Topological Betweenness Relations

Paul Bankston

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0. Background. My interest in betweenness relations was sparked a few years ago when I was visiting Dugald Macpherson. I was explaining hereditarily indecomposable continua, where the family of subcontinua have a tree structure, and he was reminded of the Adeleke-Neumann paper [*Relations related to betweenness: their structure and automorphisms*, Mem. Amer. Math. Soc. **131** (1998)] which he had reviewed some years earlier. It turns out one can describe a natural betweenness relation for the branch sets of trees, as well as for the points of a Hausdorff continuum; and in this case the two notions coincide. This encounter led me to consider betweenness from an order-theoretic point of view as well as a topological, and further led me to the formulation of the road system approach.

1. Betweenness via Road Systems. We take the intuitive view that a point c lies between points a and b exactly when every "road" allowing travel from a to b (and *vice versa*) must go through c.

This "roadblock" view of betweenness leads to the following simple abstract definition:

- A *road system* is a pair $\langle X, \mathcal{R} \rangle$, where X is a nonempty set and $\mathcal{R}$ is a family of subsets of X, called *roads*, satisfying:
  - Every singleton subset of X is a road.
  - Every doubleton subset of X is contained in at least one road.

- A road system is *additive* if the union of two overlapping roads is also a road; it is *separative* if when $a, b, c \in X$ and $b \neq c$, either there is a road containing a, b but not c, or there is a road containing a, c but not b.

If $\langle X, \mathcal{R} \rangle$ is a road system, the induced *betweenness relation* is given by stipulating $[a, c, b]_{\mathcal{R}}$ just in case $c \in R$ for each $R \in \mathcal{R}(a, b) := \{R \in \mathcal{R} : a, b \in R\}$. The $\mathcal{R}$-*interval* $[a, b]_{\mathcal{R}}$ is the set of points *between* a *and* b, defined to be the intersection $\bigcap \mathcal{R}(a, b)$.

The $\mathcal{R}$-betweenness relation $[a, c, b]_{\mathcal{R}}$ defines a ternary relation on the underlying set X; a natural question is whether one may characterize—using first-order terms involving an abstract ternary relation symbol—exactly when a ternary relation $[\ ,\ ,\ ] \subseteq X^3$ is the betweenness relation induced by a road system on X.

This question has an affirmative answer.

1.1 Theorem (Road Representation). *Let* [, ,] *be a ternary relation on a nonempty set* X. *Then* [, ,] *is induced by a road system on* X *iff the following four first-order conditions hold:*

R1 (Symmetry) $[a, c, b] \to [b, c, a]$.

R2 (Reflexivity) $[a, b, b]$.

R3 (Minimality) $[a, c, a] \to c = a$.

R4 (Convexity) $([a, c, b] \wedge [a, d, b] \wedge [c, x, d]) \to [a, x, b]$.

A ternary relaton satisfying R1—R4 is called an *R-relation*.

1.2 Theorem (Road Representation Continued). *Additive road systems also satisfy*

R5 (Disjunctivity) $[a, x, b] \rightarrow ([a, x, c] \vee [c, x, b])$,

and separative road systems also satisfy

R6 (Antisymmetry) $([a, c, b] \wedge [a, b, c]) \rightarrow b = c$.

If $\langle X, [\ ,\ ,\ ] \rangle$ *is an R-relation that is disjunctive (resp., antisymmetric, both disjunctive and antisymmetric), then it is induced by a road system that is additive (resp., separative, both additive and separative).*

2. Betweenness in Trees. A partial ordering $\langle Y, \leq \rangle$ is a *tree ordering* if each pair of elements has a common lower bound, and no two $\leq$-incomparable elements have a common upper bound. If $d \leq e$ in Y, let $[d, e]_O$ be $\{y \in Y : d \leq y \leq e\}$. Then we define $\mathcal{R}(a, b)$ to consist of sets $V(a, b, d) := [d, a]_O \cup [d, b]_O$, where $d \leq a, b$; and the resulting family $\mathcal{R}$ turns out to be an additive, separative road system.

Another betweenness relation—and this was studied extensively by Adeleke-Neumann—results when we take X to be the set of branches of a tree Y. If $\alpha, \beta, \gamma \in X$, then define $[\alpha, \gamma, \beta]$ to hold just in case $\alpha \cap \gamma \supseteq \alpha \cap \beta$. In terms of road systems, we define $\mathcal{R}$ to consist of sets of the form $V(a) := \{\alpha \in Y : a \in \alpha\}$, for $a \in Y$. Here $\mathcal{R}$ is an additive road system that generally fails to be separative.

