

# **Betweenness in a Continuum: Lessons from the Crooked Annulus**

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1. Betweenness via Road Systems
2. Subcontinuum Road Systems
3. A Characterization Problem
4. The Crooked Annulus
5. Proof Outline for 3.1
6. Summary

**1. Betweenness via Road Systems.** We take the intuitive view that point  $c$  lies between points  $a$  and  $b$  exactly when every “road” allowing travel from  $a$  to  $b$  (and *vice versa*) must go through  $c$ .

This “roadblock” vision of betweenness has led to the following simple abstract definition:

- A *road system* is a pair  $\langle X, \mathcal{R} \rangle$ , where  $X$  is a nonempty set and  $\mathcal{R}$  is a family of subsets of  $X$ , called *roads*, satisfying:
  - Every singleton subset of  $X$  is a road.
  - Every doubleton subset of  $X$  is contained in at least one road.
  - (Additivity Condition): The union of two intersecting roads is a road.

If  $\langle X, \mathcal{R} \rangle$  is a road system and  $a, b \in X$ , the set of points  $c$  *between*  $a$  and  $b$  is denoted  $[a, b]$  and is the set  $\bigcap \{R \in \mathcal{R} : a, b \in R\}$ .

The *interval membership relation*  $c \in [a, b]$  defines a ternary relation on the underlying set  $X$ .

A natural question is whether one may characterize—using first-order terms involving an abstract ternary relation symbol—exactly when a ternary relation  $B \subseteq X^3$  is the interval membership relation arising from a road system on  $X$ .

This question has an affirmative answer.

1.1 Theorem (Road Representation): *Let  $B$  be a ternary relation on a nonempty set  $X$ . Then there is a road system  $\mathcal{R}$  on  $X$  with interval membership relation  $B$  iff  $B$  satisfies the following five first-order conditions:*

R1 (Symmetry)  $B(a, c, b) \rightarrow B(b, c, a)$ .

R2 (Reflexivity)  $B(a, b, b)$ .

R3 (Minimality)  $B(a, c, a) \rightarrow c = a$ .

R4 (Convexity)  $(B(a, c, b) \wedge B(a, d, b) \wedge B(c, e, d)) \rightarrow B(a, e, b)$ .

R5 (Disjunctivity)  $B(a, x, b) \rightarrow (B(a, x, c) \vee B(c, x, b))$ .

**2. Subcontinuum Road Systems.** There are many natural situations, especially in the theory of trees and in topology, where road systems come up; the one I want to discuss today concerns roads that consist of the subcontinua of a continuum (= connected compact Hausdorff space).

In this setting  $c \in [a, b]$  means that there is no subcontinuum of  $X \setminus \{c\}$  that also contains  $\{a, b\}$ . (In particular, a point that lies between two other points in a continuum is a weak cut point of the continuum. Moreover, if  $X$  is *aposyndetic*—i.e., two points may be separated by a subcontinuum that contains one of them in its interior and misses the other—then  $c$  is actually a cut point.)

Intervals in continua are generally closed; when they're also subcontinua, we call the continuum *interval connected*.

For example, arcs are interval connected, as are dendrites in general. The  $\sin(\frac{1}{x})$ -continuum is another example. At the opposite extreme, in a simple closed curve any interval  $[a, b]$  consists of the bracketing points alone. Such intervals, when  $a \neq b$ , are called *gaps*.

Recall that a continuum is *hereditarily unicoherent* if the intersection of any two of its overlapping subcontinua is a subcontinuum.

2.1 Proposition: *A continuum is interval connected iff it is hereditarily unicoherent.*

**3. A Characterization Problem.** The issue we wish to focus on today concerns the question of characterizing—in first-order betweenness terms—the property of being interval connected.

This question is not yet answered, but here are some plausible characterization sentences, listed in order of nondecreasing logical strength.

(Gap-free Property):

$$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge c \neq a \wedge c \neq b)]$$

(Semi-strong Gap-free Property):

$$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge c \neq a \wedge b \notin [a, c])] ]$$

(Strong Gap-free Property):

$$\forall a \forall b [a \neq b \rightarrow \exists c (c \in [a, b] \wedge a \notin [c, b] \wedge b \notin [a, c])] ]$$

The gap-free property clearly follows from interval connectedness; and, using a simple “boundary bumping” argument, we can show that the semi-strong gap-free property does as well. Not so the strong gap-free property.

