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ULTRAPRODUCTS IN TOPOLOGY

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We define a reduced product (in particular an ultraproduct) construction in topology which yields a class of quotients of the box product. Retention of certain properties under the formation of ultraproducts, as well as the role of ultraproducts in the study of zero dimensional spaces are investigated. Certain analogies and disanalogies with the box product are also pointed out.

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0. Introduction

The Łoś ultraproduct construction in model theory provides motivation for defining a similar concept in general topology. This construction is shown to preserve many popular topological properties (e.g. Hausdorffness, complete regularity, non-compactness, linear orderability) and to fail to preserve many others (e.g. non-(complete regularity), normality, non-normality, compactness). In this paper we focus on preservation and develop several techniques whose applications are seen to go beyond preservation per se.

Among the results we have:

(i) Every regular space has a paracompact ultrapower.

(ii) Ultraproducts of normal spaces needn't be normal.

(iii) Any two regular perfect spaces have homeomorphic ultrapowers.

A word about set theory, notations and conventions: Firstly, our underlying set theory is ZFC (= Zermelo-Fraenkel set theory with the Axiom of Choice); secondly, cardinals are initial ordinals, where each ordinal is the set of its predecessors ($\omega = \aleph_0$ = the first infinite ordinal, $\omega_1 = \omega^+ = \aleph_1$ = the first uncountable ordinal, etc.); and lastly, we denote the end of a proof with a little box, "□".

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1. Elementary properties

Let $\langle \mathfrak{X}_i : i \in I \rangle$ be a family of topological spaces, $\mathfrak{X}_i = \langle X_i, \tau_i \rangle$, with F a filter on I. To form the *topological reduced product* $\Pi_F \mathfrak{X}_i$ we first form, as is done in model theory and algebra, the (Łoś) reduced product $\Pi_F X_i$ of the underlying sets by taking F-equivalence classes $[f]_F$ of elements of the cartesian product $\Pi_I X_i$, where $[f]_F = \{g \in \Pi_I X_i : \{i : g(i) = f(i)\} \in F\}$. We next topologize $\Pi_F X_i$ by taking, for basic open sets, all "open reduced boxes" $\Pi_F M_i = \{[f]_F : \{i : f(i) \in M_i\} \in F\}$ where $M_i \in \tau_i$. It is an easy exercise to verify that the reduced product topology can also be obtained by allowing the sets M_i to range over a basis β_i for τ_i.

Note that if $F = \{I\}$ then $\Pi_F \mathfrak{X}_i$ is the usual box product of the $\mathfrak{X}_i$'s. Note also that the natural projection $\Gamma_F : \Pi_I \mathfrak{X}_i \to \Pi_F \mathfrak{X}_i$ is always an open identification ($\Gamma_F(\Pi_I M_i) = \Pi_F M_i$; and $\Gamma^{-1}(\Pi_F M_i) = \bigcup_{J \in F} \Pi_J M_i$, where $\Pi_J M_i = \Pi_I N_i$, $N_i = \{{M_i, i \in J \atop X_i, i \notin J}$).

In the case of reduced powers there is a natural F-diagonal map $\Delta_F : \mathfrak{X} \to \Pi_F(\mathfrak{X})$ taking $x \in X$ to the F-equivalence class of the constant map at x. Δ_F is always one-one but is hardly ever continuous. In the few cases where Δ_F is continuous, however, it is an embedding. We will study this mapping in more detail later on.

We now restrict our attention to reduced products via ultrafilters. One helpful simplification which arises in the study of ultraproducts is that the complement of an ultrabox is again an ultrabox. Thus if γ_i is a basis for the closed sets of $\mathfrak{X}_i$ and U is an ultrafilter on I then $\Pi_U \gamma_i = \{\Pi_U C_i : C_i \in \gamma_i, i \in I\}$ is a basis for the closed sets of $\Pi_U \mathfrak{X}_i$.

