A Road System Interpretation of Betweenness

Paul Bankston

(joint with Aisling McCluskey and Brian McMaster)

Mathematics Seminar,
NUI Galway,
22 Nov., 2012

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1. Betweenness via Road Systems. We take the intuitive view that a point c lies between points a and b exactly when every "road" allowing travel from one point to the other must go through c. I.e., c constitutes a "complete roadblock".

An antecedent concept that unifies most of the known examples of betweenness is that of a *road system*. This is a pair $\langle X, \mathcal{R} \rangle$, where X is a nonempty set and $\mathcal{R}$ is a family of subsets of X, called *roads*, satisfying:

- Every singleton subset of X is a road. (To get from a point to itself, stay put.)

- Every doubleton subset of X is contained in at least one road. (You can always get there from here.)

Given a road system $\langle X, \mathcal{R} \rangle$ and points $a, b \in X$, let $\mathcal{R}(a,b) := \{R \in \mathcal{R} : a, b \in R\}$; i.e., the set of roads "joining" a and b. Then the system naturally induces a ternary (betweenness) relation $[\ ,\ ,\ ]_{\mathcal{R}} \subseteq X^3$, given by stipulating $[a, c, b]_{\mathcal{R}}$ just in case $c \in R$ for each $R \in \mathcal{R}(a,b)$. The $\mathcal{R}$*-interval* $[a, b]_{\mathcal{R}}$ is the set of points *between* a *and* b; i.e., $[a, b]_{\mathcal{R}} := \bigcap \mathcal{R}(a,b)$. Points a and b *bracket* the interval. (Note: an interval may have many sets of bracketing points.)

A ternary relation that is induced by a road system is called an *R-relation*.

A natural question is whether one may characterize—using first-order terms involving an abstract ternary relation symbol—exactly when a ternary relation $[\ ,\ ,\ ] \subseteq X^3$ is an R-relation.

This has an easy affirmative answer.

1.1. Theorem (Road Representation). *Let* $[\ ,\ ,\ ]$ *be a ternary relation on a nonempty set* X. *Then* $[\ ,\ ,\ ]$ *is an R-relation on* X *iff the following five first-order conditions hold:*

R1 (Symmetry) $[a, c, b] \to [b, c, a]$
($[a, b] = [b, a]$).

R2 (Reflexivity) $[a, b, b]$
($[a, b] \supseteq \{a, b\}$).

R3 (Minimality) $[a, c, a] \to c = a$
($[a, a] = \{a\}$).

R4 (Transitivity) $([a, c, b] \wedge [a, x, c]) \to [a, x, b]$
($c \in [a, b] \implies [a, c] \subseteq [a, b]$).

R5 (Convexity) $([a, c, b] \wedge [a, d, b] \wedge [c, x, d]) \to [a, x, b]$ ($c, d \in [a, b] \implies [c, d] \subseteq [a, b]$).

R4 is an easy consequence of R5 (plus R1 and R2): just replace d with a. Why we include R4 is to point out that any ternary relation $[\ ,\ ,\ ]$ satisfying R1,...,R4 gives rise to a family $\{\leq_a : a \in X\}$ of binary relations: $x \leq_a y$ holds exactly when $x \in [a, y]$. Each $\leq_a$ is reflexive (R2) and transitive (R4), and is hence a pre-order. Also the conjunction of R1, R2, and R3 says that a is the unique $\leq_a$-minimal element of X. We'll see later how these relations $\leq_a$ turn out to be partial orders.

R-relations clearly satisfy R1,...,R5; and if $[\ ,\ ,\ ] \subseteq X^3$ satisfies R1,...,R5, we simply let $\mathcal{R}$ be the set of intervals $[a, b] := \{x \in X : [a, x, b]$ holds$\}$. This works (but R1,...,R4 is not enough).