3. Betweenness in Topology. There are at least three notions of betweenness in topology that deserve mention; the *C-interpretation*, the *Q-interpretation*, and the *K-interpretation*. We briefly summarise what's going on with the first two and spend a little more time on the third.

- (The C,Q-Interpretations) For a connected topological space X, define $[a, c, b]_C$ (resp., $[a, c, b]_Q$) to hold if either $c \in \{a, b\}$ or a and b lie in different components (resp., quasicomponents) of $X \setminus \{c\}$. The C-interpretation is induced by the additive, separative road system $\mathcal{C}$ consisting of the connected subsets of X; the Q-interpretation is also induced by an additive, separative road system $\mathcal{Q}$ that is generally different from $\mathcal{C}$. (The Road Representation Theorem 1.2 comes to the rescue.) Generally the Q-interpretation is more restrictive than the C-interpretation; the two coincide in the presence of local connectedness.

A point $c \in X$ is a cut point of X precisely when it lies properly between two other points of X, in either the C- or the Q-interpretation.

- (The K-Interpretation) A connected topological space is a *continuum* if it is also compact; a *subcontinuum* of a space is a subset that is a continuum in its subspace topology. A space is *continuumwise connected* if each two of its points are contained in a subcontinuum.

  We now start with a continuumwise connected topological space X and define $[a, c, b]_K$ to hold if either $c \in \{a, b\}$ or a and b lie in different continuum components of $X \setminus \{c\}$—i.e., no subcontinuum of $X \setminus \{c\}$ contains both a and b. The K-interpretation is induced by the additive road system $\mathcal{K}$ consisting of the subcontinua of X. In continuumwise connected spaces, the K-interpretation is generally less restrictive than the C-interpretation, and generally fails to be separative. A point $c \in X$ is a *weak* cut point—i.e., its complement is not continuumwise connected—precisely when it lies properly between two other points of X, in the K-interpretation.

4. The K-Interpretation in Hausdorff Continua. If X is a Hausdorff continuum, then $(\mathcal{K})$-intervals are closed, and hence subcompacta. When they're also subcontinua, we call X *interval connected*.

For example, arcs are interval connected, as are dendrites in general. The $\sin(\frac{1}{x})$-continuum is another example. At the opposite extreme, in a simple closed curve any interval $[a, b]$ consists of the bracketing points a, b alone. Such intervals, when $a \neq b$, are called *gaps*.

Recall that a Hausdorff continuum is *hereditarily unicoherent* if the intersection of any two of its overlapping subcontinua is a subcontinuum.

4.1 Proposition: *A Hausdorff continuum is interval connected iff it is hereditarily unicoherent.*

5. A Characterization Problem. The issue we wish to focus on here concerns the question of characterizing—in first-order betweenness terms—the property of being interval connected.

This question is not yet answered, but here are some plausible characterization sentences, listed in order of nondecreasing logical strength.

(Gap-free Property)
$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge c \neq a \wedge c \neq b)]$

(Semi-strong Gap-free Property)
$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge c \neq a \wedge b \notin [a, c])]$

(Strong Gap-free Property)
$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge a \notin [c, b] \wedge b \notin [a, c])]$

The gap-free property clearly follows from interval connectedness; and, using a simple "boundary bumping" argument, we can show that the semi-strong gap-free property does as well. Not so the strong gap-free property.

5.1 Theorem. *A Hausdorff continuum satisfies the strong gap-free property iff each of its nondegenerate intervals is a decomposable subcontinuum.*

And when we strengthen gap-freeness in a completely different way, we get an even stronger condition on intervals. Recall that an R-relation is *antisymmetric* if it satisfies R6 above. In terms of intervals, this says that $[a, b] = [a, c]$ implies $b = c$.

This is clearly a first-order property, which is present in Hausdorff continua that are *aposyndetic*; i.e., any one of two points lies in the interior of a subcontinuum not containing the other.

5.2 Theorem. *A Hausdorff continuum is antisymmetric and satisfies the gap-free property iff each of its nondegenerate intervals is a generalized arc.*

6. The Crooked Annulus. A Hausdorff continuum is *hereditarily indecomposable* if the intersection of any two of its overlapping subcontinua is one or the other of them. The celebrated pseudo-arc is an example of this phenomenon. The family of subcontinua of a hereditarily indecomposable Hausdorff continuum forms a tree under reverse inclusion. The branches of this tree are the points of the continuum, and the betweenness relations, one from the tree structure, the other from the subcontinuum structure, are coincident.