3.1 Theorem: *A continuum satisfies the strong gap-free property iff each of its non-degenerate intervals is a decomposable subcontinuum.*

And when we strengthen gap-freeness in a completely different way, we get an even stronger condition on intervals. To explain this, first define a continuum (or any road system) to be *antisymmetric* if  $[a, b] = [a, c]$  implies  $b = c$ . This is clearly a first-order property, it’s present in aposyndetic continua, and we have:

3.2 Theorem: *A continuum is antisymmetric and satisfies the gap-free property iff each of its nondegenerate intervals is a generalized arc.*

**4. The Crooked Annulus.** A continuum is *hereditarily indecomposable* if the intersection of any two of its overlapping subcontinua is one or the other of them. The celebrated pseudo-arc is an example of this phenomenon.

The strong gap-free property is too strong to characterize interval connectedness in general because hereditarily indecomposable continua are hereditarily unicoherent; hence intervals are indecomposable subcontinua.

But the ever so slightly weaker semi-strong gap-free property is *too weak*.

Define a continuum  $X$  to be a *crooked annulus* if it may be decomposed as a union  $K \cup M$  of two hereditarily indecomposable subcontinua such that  $K \cap M$  has exactly two components, each nondegenerate.

4.1 Theorem: *Every crooked annulus satisfies the semi-strong gap-free property, while failing to be interval connected.*

Some remarks: Let  $X = K \cup M$ , where  $K, M$  are subcontinua such that  $K \cap M$  is a union  $A \cup B$  of disjoint nondegenerate subcontinua.

- (1) If  $a \in A$  and  $b \in B$ , then  $[a, b]$  is clearly not connected.
  
- (2) If  $H$  is a subcontinuum of  $X$  that intersects both  $K$  and  $M$ , and if  $C$  is a component of  $H$  in  $K$ , then  $C$  intersects  $M$ . (“Boundary bumping,” just uses fact that  $X = K \cup M$ .)

Now assume that both  $K$  and  $M$  are hereditarily indecomposable.

- (3) If  $H$  is a subcontinuum of  $X$  that intersects both  $A$  and  $B$ , then  $A \cup B \subseteq H$ .
- (4) Hence, if  $a \in A$  and  $b \in B$ , then  $[a, b] \supseteq A \cup B$ . (In fact, they're equal.)
- (5) In general, we show  $X$  satisfies semi-strong gap-freeness by proving that, no matter where  $a, b$  lie in  $X$ ,  $[a, b]$  is either connected, or contains two nondegenerate disjoint subcontinua, one containing  $a$ , the other containing  $b$ .
- (6) A crooked annulus also satisfies another consequence of being interval connected, namely the *centroid property*: for any  $a, b, c \in X$ ,  $[a, b] \cap [a, c] \cap [b, c] \neq \emptyset$ .

## 5. Proof Outline for 3.1.

(1) If  $[a, b]$  decomposes into  $K \cup M$ , both proper subcontinua, then any  $c \in K \cap M$  witnesses that the strong gap-free property holds.

(2) If the strong gap-free property holds and intervals are connected, then the non-degenerate ones are easily seen to be decomposable.

(3) If  $A$  and  $B$  are disjoint nonempty closed subsets of  $X$ , a Zorn's lemma argument allows you to find  $a \in A$  and  $b \in B$  such that for any  $a' \in A$ ,  $b' \in B$ , if  $[a', b'] \subseteq [a, b]$ , then  $[a', b'] = [a, b]$ . ( $a$  and  $b$  are *minimally close*).

(4) In the absence of interval connectedness, we have subcontinua  $K, M$  with  $K \cap M = A \cup B$ , where  $A$  and  $B$  are closed, nonempty, and disjoint. Let  $a \in A$  and  $b \in B$  be minimally close (relative to  $A, B$ ). If  $c \in [a, b]$ , then either  $c \in A$  or  $c \in B$ . In the first case  $[c, b] = [a, b]$ ; in the second  $[a, c] = [a, b]$ . Thus the strong gap-free property fails for  $X$ .

## 6. Summary.

Call a property  $\mathfrak{P}$  of continua *B-definable* if there is a first-order sentence  $\phi$  in an alphabet with equality and one ternary predicate symbol, such that a continuum is in class  $\mathfrak{P}$  iff the corresponding interval membership relation satisfies  $\phi$ .

Examples of properties that are B-definable include:

- Having every nondegenerate interval a decomposable continuum.
- Having every nondegenerate interval a generalized arc.
- Being hereditarily indecomposable.
- Being irreducible.

Examples of properties that are *not* B-definable include:

- Being of dimension  $n$ .
- Being chainable.
- Being homogeneous.
- Being self-similar.

And in addition to our focus question of whether being interval connected (= hereditarily unicoherent) is B-definable, here are some properties for which B-definability is unknown:

- Being indecomposable. [B-definable when we restrict to metric continua.]
- Having every interval an indecomposable continuum.

**GO RAIBH MILE MAITH AGAIBH!**