A Framework for Characterising Topological Spaces

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14th Galway Topology Colloquium,
Queen's University Belfast, 15–17 August, 2011

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1. Characterisation Frameworks. Many well-known concrete geometric objects have been characterised (up to homeomorphism) in terms of abstract topological properties. For example, among the metric (=metrisable) spaces, we have:

1.1. Theorem:

[L. E. J. Brouwer, 1910; M. H. Stone, 1936] *The Cantor (middle-thirds) set* $\mathbb{C}$ *is unique in being compact, totally disconnected (aka, anticonnected), and having no isolated points.*

[R. L. Moore, 1920] *The arc (closed unit interval)* $\mathbb{A}$ *is unique in being compact, connected, and having exactly two noncut points.*

[R. H. Bing, 1951] *The (Knaster-Moise) pseudo-arc* $\mathbb{P}$ *is unique in being compact, chainable, and hereditarily indecomposable.*

In each of these characterisations, the space in question is compared to other members of a *class of peers* (the metric spaces, in this case) and is characterised in terms of "purely" topological properties defined in informal mathematical language.

The distinction between "class of peers" and "terms of characterisation," while largely arbitrary at the informal level, becomes less so when we formalise certain topological descriptions. In this talk we present a (painless) mathematical logic framework to do just this.

One advantage of this approach is that it places under mathematical scrutiny not only the objects of topological interest, but also the formulation of the properties they exhibit.

For this talk, a *characterisation framework* is a triple $\langle \mathcal{P}, \mathsf{L}_*, (\ )_{\mathsf{L}} \rangle$, where:

(1) $\mathcal{P}$ is a class of topological spaces, the *peers* to which a given member of $\mathcal{P}$ is to be compared for the purposes of characterisation.

(2) L_* is a *characterisation language*, consisting of formal logical sentences over the *characterisation alphabet* L of finitary relation and function symbols, as well as a precise notion of when an L-structure *satisfies* a sentence of L_*. Such a notion should automatically ensure that isomorphic L-structures satisfy the same L_*-sentences.

(3) $(\ )_{\mathsf{L}}$ is a *structure assignment* $X \mapsto X_{\mathsf{L}}$ from spaces in $\mathcal{P}$ to L-structures, such that homeomorphic spaces are assigned isomorphic structures.

For any $X \in \mathcal{P}$, a set $\Sigma \subseteq \mathsf{L}_*$ of sentences over L is an L_*-*characterisation* of X, relative to $\mathcal{P}$, if: (i) $X_{\mathsf{L}} \models \Sigma$; and (ii) whenever $Y \in \mathcal{P}$ is such that $Y_{\mathsf{L}} \models \Sigma$, it follows that Y is homeomorphic to X.

We then say that $X \in \mathcal{P}$ is L_*-*characterisable* if such a Σ exists.

In this talk we concern ourselves with characterising the Cantor set $\mathbb{C}$, the arc $\mathbb{A}$, and the pseudo-arc $\mathbb{P}$ within various characterisation frameworks centered around the lattice alphabet $\mathsf{L} = \{\sqcup, \sqcap, 0, 1\}$ and the structure assignment that takes a space X to its bounded lattice X_{L} of closed subsets. Not surprisingly, restriction of the peer class of a space makes characterisation of the space easier; restriction of the characterisation language, on the other hand, makes life harder. All bets are off, however, if we make both restrictions simultaneously.

2. The Cantor Set. In this section we let our characterisation language be $\mathsf{L}_{\omega\omega}$, the finitary first-order language over L. Neither metrisability nor compactness are expressible in finitary first-order terms, so we define $\mathcal{P}$ to be the peer class of metric compacta. The two remaining conditions from Brouwer's characterisation are easily couched in first-order terms:

Total Disconnectedness (expressed equivalently as zero-dimensionality):
$\forall xy[(x \sqcap y = 0) \rightarrow$
$\exists uvu'v'((x \sqcap u = x) \wedge (y \sqcap v = y) \wedge (u \sqcap v = 0)$
$\wedge (u \sqcap u' = v \sqcap v' = 0) \wedge (u \sqcup u' = v \sqcup v' = 1))]$

No isolated points:
$\forall xx'[((x \neq 0) \wedge (x \sqcap x' = 0) \wedge (x \sqcup x' = 1)) \rightarrow$
$\exists y((y \neq 0) \wedge (y \sqcap x = y) \wedge (y \neq x))]$

With Σ comprising the two sentences above, the following is straightforward.

2.1. Theorem: Σ *is an* $\mathsf{L}_{\omega\omega}$*-characterisation of* $\mathbb{C}$*, relative to the class of metric compacta*.

3. The Arc. An analogue of Theorem 2.1 for the arc springs from the Moore characterisation given in Theorem 1.1. Connectedness, plus the fact that there are only two noncut points, are easily $\mathsf{L}_{\omega\omega}$-expressible; hence one may define the two separation orders. We then use the fact that a connected linearly ordered space with end points is automatically compact to allow the omission of compactness in the specification of the peer class.

