An Ultracoproduct Mapping Theorem, with Applications to Continua

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1. The Ultracoproduct Construction (topological version)

• Begin with an indexed collection \( \langle X_i : i \in I \rangle \) of compact Hausdorff spaces, and an ultrafilter \( \mathcal{D} \) on \( I \).

• Let \( X := \bigcup_{i \in I} (X_i \times \{i\}) \) (=disjoint union), with \( q : X \to I \) the obvious projection map onto the discrete space \( I \).

• Let \( \beta(q) : \beta(X) \to \beta(I) \) be the Stone-Čech lift of \( q \).

• The \( \mathcal{D} \)-ultracoproduct \( \sum_{\mathcal{D}} X_i \) is then the pre-image of \( \mathcal{D} \in \beta(I) \) under \( \beta(q) \). If the spaces \( X_i \) are continua, the ultracoproducts constitute the components of \( \beta(X) \).
2. The Ultracoproduct Construction (model-theoretic version)

- Begin with an indexed collection \( \langle X_i : i \in I \rangle \) of compact Hausdorff spaces, and an ultrafilter \( D \) on \( I \).

- For each \( i \in I \), let \( A_i \) be a lattice base for \( X_i \); i.e., a base for, as well as a sublattice of, the closed set lattice \( F(X_i) \).

- Let \( A \) be the model-theoretic ultraprod-\( uct \prod_D A_i \), a distributive lattice that satisfies the conditions of normality (i.e., usual topological normality expressed in closed-set terms) and disjunctiveness (i.e., for any two elements, there is a third, not bottom, that is contained in one and disjoint from the other).

- \( \sum_D X_i \) is then the maximal spectrum space \( S(A) \). [This definition is independent of original choice of lattice bases \( A_i \).]
3. Historical Remarks

- We first conceived of the ultracoproduct in the mid-'70s as an inverse limit of coproducts (in the category of compact Hausdorff spaces and continuous maps); that is, spaces of the form $\beta(\bigcup_{i \in J}(X_i \times \{i\}))$. This presented the construction as a dual to the ultraproduct, viewed by many algebraists as a direct limit of products. We eventually came to the versions outlined above, focusing mainly on the one involving ultraproducts of lattices.
At about the same time (mid ’70s), Jerzy Mioduszewski conceived of the ultracoproduct in its “β-version” above; especially in the case where $I$ is countable and each $X_i$ is a copy of the closed unit interval. This idea proved valuable in the study of the continuum structure of the Stone-Čech remainder of the half-open interval. Several authors, including Eric van Douwen, Michel Smith, and Jian-Ping Zhu, have made important contributions to this study using (what we call) ultracoproducts. [Reference: K. P. Hart, “The Čech-Stone compactification of the Real line”, 1992]
4. Ultracopowers and Co-existential Maps

• Let $X$ be a compact Hausdorff space, $I$ an infinite set, $\mathcal{D}$ an ultrafilter on $I$. Then we have the ultracopower $X I \backslash \mathcal{D} = (\beta(q))^{-1}[\mathcal{D}]$, where (as above) $q : X \times I \to I$ is projection. (So all the spaces $X_i$ are just $X$.)

• We also have the projection $p : X \times I \to X$, and the restriction $p_{\mathcal{D}}$ of $\beta(p)$ to the subspace $X I \backslash \mathcal{D}$ is a continuous map onto $X$. This is the canonical co-diagonal map and is the dual of the diagonal embedding from any relational structure into its ultrapower.

• A continuous map $f : X \to Y$ between compact Hausdorff spaces is called co-existential if there is some continuous surjection $g : Y I \backslash \mathcal{D} \to X$ such that $f \circ g = p_{\mathcal{D}}$. 
5. Basic Properties of Co-existentia Maps

- A co-existential map \( f : X \to Y \) between compacta is weakly confluent. Moreover, if \( Y \) is locally connected, then \( f \) is monotone.

- Co-existential maps between compacta never raise covering dimension.

- Co-existential maps between continua preserve: being indecomposable, being hereditarily indecomposable, being hereditarily decomposable; but not being decomposable.

- Let \( X \) be a co-existentially closed continuum; i.e., a continuum that is only a co-existentual image of other continua. Then \( X \) is hereditarily indecomposable, of covering dimension one. [Every nondegenerate continuum is a continuous image of a co-existentially closed continuum of the same weight.]
6. Some Open Questions

- Are co-existential maps between compacta confluent?

- Is being one-dimensional and hereditarily indecomposable sufficient for a continuum to be co-existentially closed?

- Is the pseudo-arc a co-existentially closed continuum?
7. The Advertised Mapping Theorem

- Let $X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \ldots$ be an $\omega$-indexed inverse system of compacta and surjective bonding maps, and let $\mathcal{D}$ be a nonprincipal ultrafilter on $\omega$. Then $\lim_{\mathcal{D}} X_n$ is a co-existential image of $\sum_{\mathcal{D}} X_n$.

- In the early 1990s, both M. Smith and J.-P. Zhu independently proved that ultracopowers of $[0,1]$ via nonprincipal ultrafilters on $\omega$ contain (an abundance of) indecomposable subcontinua. What follows is a generalization of a somewhat weaker version of this result.
8. An Application

- Let $X$ be any nondegenerate Peano continuum, $\mathcal{D}$ a nonprincipal ultrafilter on $\omega$. Then $X\omega\setminus\mathcal{D}$ contains indecomposable subcontinua.

Proof: Let $f : X \to [0, 1]$ be a continuous surjection, let $g : [0, 1] \to [0, 1]$ be a “tent” map, and let $h : [0, 1] \to X$ be another continuous surjection (because $X$ is a Peano continuum). Then $k = h \circ g \circ f : X \to X$ is an indecomposable map; i.e., if $X = A \cup B$ is a decomposition of $X$ into subcontinua, then one of $k[A], k[B]$ is all of $X$. Thus if $Y$ is the inverse limit of the system $X \xleftarrow{k} X \xleftarrow{k} \ldots$, then $Y$ is an indecomposable continuum. On the other hand, we have a co-existential map from $X\omega\setminus\mathcal{D}$ to $Y$. Since co-existential maps are weakly confluent, and so preserve hereditary decomposability, we conclude that the ultracopower contains indecomposable subcontinua. \qed
9. Some Closing Remarks

- So $X_\omega \setminus \mathcal{D}$ contains an indecomposable sub-continuum if $X$ admits an indecomposable self-map. A new characterization of being an indecomposable continuum is being a continuum that admits an indecomposable self-map that is co-existential.

- The mapping theorem presented above has a version that covers systems $X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \ldots$ where the bonding maps are co-existential. In such a situation, the covering dimension of the inverse limit is the supremum of the covering dimensions of the individual spaces.