Identifying a Topological Graph Using First-order Lattice Properties

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1. Introduction

1.1. Lattice Bases

A base for the closed sets of a topological space is a lattice base if it is also closed under finite unions and intersections.

In this talk, our interest is in recovering a compact Hausdorff space (compactum) from the first-order information about any one of its lattice bases.
1.2. A Representation Theorem

The model theory of lattices works remarkably well in the topological setting of compact Hausdorff spaces, largely because of the work of M. H. Stone in the 1930s.

(Representation Theorem) 
*The class of (isomorphic copies of) lattice bases for compacta is definable by a single first-order sentence in the signature of bounded lattices.*
1.3. Elementary Similarity

Two lattices are *elementarily equivalent* if they satisfy the same first-order sentences (properties).

By the representation theorem above, if $A$ is a lattice base for a compactum and $B$ is elementarily equivalent to $A$, then $B$ is also (isomorphic to) a lattice base for a compactum. (A lattice base determines a compactum, but not conversely.)

Two spaces are *elementarily similar* if some lattice base for one is elementarily equivalent to some lattice base for the other. (We don’t claim elementary similarity is an equivalence relation.)
2. The Main Theorem

Let $G$ be a topological graph; suppose $X$ is a locally connected metrizable compactum that is elementarily similar to $G$. Then $X$ and $G$ are homeomorphic.

The proof for arbitrary graphs is fairly technical, involving the ultracoproduct construction; as well as the notion of “$G$-cover,” designed to simulate the decomposition of a graph into constituent arcs. Details are contained in my yet-to-appear paper, “Categoricity and topological graphs.”
3. An Illustrative Special Case: Arcs

3.1. Why Local Connectedness?

In [“Reduced coproducts of compact Hausdorff spaces,” 1987] I posed the question whether any metrizable compactum elementarily similar to an arc is itself an arc. (The statement is true for Cantor sets.)

The answer, in the negative, was not long in coming.

Reuven Gurevič [“On ultracoprodacts of compact Hausdorff spaces,” 1988]:

*There is a metrizable continuum, elementarily similar to an arc, which is not locally connected.*
3.1. Why local connectedness (cont.)

But there is a positive answer if local connectedness is assumed.

[“Model-theoretic characterizations of arcs and simple closed curves,” 1988]

Let $G$ be either an arc or a simple closed curve; suppose $X$ is a locally connected metrizable compactum that is elementarily similar to $G$. Then $X$ and $G$ are homeomorphic.
3.1. Why local connectedness (cont.)

(Remark) A major step in the proof of this result is based on work of R. L. Moore in the 1920s:

[“Foundations of Point Set Theory,” 1962]

If $X$ is a locally connected metrizable continuum that is neither an arc nor a simple closed curve, then $X$ contains a simple triod.
3.2. Why Metrizability?

In model theory there is the famous result of Löwenheim-Skolem, an “upward” form of which is:

If $A$ is a countably infinite relational structure, then there is an uncountable relational structure $B$ that is elementarily equivalent to $A$. 
3.2. Why metrizability (cont.)

In the compact Hausdorff situation, the right analogue for cardinality of underlying set is weight; and countable weight in this context means metrizability.

There are contexts in model theory where other impositions on structure obviate the need for restrictions on cardinality.

For example, if \( L \) is any wellordering that is elementarily equivalent to the order type \( \omega \) of the natural numbers, then \( L \) is order isomorphic to \( \omega \).
3.2. Why metrizability (cont.)

In view of this, it makes sense to ask whether, in the presence of local connectedness, metrizability is a necessary condition to add.

It is.

[“Co-elementary equivalence, co-elementary maps, and generalized arcs,” 1997]

Any continuum with exactly two noncut points is locally connected, but is also elementarily similar to an arc. (There are easy examples of such spaces of any desired weight.)
4. Dealing with Branch Points

Now, with the necessity of both metrizability and local connectedness established so that compacta may be realistically compared to a given graph, the main difficulty in going beyond arcs and simple closed curves is the issue of branch points.

The next breakthrough came with the study of “co-elementary” maps, topological analogues of elementary embeddings in model theory.
4. Dealing with branch points (cont.)

[“Dendrites, graphs, and 2-dominance,” 200?]

Let $G$ be a simple $n$-od ($n \geq 3$); suppose $X$ is a locally connected metrizable compactum that is elementarily similar to $G$. Then $X$ and $G$ are homeomorphic.
5. Open Questions

5.1. Extending the Results

Can one extend the “categoricity” results above to other locally connected compacta? For example, the techniques developed so far do not even handle dendrites, let alone such familiar higher-dimensional continua as spheres and cells.

In the case of the 2-sphere and the 2-cell, there are well-known characterizations in terms of everyday topological concepts. Can these be translated into first-order conditions on lattice bases?
5.2. Failure of Categoricity

We are still lacking an example of two elementarily similar locally connected metrizable continua that fail to be homeomorphic.
Given the difficulty of proving positive results for spaces as rudimentary as graphs, it shouldn’t be so hard to find such an example.