A *topological class* is a class of spaces closed under homeomorphic images. Such a class will be called *closed* if ultraproducts of spaces from the class are again in the class. A class is *open* if its complement is closed; and is *clopen* if it is both open and closed. The central theme of this paper is to inventory some of the more popular topological classes vis à vis their status as closed (etc.) classes. Exemplary results involving only elementary methods follow:

1.1 Examples. *The classes of discrete,* T_0, T_1, T_2 *and* T_3 *spaces are clopen.*

Proof. We illustrate the kind of argument by showing that the class T_3 (i.e. of regular T_1 spaces) is closed. Let $[f]_U \in \Pi_U X_i$ where each X_i is T_3, and let $\Pi_U C_i$ be a basic closed set not containing $[f]_U$. Then $J = \{i : f(i) \notin C_i\} \in U$. For $i \in J$ let M_i, N_i be open such that $f(i) \in M_i$, $C_i \subseteq N_i$, and $M_i \cap N_i = \emptyset$, and for $i \notin J$ let $M_i = N_i = X_i$. Then $[f]_U \in \Pi_U M_i$, $\Pi_U C_i \subseteq \Pi_U N_i$, and $\Pi_U M_i \cap \Pi_U N_i = \emptyset$. □

1.2 Example. *The class* $T_{3.5}$ *of completely regular-spaces is closed.*

Proof. There are several ways of proving this. For our purposes the most convenient is to use the characterization (due to Frink, see [2]) of $T_{3.5}$ spaces as those which possess normal bases for closed sets (i.e. bases γ for closed sets which are closed under finite unions and finite intersections, and which have the property that whenever $C_1, C_2 \in \gamma$ are disjoint there are disjoint open sets $M_1 \supseteq C_1$, $M_2 \supseteq C_2$ such that the complements M_1^c, M_2^c are members of γ). So assume each $\mathfrak{X}_i$ has a normal basis γ_i. Then $\Pi_U \gamma_i$ is a normal basis for $\Pi_U \mathfrak{X}_i$. □

1.3 Example. Suppose for each $i \in I$ we are given a function $\phi_i : \mathfrak{X}_i \to \mathfrak{Y}_i$. If U is an ultrafilter on I then the ultraproduct of the maps ϕ_i, denoted by $\Pi_U \phi_i$, is the assignment $[f]_U \to [g]_U$ where $g(i) = \phi_i(f(i))$. Many of the standard properties (e.g. injectivity, surjectivity, continuity, openness) which one can express about maps hold for $\Pi_U \phi_i$ iff they hold for almost all (modulo U) ϕ_i. Thus in particular *the class of homogeneous spaces is closed while the class of spaces with the fixed point property is open.*

1.4 Example. *The class of (locally) compact spaces is open.*

Proof. Let $\langle \mathfrak{X}_i : i \in I \rangle$ be a family of noncompact spaces. For each $i \in I$ let $\mathfrak{M}_i$ be an open cover with no finite subcover. Then $\Pi_U \mathfrak{M}_i$ is an open cover of $\Pi_U \mathfrak{X}_i$ with no finite subcover. For suppose $\{\Pi_U M_{i,0}, \ldots, \Pi_U M_{i,n}\}$ were such a finite subcover. Then $\Pi_U M_{i,0} \cup \ldots \cup \Pi_U M_{i,n} = \Pi_U (M_{i,0} \cup \ldots \cup M_{i,n})$ because U is an ultrafilter. Thus for almost all $i \in I$ (modulo U) $\{M_{i,0}, \ldots, M_{i,n}\}$ is a finite subcover for $\mathfrak{X}_i$. The argument for local compactness is just as easy. □

1.5 Example. *The class of spaces with the countable chain condition (c.c.c.) is open.*

1.6 Example. *The class of connected spaces is open; the classes of totally disconnected and zero dimensional (in the sense of weak inductive dimension) spaces are closed.*