2. Classical Antecedents of Betweenness.

2.1. Doubly Directed Partial Orders. A partially ordered set $\langle X, \leq \rangle$ is *doubly directed* if each pair of elements is commonly bounded both above and below. The *D-interpretation of betweenness* in a doubly directed set is given by saying $[a, c, b]_D$ just in case $d \leq c \leq e$ for every $d \leq a, b$ and $e \geq a, b$. By setting $\mathcal{R}$ to comprise the order intervals $[d, e]_O = \{x \in X : d \leq x \leq e\}$ for $d \leq e$ in X, we immediately have $[\ ,\ ,\ ]_{\mathcal{R}} = [\ ,\ ,\ ]_D$.

2.2. Trees. A partially ordered set $\langle X, \leq \rangle$ is a *tree ordering* if each pair of elements has a common lower bound, and no two $\leq$-incomparable elements have a common upper bound. The *T-interpretation of betweenness* is given by saying $[a, c, b]_T$ holds just in case there is a common lower bound $d \leq a, b$ such that $d \leq c \leq a$ or $d \leq c \leq b$. This is clearly an R-relation; it is induced by the road system $\mathcal{R}$, where $\mathcal{R}(a, b)$ comprises the sets $V(a, b, d) := [d, a]_O \cup [d, b]_O$, $d \leq a, b$.

2.3. Real Vector Spaces. If X is a vector space over the reals and $a, b \in X$, the *S-interpretation of betweenness* is given by saying $[a, c, b]_S$ holds just in case $c = ta + (1 - t)b$ for some $0 \leq t \leq 1$. The segments $[a, b]_S$ constitute a road system inducing $[\ ,\ ,\ ]_S$.

2.4. Metric Spaces. If $\langle X, \rho \rangle$ is a metric space, then the *M-interpretation of betweenness* has $[a, c, b]_M$ holding just when $\rho(a, b) = \rho(a, c) + \rho(c, b)$. The collection $\mathcal{R}$, consisting of the metric intervals $[a, b]_M$, is clearly a road system, but it may fail to induce $[\ ,\ ,\ ]_M$: K. Menger (1928) showed that, while R1,...,R4 hold for the M-interpretation, R5 does not; it is possible for $\bigcap \mathcal{R}(a, b)$ to be properly contained in $[a, b]_M$.

In a 1943 paper, M. F. Smiley considers betweenness relations that arise from multiple antecedent structures on the same underlying set. For instance, if a real vector space X has a norm $\| \ \|$, then X has an S-interpretation as well as an M-interpretation of betweenness (where $\rho(a, b) = \|a - b\|$).

Betweenness in the S-sense generally implies betweenness in the M-sense, but the two notions agree exactly when the normed space is *rotund*: $\|a + b\| = \|a\| + \|b\|$ holds for nonzero a, b iff one of the two is a nonnegative scalar multiple of the other.

3. Additive Road Systems. A road system is *additive* if the union of two intersecting roads is a road. Additive road systems satisfy:

R6 (Disjunctivity) $[a, x, b] \to ([a, x, c] \vee [c, x, b])$ $([a, b] \subseteq [a, c] \cup [c, b])$.

(To see this, suppose $\langle X, \mathcal{R} \rangle$ is additive and $x \notin [a, c]_{\mathcal{R}} \cup [c, b]_{\mathcal{R}}$. Then there are $R \in \mathcal{R}(a, c)$ and $S \in \mathcal{R}(c, b)$ such that $x \notin R \cup S$. But $c \in R \cap S$, so $R \cup S \in \mathcal{R}(a, b)$. Hence $x \notin [a, b]_{\mathcal{R}}$.)

We also have a converse:

3.1. Theorem (Road Representation Continued). *An R-relation is disjunctive if and only if each of its inducing road systems is contained in an inducing road system that is additive.*

Here's what's going on with the proof:

Given a road system $\langle X, \mathcal{R} \rangle$, one may define $\mathcal{R}^*$ to consist of all nonempty finite unions $\bigcup_{i \in I} R_i$ of roads in $\mathcal{R}$ such that whenever $I = J \cup K$ with $J \neq \emptyset \neq K$, we have $(\bigcup_{i \in J} R_i) \cap (\bigcup_{i \in K} R_i) \neq \emptyset$. Since $\mathcal{R}^* \supseteq \mathcal{R}$ and every superset of a road system is a road system, it's clear that the R-relation $[\ ,\ ,\ ]_{\mathcal{R}^*}$ is generally more restrictive than $[\ ,\ ,\ ]_{\mathcal{R}}$. Indeed, the following says that $\mathcal{R}^*$ is the *additive closure* of $\mathcal{R}$:

3.2. Theorem. *For any road system $\mathcal{R}$, $\mathcal{R}^*$ is an additive road system that contains $\mathcal{R}$. Moreover, if $\mathcal{S}$ is any additive road system containing $\mathcal{R}$, then $\mathcal{R}^* \subseteq \mathcal{S}$. Finally, $[\ ,\ ,\ ]_{\mathcal{R}^*} = [\ ,\ ,\ ]_{\mathcal{R}}$ iff $[\ ,\ ,\ ]_{\mathcal{R}}$ is disjunctive.*

An R-relation $[\ ,\ ,\ ]$ on X is *weakly disjunctive* if $[a,b] \subseteq [a,c] \cup [c,b]$ for all $c \in [a,b]$. (The reverse inclusion is always true, so this means equality.) Here's how some of our classical antecedent structures stack up vis à vis (weak) disjuntivity:

- The D-interpretation for lattices is weakly disjunctive iff the lattice is a chain: For the "hard" direction, suppose a and b are incomparable. Then $a \sqcup b \in [a,b]_D$. However, $a \sqcap b \in [a,b]_D \setminus ([a, a \sqcup b]_D \cup [a \sqcup b, b]_D)$.

- The S-interpretation for real vector spaces is weakly disjunctive, but is disjunctive iff the vector space dimension is one: Weak disjunctivity is a simple exercise in parameterization juggling; as for non-disjunctivity when the dimension is ≥ 2, it is easy to show that $\mathcal{R}^*$-intervals are trivial—i.e., consist of their bracket points only—whenever $\mathcal{R}$ is a road system inducing $[\ ,\ ,\ ]_S$.

- The T-interpretation for trees is disjunctive; this is not completely obvious.

4. Separative Road Systems. A road system $\langle X, \mathcal{R}\rangle$ is *separative* if for any $a, b, c \in X$ with $b \neq c$, there is some $R \in \mathcal{R}$ such that either $a, b \in R$ but $c \notin R$ or $a, c \in R$ but $b \notin R$. Recall from above the pre-orders $\leq_a$ on R-relations; reflexive because of R2 and transitive because of R4. They become antisymmetric—and hence partial orders—in the presence of the following condition.

R7 (Antisymmetry) $([a, b, c] \wedge [a, c, b]) \rightarrow b = c$ $([a, b] = [a, c] \implies b = c)$.

4.1. Proposition. *A road system is separative iff its induced R-relation is antisymmetric.*

4.2. Corollary. *An R-relation is disjunctive and antisymmetric iff each of its inducing road systems is contained in an inducing road system that is additive and separative.*

4.3. Proposition. *If $\langle X, [\ ,\ ,\ ]\rangle$ is an R-relation that is weakly disjunctive and antisymmetric, then for each $a \in X$ the ordering $\leq_a$ is a tree ordering with least element (root) a.*

In view of Proposition 4.3, each tree ordering $\leq_a$ gives rise to a T-interpretation $[\ ,\ ,\ ]_a$ of betweenness on X, and it is natural to ask how these T-interpretations relate to one another and to $[\ ,\ ,\ ]_{\mathcal{R}}$.

4.4. Theorem. *Let $\langle X, [\ ,\ ,\ ]\rangle$ be an antisymmetric, disjunctive R-relation, with root point $a \in X$. Then each interval $[b, c]$ is contained in the corresponding tree interval $[b, c]_a$. Furthermore, if the* centroid *$[a, b] \cap [a, c] \cap [b, c]$ is nonempty, then $[b, c] = [b, c]_a$.*

In weakly disjunctive, antisymmetric R-relations, centroids are unique when they exist. The following example shows that the centroid existence hypothesis cannot be dropped from Theorem 4.4.