3.1. Theorem [C. W. Henson, et al, 1979]: $\mathbb{A}$ *is* $\mathsf{L}_{\omega\omega}$*-characterisable, relative to the class of metric spaces.*

In the remainder of this talk we concentrate on peer classes contained within the compact metric spaces, and consider some other characterisation languages.

Many topological properties have textbook definitions that are expressible in terms of not only the closed-set lattice, but of any *lattice base* for a space; i.e., any sublattice that serves as a base for the closed sets. For example,

Connectedness
$\forall xy[((x \sqcap y = 0) \wedge (x \sqcup y = 1)) \rightarrow ((x = 0) \vee (x = 1))]$

is true in one lattice base for a compactum iff it's true in every other lattice base for that compactum. An example of an $L_{\omega\omega}$-sentence that is true for closed-set lattices but not generally for lattice bases is the one that says the lattice is atomic.

So define an L_*-sentence φ to be *base free* if whenever X is a compactum (=compact Hausdorff space) and $\mathcal{A}$ is a lattice base for X, then $\mathcal{A} \models \varphi$ iff $X_{\mathsf{L}} \models \varphi$. We denote by L_*^b the set of sentences of L_* that are base free. For example, it is easy to show that clopen sets belong to every lattice base of a compactum, and hence that the two sentences constituting Σ above are base free. This allows for an immediate strengthening of Theorem 2.1.

3.2. Theorem. *$\mathbb{C}$ is $\mathsf{L}_{\omega\omega}^b$-characterisable, relative to the class of metric compacta.*

Can we do the same for $\mathbb{A}$? (We would be looking for an analogue of Theorem 3.1, with restrictions being made to *both* the peer class and the characterisation language.)

The answer to this question is *no*; first-order base-free sentences are not enough to nail down the arc.

3.3. Theorem [R. Gurevič, 1988]: $\mathbb{A}$ *is not* $\mathsf{L}^b_{\omega\omega}$*- characterisable, relative to the class of metric continua.*

Define two compacta to be b*-replicas* of one another if there is a lattice base of one and a lattice base of the other, both satisfying the same first order sentences. The important thing to note about lattice bases of compacta is that they can be specified in $\mathsf{L}_{\omega\omega}$. So what Gurevič did in Theorem 3.3 was to show, using a combination of ultrapowers and the Löwenheim-Skolem theorem–processes that preserve first order conditions–that the arc, indeed, any non-degenerate metric continuum, has a metric b-replica that is not locally connected.

So what happens if we "cheat" a little and include local connectedness in the specification of the peer class? Call a compactum *Peano* if it's both metric and locally connected.

3.4. Theorem [P. B., 1988]: $\mathbb{A}$ *is* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of Peano compacta.*

A key idea that helped this work is a classification result of R. L. Moore: *If a nondegenerate Peano continuum does not contain a simple triod, then it is either an arc or a simple closed curve.*

Theorem 3.4 raised the question of whether other Peano continua can be characterised in a similar way, and several small extensions of its technique culminated in

3.5. Theorem [P. B., 2011]: *Any topological graph is* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of Peano compacta.*

Since the Cantor set is $\mathsf{L}^b_{\omega\omega}$-characterisable relative to the peer class of compact metric spaces, and the Gurevič result 3.2 says this cannot be the case for any nondegenerate Peano continuum, it is natural to ask about base-free characterisations of metric continua that are not locally connected.

This brings us to a consideration of the pseudo-arc.

4. The Pseudo-Arc. The Bing characterisation of the pseudo-arc $\mathbb{P}$ (Moise coined the term in 1948, but earlier constructions of the space go back to Knaster, 1922) involves two notions that may not be familiar to the general topological audience.

Chainable: A space is chainable if every finite open cover refines to a finite cover that is like a "string of beads": the first set intersects the second set only; the second set intersects just the first and the third sets, etc. (Chainability automatically entails connectedness.)

Hereditarily Indecomposable: A space is hereditarily indecomposable if any two of its subcontinua are either disjoint or set-theoretically comparable.

J. Krasinkiewicz and P. Minc (1977) gave a "crookedness" characterisation of hereditary indecomposability which, in a 2000 paper, K. P. Hart, J. van Mill, and R. Pol noticed to be expressible in first-order base-free terms. In light of this observation, we may paraphrase the K-M result as follows.

4.1. Theorem: *Let X be a continuum; the following are equivalent:*

(a) *X is hereditarily indecomposable.*

(b) *For any lattice base $\mathcal{A} \subseteq X_{\mathsf{L}}$, the following holds: If $A, B \in \mathcal{A}$ are disjoint and U, V are such that $X \setminus U, X \setminus V \in \mathcal{A}$, $A \subseteq U$ and $B \subseteq V$, then there exist $M, N, P \in \mathcal{A}$ such that $A \subseteq M$, $B \subseteq N$, $M \cap N = \emptyset$, $M \cap P \subseteq V$, $N \cap P \subseteq U$, and $M \cup N \cup P = X$.*

From this, the following is a direct consequence.