Proof. We illustrate by proving that the zero dimensional spaces form a closed class. Suppose each $\mathfrak{X}_i$ has a basis β_i such that each member of β_i is clopen. Then it is easy to see that $\Pi_U \beta_i$ is just such a basis for $\Pi_U \mathfrak{X}_i$. □

1.7 Example. *The class of extremally disconnected spaces is open.*

Proof. This contrasts in a sense with (1.6). Suppose each $\mathfrak{X}_i$ has an open set M_i whose closure is not open. Then $\Pi_U M_i$ is an open set in $\Pi_U \mathfrak{X}_i$ whose closure is not open. Indeed $\overline{(\Pi_U M_i)} = \Pi_U \overline{M_i}$. Since for (almost) every $i \in I$ we can find a point $x_i \in \overline{M_i} - M_i$ it follows that, by letting $f(i) = x_i$, $[f]_U \in \Pi_U \overline{M_i} - \Pi_U M_i$. □

1.8 Example. *The class of non-Archimedean spaces is closed.*

Proof. For each $\mathfrak{X}_i$ let β_i be a basis with the defining property that each pair of elements is either disjoint or related under inclusion. Then, since U is maximal, $\Pi_U \beta_i$ has the same property. □

1.9 Example. *The class of separable spaces is open.*

Proof. Suppose $\Pi_U\mathcal{X}_i$ is separable but that the $\mathcal{X}_i$'s are not (either almost all $\mathcal{X}_i$'s are separable or almost all $\mathcal{X}_i$'s are nonseparable). Let $D \subseteq \Pi_U\mathcal{X}_i$ be countable dense, say $D = \{[f_0]_U, \ldots, [f_n]_U, \ldots\}$; let $D_i = \{f_1(i), \ldots, f_n(i), \ldots\}$; and let $J = \{i : D_i$ is not dense$\} \in U$. Then clearly $\Pi_U D_i$ is not dense in $\Pi_U\mathcal{X}_i$. But $D \subseteq \Pi_U D_i$. $\square$

1.10 Example. *Many classes of topological algebras are closed.* One can prove easily that ultraproducts commute with finite cartesian products (i.e. $\Pi_U\mathcal{X}_i \times \mathcal{Y}_i$ and $\Pi_U\mathcal{X}_i \times \Pi_U\mathcal{Y}_i$ are naturally homeomorphic). Thus if each $\mathcal{X}_i$ is equipped with, say, a continuous binary operation $\mu_i : \mathcal{X}_i \times \mathcal{X}_i \to \mathcal{X}_i$ then it makes sense to talk of the "lifting" $\Pi_U\mu_i : \Pi_U\mathcal{X}_i \times \Pi_U\mathcal{X}_i \to \Pi_U\mathcal{X}_i$, again a continuous binary operation. In this way it can be easily shown that *ultraproducts of topological groups (rings, fields) are again topological algebras of the same (first order) axiomatic type.*

1.11 Example. Suppose each $\mathcal{X}_i$ is equipped with a linear order structure $\leqslant_i$. Then the ultraproduct $\Pi_U \leqslant_i$ is again a linear ordering on $\Pi_U X_i$. If, in addition, τ_i is basically generated by open $\leqslant_i$-intervals then $\Pi_U\tau_i$ is basically generated by open $\Pi_U\leqslant_i$-intervals. Thus *ultraproducts of linearly orderable spaces are also linearly orderable spaces.*

1.12 Examples. As another example of extra-topological structure let us consider uniform spaces. We say a uniformity $\mathfrak{U}$ on X is *linear* if there is a basis for $\mathfrak{U}$ (considered as a filter on $X \times X$) whose elements are linearly ordered by inclusion. A space $\mathcal{X}$ is *linearly uniformizable* if there is a linear uniformity on X which generates the topology. For every infinite cardinal κ, the κ*-metrizable* spaces are those linearly uniformizable spaces with a linearly ordered basis of cofinality κ. The ω-metrizable spaces are the metrizable ones. $\mathcal{X}$ is *ultrametrizable* if $\mathcal{X}$ is linearly uniformizable but not metrizable. As a word of preview it turns out that linearly uniformizable spaces are (hereditarily) paracompact; and that in addition ultrametrizable spaces are non-Archimedean (see [14]). We will show presently that *the class of linearly uniformizable spaces is closed.*