4.5 Example. *Let* X *be the unit circle, with topology inherited from the euclidean plane. Let* $\mathcal{R}$ *consist of the connected subsets of* X*. Then* $\mathcal{R}$ *is an additive, separative road system, and all* $\mathcal{R}$*-intervals are trivial. However, if* $a \in X$ *is any root point, the tree order* $\leq_a$ *is described by saying* $x \leq_a y$ *precisely when either* $x = y$ *or* $x = a$*. If* $b, c \in X \setminus \{a\}$*, then* $[b, c]_{\mathcal{R}} = \{b, c\}$*, while* $[b, c]_a = \{a, b, c\}$

5. Topological Antecedents of Betweenness. There are at least three notions of betweenness in topology that deserve mention.

5.1. The C- and Q-Interpretations For a connected topological space X, define $[a, c, b]_C$ (resp., $[a, c, b]_Q$) to hold if either $c \in \{a, b\}$ or a and b lie in different components (resp., quasicomponents) of $X \setminus \{c\}$. The C-interpretation is induced by the additive, separative road system $\mathcal{C}$ consisting of the connected subsets of X; the Q-interpretation is also induced by an additive, separative road system $\mathcal{Q}$ that is generally different from $\mathcal{C}$, but which may be taken to contain $\mathcal{C}$. (Theorems 3.1 and 3.2 come to the rescue, though proving that $[\ ,\ ,\ ]_Q$ satisfies R1,...,R7 takes a bit of work.) Generally the Q-interpretation is more restrictive than the C-interpretation; the two coincide in the presence of local connectedness. A point $c \in X$ is a cut point of X precisely when it lies properly between two other points of X, in either the C- or the Q-interpretation.

5.2. The General K-Interpretation A connected topological space is a *continuum* if it is also compact; a *subcontinuum* of a space is a subset that is a continuum in its subspace topology. A space is *continuum-wise connected* if each two of its points are contained in a subcontinuum.

We now start with a continuumwise connected topological space X and define $[a,c,b]_K$ to hold if either $c \in \{a,b\}$ or a and b lie in different continuum components of $X \setminus \{c\}$—i.e., no subcontinuum of $X \setminus \{c\}$ contains both a and b. The K-interpretation is induced by the additive road system $\mathcal{K}$ consisting of the subcontinua of X. In continuumwise connected spaces, the K-interpretation is less restrictive than the C-interpretation, and generally fails to be separative. A point $c \in X$ is a *weak* cut point—i.e., its complement is not continuumwise connected—precisely when it lies properly between two other points of X, in the K-interpretation.

5.3 The K-Interpretation in Hausdorff Continua. If X is a Hausdorff continuum, then $\mathcal{K}$-intervals are closed, and hence subcompacta. When they're also subcontinua, we call X *interval connected*.

For example, arcs are interval connected, as are dendrites in general. The $\sin(\frac{1}{x})$-continuum is another example. At the opposite extreme, in a circle (simple closed curve) any interval is trivial. Trivial intervals with at least two points are called *gaps*.

Recall that a Hausdorff continuum is *hereditarily unicoherent* if the intersection of any two of its overlapping subcontinua is a subcontinuum.

5.3.1. Proposition. *A Hausdorff continuum is interval connected iff it is hereditarily unicoherent.*

A primary focus in this study is characterizing being interval connected purely in first-order betweenness terms.

This issue is not yet settled, but some progress has been made.

Here are some plausible characterization sentences, listed in order of nondecreasing logical strength.

(Gap-free Property)
$\forall a \forall b[a \neq b \rightarrow \exists c(c \in [a,b] \wedge c \neq a \wedge c \neq b)]$

(Semi-strong Gap-free Property)
$\forall a \forall b[a \neq b \rightarrow \exists c(c \in [a,b] \wedge c \neq a \wedge b \notin [a,c])]$

(Strong Gap-free Property)
$\forall a \forall b[a \neq b \rightarrow \exists c(c \in [a,b] \wedge a \notin [c,b] \wedge b \notin [a,c])]$

The gap-free property clearly follows from interval connectedness; and, using a simple "boundary bumping" argument, we can show that the semi-strong gap-free property does as well. Not so the strong gap-free property.