4.2. Theorem: $\mathbb{P}$ *is* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of chainable metric compacta.*

Meanwhile, in [T. Banach, P. B., B. Raines, W. Ruitenburg, 2006], it was possible to obtain an analogue of the Gurevič result to show that every nondegenerate metric continuum has a metric b-replica that is not chainable. As a direct consequence of this, we have:

4.3. Theorem: $\mathbb{P}$ *is not* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of metric continua.*

The final *coup de grâce* to the search for base free characterisable continua was delivered in a recent (2010) paper by K. P. Hart, who proved that every nondegenerate metric continuum has a metric b-replica that is topologically distinct. His proof made essential use of the famous result (1934) of Z. Waraszkiewicz, to the effect that no metric continuum has every metric continuum as a continuous image.

Hart proved that if X and Z are metric continua, with X nondegenerate, then X has a metric b-replica Y, having Z as a continuous image.

So, given nondegenerate X, use Waraszkiewicz to find a metric continuum Z not a continuous image of X. Then the continuum Y that Hart provides is a topologically distinct metric b-replica of X.

In summary, we have

4.4. Theorem [K. P. Hart, 2010]: *No non-degenerate metric continuum is* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of metric continua.*

5. Some Open Questions.

5.1. Question: *If compact metric X is* $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of metric compacta, what may we conclude about X?*

Comments: (1) From Hart's Theorem 4.4, we know X cannot be connected; by extending the methods of proof, we can show more—e.g., that it has every metric continuum as a continuous image, but also cannot be a "bar code;" i.e., the cartesian product of an arc and a Boolean space.

(2) In a private conversation in the mid 1990s, C. W. Henson told us that he was able to show—using a version of the Ryll-Nardzewski categoricity theorem from Banach model theory—that X would have the following property: There is a natural number n such that no family of pairwise disjoint proper subcontinua of X, each with nonempty interior, can have cardinality $> n$.

(3) We strongly suspect that the ultimate consequence if this broad form of $\mathrm{L}^{b}_{\omega\omega}$-characterisability is total disconnectedness; and hence—by the model theory of Boolean algebras—being either finite or the free union of a Cantor set with a finite set.

5.2. Question: *Is* $\mathbb{A}$ L^b_**-characterisable, relative to the class of metric compacta, for a suitable* $\mathsf{L}_* \supseteq \mathsf{L}_{\omega\omega}$*?*

Comments: (1) In a recent (2011) paper, we were able to show that chainability can be couched in the language $\mathsf{L}^b_{\omega_1\omega}$, where $\mathsf{L}_{\omega_1\omega}$ allows for countably infinite disjunctions. Hence, in light of Theorem 4.2, we know that $\mathbb{P}$ is $\mathsf{L}^b_{\omega_1\omega}$-characterisable, relative to the class of metric compacta.

(2) The $\sin\frac{1}{x}$-curve is a chainable continuum, quite distinct from the arc. However, we have yet to determine whether the two are b-replicas of one another.

5.3. Question: *Is there an example of a Peano continuum that is* not $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of Peano continua*?

Comment: Considering the difficulties encountered in trying to prove Theorem 3.5—that topological graphs are so characterisable—this should be relatively easy to answer. The hitch lies in the fact that, while we know how to destroy local connectedness (and chainability) in producing b-replicas, we have no way of preserving it while still producing something topologically distinct. (One is reminded of the Humpty Dumpty tale.)

5.4. Question: *Is* $\mathbb{P}$ $\mathsf{L}^b_{\omega\omega}$*-characterisable, relative to the class of weakly chainable metric compacta*?

Comments: (1) This would be a substantial strengthening of Theorem 4.2, and would stand in closer parallel with the $\mathsf{L}^b_{\omega\omega}$-characterisations of $\mathbb{C}$ and $\mathbb{A}$ above: What $\mathbb{C}$ and $\mathbb{A}$ share in common is the fact that they are both $\mathsf{L}^b_{\omega\omega}$-characterisable, relative to their respective classes of Hausdorff continuous images. In 1963, A. Lelek defined the class of *weakly chainable* compacta (automatically continua) and proved–*à la* Hahn-Mazurkiewicz–that this class comprises the Hausdorff continuous images of the pseudo-arc.

(2) The compactum $\mathbb{A}\times\mathbb{C}$ is *not* $\mathsf{L}^b_{\omega\omega}$-characterisable, relative to the class of metric compacta. Hence it stands as an example of an infinite compactum that is not so characterisable relative to its class of Hausdorff continuous images.

THAT'S ALL, FOLKS,
GO RAIBH MILE MAITH AGAIBH!