Proof. Let $\langle \mathcal{X}_i : i \in I\rangle$ be a family of linearly uniformizable spaces; and for each $i \in I$ let $\mathfrak{B}_i$ be a linearly ordered basis for a uniformity which generates τ_i. Then $\mathfrak{B} = \Pi_U\mathfrak{B}_i$ is a linearly ordered basis for a uniformity which generates $\Pi_U\tau_i$ (strictly speaking, since $\mathfrak{B}_i \subseteq \mathbf{P}(X_i \times X_i)$, we have that $\Pi_U\mathfrak{B}_i \subseteq \Pi_U\mathbf{P}(X_i \times X_i)$. But $\Pi_U\mathbf{P}(X_i \times X_i)$ naturally embeds within $\mathbf{P}(\Pi_U X_i \times \Pi_U X_i)$ so we think of $\Pi_U\mathfrak{B}_i$ in this light, wilfully abusing notation). To see this, it is first clear that $\mathfrak{B}$ is indeed linearly ordered since U is an ultrafilter. That $\mathfrak{B}$ is a basis for a uniformity is equally straightforward. Now suppose β_i is the topological basis generated by $\mathfrak{B}_i$. Then every open ultrabox of $\Pi_U\beta_i$ arises as a $\mathfrak{B}$-neighborhood. Conversely every $\mathfrak{B}$-neighborhood arises as a $\Pi_U\beta_i$-ultrabox. The details are completely straightforward. $\square$

2. Ultrapowers of Euclidean spaces

This paper is not about nonstandard analysis; nor is it about nonstandard topology. Both of these topics are treated in [21] as well as in other excellent references. What does concern us partly is the *topology* of the nonstandard extensions of Euclidean space (what Robinson calls the "Q-topology" in [21] (see also [4, 5])). We would indeed be doing nonstandard topology if we confined our attention to "internal" open sets etc. (i.e. the open ultraboxes).

Let U be a nonprincipal (= free) ultrafilter on a countable set, and let $^*\mathbf{R}$ denote $\Pi_U(\mathbf{R})$ (the topology being, as we have seen, the ultraproduct order topology). Since ultraproducts commute with finite cartesian products we can identify $^*(\mathbf{R}^2)$ with $^*\mathbf{R} \times {}^*\mathbf{R}$, etc. Although questions arising as to dimensionality and invariance of domain (à la Brouwer) can be better dealt with once we develop some more machinery, let us for the present merely preview some of the results in this connection and then return to more mundane tasks.

Most of the following remains true if we do not assume the Continuum Hypothesis (CH); however certain adjustments have to be made regarding the combinatorial nature of U. So for convenience assume CH.

2.1 Fact. If V is another free ultrafilter on ω then $\Pi_U(\mathbf{R})$ and $\Pi_V(\mathbf{R})$ are order isomorphic (algebraically isomorphic as well); hence, in particular, homeomorphic.

2.2 Fact. $^*\mathbf{R}$ and $(^*\mathbf{R})^2$ are (noncanonically) homeomorphic. Thus Brouwer invariance of domain fails for nonstandard Euclidean space. In fact every U-nonstandard Euclidean space without isolated points is homeomorphic to the U-nonstandard rationals $^*\mathbf{Q}$; and all U-nonstandard Euclidean spaces are ultrametrizable and hence have dimension zero (in every known sense of the term).