5.3.2. Theorem. *A Hausdorff continuum satisfies the strong gap-free property iff each of its nondegenerate intervals is a decomposable subcontinuum.*

And when we strengthen gap-freeness in a completely different way, we get an even stronger condition on intervals. Recall that an R-relation is *antisymmetric* if it satisfies R7 above. In terms of intervals, this says that $[a, b] = [a, c]$ implies $b = c$.

This first-order property is present in Hausdorff continua that are *aposyndetic*; i.e., any one of two points lies in the interior of a subcontinuum not containing the other.

5.3.3. Theorem. *A Hausdorff continuum is antisymmetric and satisfies the gap-free property iff each of its nondegenerate intervals is a generalized arc.*

So the strong gap-free property, as well as the conjunction of gap-freeness and anti-symmetry, are too strong for mere interval connectedness. On the other hand, simple gap-freeness—even semi-strong gap-freeness and the existence of centroids—is too weak.

By a *crooked annulus* we mean a continuum X that may be decomposed as a union $X = K \cup M$, where K and M are hereditarily indecomposable continua and $K \cap M$ has precisely two components, each nondegenerate.

5.3.4. Theorem. *Every crooked annulus satisfies the semi-strong gap-free property, as well as the existence of centroids, while failing to be interval connected.*

6. Setwise Betweenness. Sometimes one roadblock is not enough; it may take many to prevent the bankrobbers' escape from San Diego across the border to Tijuana.

In the language of road systems, we define a *set* C to *lie between* points $a, b \in X = \langle X, \mathcal{R} \rangle$ just in case $C \cap R \neq \emptyset$ for all $R \in \mathcal{R}(a, b)$. (Note: it can still happen that $C \cap [a, b] = \emptyset$.)

In the setting of Hausdorff continua, we restrict attention to subsets C that are closed, indeed finite.

For $1 \leq n < \omega$, we denote by $\mathcal{F}_n(X)$ the collection of nonempty subsets of X of cardinality $\leq n$, equipped with the Vietoris topology: For any set $A \subseteq X$, let $\langle A \rangle_n^+ := \{C \in \mathcal{F}_n(X) : C \cap A \neq \emptyset\}$, $\langle A \rangle_n^- := \{C \in \mathcal{F}_n(X) : C \subseteq A\}$. The sets $\langle U \rangle_n^+$ and $\langle U \rangle_n^-$, where U is open in X, constitute a subbase for a topology that makes $\mathcal{F}_n(X)$ into a Hausdorff continuum whenever the same is true for X.

We then define $[a,b]_n := \{C \in \mathcal{F}_n(X) : C \cap R \neq \emptyset$ for all $R \in \mathcal{K}(a,b)\}$. A Hausdorff continuum is *n-interval connected* if $[a,b]_n$ is connected for all $a, b \in X$.

The problem we are trying to solve is to characterize when a Hausdorff continuum is n-interval connected. Can this characterization be accomplished in first-order terms involving the obvious (n+2)-ary predicate? (We don't even know the answer for $n = 1$!)

Things we do know:

- For $n \geq 2$ and $A \subseteq X$, $\langle A \rangle_n^+$ is always connected. (Proof by transfinite induction.)

- $[a,b]_n$ always contains $\langle [a,b] \rangle_n^+$, which is itself connected when $n \geq 2$. The connectedness of $[a,b]$ is sufficient for equality to hold, but not necessary.

- Being 1-interval connected imples being n-interval connected for $n \geq 1$, and is the same as being hereditarily unicoherent.

- Being 2-interval connected does not imply being 1-interval connected: The unit circle is an example. (Indeed, $[a, b]_2$ $(a \neq b)$ is connected, while properly containing $\langle [a, b] \rangle_2^+ = \langle \{a, b\} \rangle_2^+$.)

- For $n \geq 2$, being n-interval connected does not imply being $(n+1)$-interval connected: If E_n is the union of n figure-eights joined at the tops and bottoms, then E_{n+1} is not $(n+1)$-interval connected. However it is n-interval connected exactly when $n \geq 2$. (Indeed, $[a, b]_n = \langle \{a, b\} \rangle_n^+$.)

A WELSH THANKYOU:
DIOLCH YN FAWR AM WRANDO!