $b$. Then the *K-interval* $[a, b]_K$ *bracketed by* $a$ *and* $b$ is defined to be $\bigcap \mathcal{K}(a, b)$. Hence $[a, c, b]_K$ holds iff $c \in [a, b]_K$.

3.1 Proposition. *A continuum is hereditarily unicoherent iff each of its K-intervals is a subcontinuum.* $\square$

Hereditary unicoherence clearly implies K-gap freeness, and it is natural to ask whether this weakening of the connected intersection property is actually a characterization.

The answer turns out to be no.

A continuum $X$ is a *crooked annulus* if it has a decomposition $X = M \cup N$ into subcontinua such that:

- Both $M$ and $N$ are hereditarily indecomposable; and

- $M \cap N = A \cup B$, where $A$ and $B$ are disjoint nondegenerate subcontinua.

3.2 Theorem. *A crooked annulus is K-gap free without being even unicoherent, let alone hereditarily so.* $\Box$

In a crooked annulus one can show that each nondegenerate K-interval $[a, b]_K$ contains two nondegenerate subcontinua, one containing $a$ and the other containing $b$. (E.g., if $a \in A$ and $b \in B$, then $[a, b]_K = A \cup B$.) This clearly gives us K-gap freeness.

## 4. Strong K-gap Freeness.

Recall the first-order statement of gap freeness from above.

- Gap Freeness:
  $\forall ab(a \neq b \rightarrow \exists x([a,x,b] \wedge x \neq a \wedge x \neq b))$

If we replace negations of equality in the conclusion with negations of betweenness, we obtain a stronger property (when betweenness is interpreted properly).

- Strong Gap Freeness:
  $\forall ab(a \neq b \rightarrow \exists x([a,x,b] \wedge \neg[x,a,b] \wedge \neg[a,b,x]))$

With the Q- and the C-interpretations, strong gap freeness is not really stronger than gap freeness because these interpretations satisfy

- Antisymmetry:
  $\forall abc(([a,b,c] \wedge [a,c,b]) \to b = c)$

To see this, suppose $[a,c,b]_C$ and $b \neq c$. If $c = a$ then clearly $\neg[a,b,c]_C$; so assume $c \notin \{a,b\}$. Then there are components $A, B$ of $X \setminus \{c\}$ with $a \in A$ and $b \in B$. Thus $X \setminus B$ is a connected subset of $X \setminus \{b\}$ containing $a$ and $c$; so $\neg[a,b,c]$. The Q-interpretation is antisymmetric as well because it is finer than the C-interpretation.

The *topologist's sine curve* is not K-antisymmetric: if $a$ is any point on the graph of $\sin(1/x)$, $0 < x \leq 1$, and $b$ and $c$ are any two points on the line segment $\{0\} \times [-1,1]$, then both $[a,c,b]_K$ and $[a,b,c]_K$ hold.

By Proposition 2.1, aposyndetic continua are K-antisymmetric. However, the *comb space* is K-antisymmetric without being aposyndetic.

Recall Ward's result that Q-gap freeness in continua is equivalent to the connected intersection property. This property automatically implies both local connectedness and hereditary decomposability, but its weaker cousin hereditary unicoherence does not. (E.g., any *pseudo-arc* is hereditarily unicoherent.) And while it is an open problem whether hereditary unicoherence has a first-order characterization ever so slightly stronger than K-gap freeness, we have the following.

4.1 Theorem. *Strong K-gap freeness in a continuum is equivalent to the continuum's being both hereditarily unicoherent and hereditarily decomposable.*

Proof. The easy direction is to assume the conjunction of hereditary unicoherence and hereditary decomposability. For then each nondegenerate K-interval is connected, by Proposition 3.1, hence it is a decomposable continuum. Any point in the intersection of a decomposition of a K-interval witnesses strong K-gap freeness.

For the opposite direction, let $X$ be a strongly K-gap free continuum. If $M$ and $N$ are subcontinua with $M \cap N = A \cup B$, where $A$ and $B$ are nonempty, closed, and disjoint, we use Zorn's lemma to find $a \in A$, $b \in B$ such that if $a' \in A$, $b' \in B$, and $[a', b']_K \subseteq [a, b]_K$, then $[a', b']_K = [a, b]_K$.

So we use strong K-gap freeness to find $c \in [a, b]_K$ such that both $[a, c]_K$ and $[c, b]_K$ are proper subsets of $[a, b]_K$. But either $c \in A$ or $c \in B$, and this contradicts the minimality of $[a, b]_K$.

Thus we infer that $X$ is hereditarily unicoherent; next we tackle hereditary decomposability.

Suppose $Y$ is a nondegenerate indecomposable subcontinuum of $X$. Then, by a result of D. Bellamy, $Y$ contains an indecomposable subcontinuum with more than one composant; hence we may (WLOG) assume $Y$ itself is irreducible about some doubleton set $\{a, b\}$. But $[a, b]_K$ is a subcontinuum, by hereditary unicoherence, so $Y = [a, b]_K$. Now, by strong K-gap freeness, there is some $c \in [a, b]_K$ such that both $[a, c]_K$ and $[c, b]_K$ are proper subsets of $[a, b]_K$. By hereditary unicoherence again, we infer that $Y$ is decomposable, a contraduction. $\square$

## 5. Extra Strong K-gap Freeness.

By *extra strong gap freeness* in an interpretation of betweenness we mean that both gap freeness and antisymmetry hold.

5.1 Theorem. *Extra Strong K-gap freeness in a continuum is equivalent to saying that all the continuum's nondegenerate K-intervals are (Hausdorff) arcs.* $\square$