Characterizing the Arc

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Spring Topology and Dynamical Systems Conference,
2011, Tyler, TX
17–19 March, 2011

1. The Framework

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1. The Framework. Arcs, as well as other well known topological spaces, have been characterized in many ways; e.g.

Theorem 1.1: [R. L. Moore, 1920] *The arc is unique among metrizable continua with exactly two noncut points.*

In this famous characterization, the arc is compared to other members of a *class of peers* (the metrizable continua) and is characterized in *terms* of cut points. In this talk we present a first-order logic framework in which we may specify the "terms of characterization." One advantage of this extra precision is that we may also make sense of when an object is *not* characterizable in certain terms, relative to a given peer class.

The main ingredients in our framework are the following:

(1) A **class** $\mathcal{P}$ of topological spaces, the *peers* to which a given member of $\mathcal{P}$ is to be compared for the purposes of characterization.

(2) An **alphabet** L of finitary relation and function symbols, with an assignment

$$Y \mapsto Y_{\mathsf{L}}$$

from spaces in $\mathcal{P}$ to L structures, such that homeomorphic spaces are assigned isomorphic structures.

(3) A **characterization language** Ψ of first-order sentences over L.

For any $X \in \mathcal{P}$, a set Φ_X of first-order sentences over L is a Ψ *characterization* of X, relative to $\mathcal{P}$, if: (i) $\Phi_X \subseteq \Psi$; (ii) $X_{\mathsf{L}} \models \Phi_X$; and (iii) whenever $Y \in \mathcal{P}$ is such that $Y_{\mathsf{L}} \models \Phi_X$, it follows that Y is homeomorphic to X.

We then say that $X \in \mathcal{P}$ is Ψ *characterizable* if such a Φ_X exists.

In this talk we are concerned mainly with characterizing the arc relative to various peer classes, using various characterization languages. Not surprisingly, restriction of the peer class of a space makes characterization of the space easier; restriction of the characterization language, on the other hand, makes life harder.

2. Results Involving Lattices. In this section L is the alphabet of bounded lattices, and X_{L} is the closed set lattice of a space X. The first result known to us, couchable in our framework, is the following, where the characterization language consists of all first-order sentences.

Theorem 2.1 [C. W. Henson, et al, 1979]: *The arc is first-order lattice characterizable, relative to the class of metrizable spaces.*

In the remainder of this section, we restrict both the peer class and the characterization language, and give both positive and negative results on characterizing the arc.

A *lattice base* for a space X is a closed set base that is also a sublattice of X_{L}. An L sentence φ is *base free* if whenever X is a compactum and $\mathcal{A}$ is a lattice base for X, then $\mathcal{A} \models \varphi$ iff $X_{\mathsf{L}} \models \varphi$.

In contrast to Theorem 2.1, we have:

Theorem 2.2 [R. Gurevič, 1988]: *The arc is not base free lattice characterizable, relative to the class of metrizable continua.*

Define two compacta to be *lattice base related* if there is a lattice base of one and a lattice base of the other, both satisfying the same first-order sentences. What Gurevič did in Theorem 2.2 was to show that the arc, as well as every nondegenerate metrizable continuum, is lattice base related to a metrizable continuum that is not locally connected. This led to the following positive result.

Theorem 2.3 [PB, 1988]: *The arc is base free lattice characterizable, relative to the class of locally connected metrizable compacta.*

The positive result in Theorem 2.3 raised the question of whether other Peano continua could be so characterized, and several small extensions of the technique culminated in

Theorem 2.4 [PB, 2011]: *Any topological graph is base free lattice characterizable, relative to the class of locally connected metrizable compacta.*

The Gurevič result 2.2 also suggested that certain non-locally connected continua might still be base free characterizable; a natural candidate was the pseudo-arc, famously characterized by R. H. Bing as being unique among the nondegenerate hereditarily indecomposable chainable metrizable continua.

J. Krasinkiewicz and P. Minc (1977) gave a characterization of hereditary indecomposability which, years later, K. P. Hart noticed to be expressible as a base free condition. From this, the following is a direct consequence.

Theorem 2.4: *The pseudo-arc is base free lattice characterizable, relative to the class of chainable metrizable continua.*

Meanwhile, in [T. Banach, PB, B. Raines, W. Ruitenburg, 2006] it was possible to obtain an analogue of the Gurevič result to show that every nondegenerate metrizable continuum is lattice base related to a metrizable continuum that is not chainable. As a direct consequence of this, we have:

Theorem 2.5 *The pseudo-arc is not base free lattice characterizable, relative to the class of metrizable continua.*

The final *coup de grâce* to the search for base free characterizable continua was delivered in a recent (2010) paper by K. P. Hart, who proved that every nondegenerate metrizable continuum is lattice base related to a topologically distinct metrizable continuum. His proof made essential use of the famous result (1934) of Z. Waraszkiewicz, to the effect that no metrizable continuum has every metrizable continuum as a continuous image. So immediately, we have

Theorem 2.6 [K. P. Hart, 2010]: *No nondegenerate metrizable continuum is base free lattice characterizable, relative to the class of metrizable continua.*

The results so far, together with the fact that the Cantor space is a metric compactum whose classic characterization as a zero-dimensional metrizable compactum with no isolated points easily translates into base free terms, suggests the following

Conjecture: *An infinite metrizable compactum is base free lattice chacterizable relative to the class of metrizable compacta if and only of that compactum is the free union of a Cantor set with a finite set.*

3. Results Involving Betweenness. In this section L consists of one ternary relation symbol, along with equality, and is what we call the alphabet of betweenness. For each continuum X, X_{L} is the betweenness structure whose elements are the points of X; and we say that the triple $\langle a, c, b\rangle \in X^3$ satisfies the condition that c *lies between* a *and* b just in case every connected closed subset of X containing both a and b contains c as well. We now consider characterizations in terms of first-order sentences from this alphabet.

Characterizations involving betweenness, and the study of betweenness in a topological setting in general, are in fairly preliminary stages, but we can announce the following analogues of Theorems 2.1 and 2.5 above.

Theorem 3.1: *The arc is first-order betweenness characterizable, relative to the class of metrizable spaces. The pseudo-arc, however, is not.*

THANK YOU