The strong gap-free property is too strong to characterize interval connectedness in general because hereditarily indecomposable continua are hereditarily unicoherent; hence intervals are indecomposable subcontinua.

But the ever so slightly weaker semi-strong gap-free property is *too* weak.

Define a continuum X to be a *crooked annulus* if it may be decomposed as a union $K \cup M$ of two hereditarily indecomposable subcontinua such that $K \cap M$ has exactly two components, each nondegenerate.

6.1 Theorem. *Every crooked annulus satisfies the semi-strong gap-free property, while failing to be interval connected.*

Some remarks: Let $X = K \cup M$, where K, M are subcontinua such that $K \cap M$ is a union $A \cup B$ of disjoint nondegenerate subcontinua.

(1) If $a \in A$ and $b \in B$, then $[a, b]$ is clearly not connected.

(2) If H is a subcontinuum of X that intersects both K and M, and if C is a component of H in K, then C intersects M. ("Boundary bumping," just uses fact that $X = K \cup M$.)

Now assume that both K and M are hereditarily indecomposable.

(3) If H is a subcontinuum of X that intersects both A and B, then $A \cup B \subseteq H$.

(4) Hence, if $a \in A$ and $b \in B$, then $[a, b] \supseteq A \cup B$. (In fact, they're equal.)

(5) In general, we show X satisfies semi-strong gap-freeness by proving that, no matter where a, b lie in X, $[a, b]$ is either connected, or contains two nondegenerate disjoint subcontinua, one containing a, the other containing b.

(6) A crooked annulus also satisfies another consequence of being interval connected, namely the *centroid property*: for any $a, b, c \in X$, $[a, b] \cap [a, c] \cap [b, c] \neq \emptyset$.

7. Proof Outline for 5.1.

(1) If $[a, b]$ decomposes into $K \cup M$, both proper subcontinua, then any $c \in K \cap M$ witnesses that the strong gap-free property holds.

(2) If the strong gap-free property holds and intervals are connected, then the non-degenerate ones are easily seen to be decomposable.

(3) If A and B are disjoint nonempty closed subsets of X, a Zorn's lemma argument allows you to find $a \in A$ and $b \in B$ such that for any $a' \in A$, $b' \in B$, if $[a', b'] \subseteq [a, b]$, then $[a', b'] = [a, b]$. (a and b are *minimally close*).

(4) In the absence of interval connectedness, we have subcontinua K, M with $K \cap M = A \cup B$, where A and B are closed, nonempty, and disjoint. Let $a \in A$ and $b \in B$ be minimally close (relative to A, B). If $c \in [a, b]$, then either $c \in A$ or $c \in B$. In the first case $[c, b] = [a, b]$; in the second $[a, c] = [a, b]$. Thus the strong gap-free property fails for X.

8. Setwise Betweenness. This work is ongoing, and joint with Aisling McCluskey and Brian McMaster.

Suppose there are many roads from point a to point b. One roadblock may not be enough to prevent "the bankrobbers' getting from San Diego to Tijuana".

So, in the language of road systems, we define a *set* C to lie between points $a, b \in X = \langle X, \mathcal{R} \rangle$ just in case $C \cap R \neq \emptyset$ for all $R \in \mathcal{R}(a, b)$.

In the setting of Hausdorff continua, we restrict attention to subsets C that are closed. For example, suppose we consider only sets C, where $1 \leq |C| \leq 2$.

We denote by $\mathcal{F}_2(X)$ the collection of nonempty subsets of X of cardinality ≤ 2, equipped with the Vietoris topology: For any set $A \subseteq X$, let $\langle A \rangle^+ := \{C \in \mathcal{F}(X) : C \cap A \neq \emptyset\}$, $\langle A \rangle^- := \{C \in \mathcal{F}(X) : C \subseteq A\}$. The sets $\langle U \rangle^+$ and $\langle U \rangle^-$, where U is open in X, constitute a subbase for a topology that makes $\mathcal{F}_2(X)$ into a Hausdorff continuum whenever the same is true for X.

We then define $[a, b]_2 := \{C \in \mathcal{F}(X) : C \cap R \neq \emptyset$ for all $R \in \mathcal{K}(a, b)\}$. A Hausdorff continuum is *2-interval connected* if $[a, b]_2$ is connected for all $a, b \in X$.

The current problem we are trying to solve is to characterize when a Hausdorff continuum is 2-interval connected. Can this characterization be accomplished in first-order terms involving the obvious quaternary predicate?

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