We now rejoin the search for closed classes and use the U-nonstandard unit internal $^*[0,1]$ to refute some possible conjectures. Let $t \in [0,1]$. We define the *monad* $\mu(t) = \mu_U(t)$ to be the set $\{[f]_U : \text{for all } 0 < n < \omega, \{i : |f(i) - t| < 1/n\} \in U\}$. We call members of $\mu(0)$ *infinitesimals* and use letters ε, δ to denote them. For $[f]_U \in {}^*[0,1]$ we let $\lim_U([f]_U)$ be the unique $t \in [0,1]$ with $[f]_U \in \mu(t)$.

Now each monad, being a union of open intervals, is open. Consequently the set $\mathfrak{M} = \{\mu(t) : t \in [0,1]\}$ forms an uncountable clopen partition of $^*[0,1]$; whence $^*[0,1]$ is not connected, path connected, second countable, separable, Lindelöf, c.c.c., or compact; all properties enjoyed by $[0,1]$. These classes are hence not closed. Not quite as obvious is the fact that, although $[0,1]$ has the fixed point property, $^*[0,1]$ does not. To see this let $\varepsilon > 0$ be an infinitesimal and define ϕ by the rule

$$\phi([f]_U) = \begin{cases} [f]_U + \varepsilon & \text{if } \lim_U([f]_U) < 1 \\ [f]_U - \varepsilon & \text{if } \lim_U([f]_U) = 1. \end{cases}$$

ϕ is defined piecewise on an open partition of $^*[0,1]$ and is continuous on each of the pieces. Thus ϕ is continuous everywhere but cannot have a fixed point.

3. Countably complete ultraproducts

Let $\kappa \geq \omega$ be a cardinal. An ultrafilter U is *κ-complete* if whenever $E \subseteq U$ has cardinality $< \kappa$ then $\bigcap E \in U$. U is *countably complete* if U is ω_1-complete. Although the main emphasis of this paper is on ultraproducts via countably incomplete ultrafilters, there do arise instances where in order to show that a particular class is closed it seems necessary to cleave the proof into two parts dictated by whether or not the ultrafilter in question is countably complete.

By way of reference we list some of the salient facts relating to countably complete ultrafilters and ultraproducts. All of the proofs can be found in [8]. However in some cases we include proof sketches.

We first define a cardinal $\kappa > \omega$ to be *measurable* if there is a free κ-complete ultrafilter on κ. Measurable cardinals are quite large.

3.1 Fact. If κ is measurable then there is an increasing sequence $\langle \lambda_\xi : \xi < \kappa \rangle$ of strongly inaccessible cardinals, each of which is less than κ. In particular κ is strongly inaccessible and so exceeds the power of the continuum.

3.2 Fact (D. Scott). If there exists a measurable cardinal then Gödel's Axiom of Constructibility $(V = L)$ fails.

3.3 Fact. Let U be any free ultrafilter. Then there is a greatest cardinal κ such that U is κ-complete. And if $\kappa > \omega$ then κ is measurable (if there is an ω_1-complete free ultrafilter U on λ, we say λ is *Ulam-measurable*. Thus λ is Ulam-measurable iff $\lambda \geq$ the first measurable cardinal).

Proof, sketch. Since U is free there is a least cardinal λ such that U is not λ-complete. λ cannot be a limit cardinal since if it were then U would clearly be λ-complete. Thus $\lambda = \kappa^+$ for some κ. Since U is not κ^+-complete there is a partition $\langle J_\xi : \xi < \kappa \rangle$ of I so that no J_ξ is in U. Let $f : I \to \kappa$ take i to ξ if $i \in J_\xi$. Then $V = \{K \subseteq \kappa : f^{-1}(K) \in U\}$ is a κ-complete ultrafilter on κ, showing that either $\kappa = \omega$ or κ is measurable. $\square$

A corollary of (3.2, 3.3) is the following:

3.4 Fact. If $V = L$ then every free ultrafilter is countably incomplete.

3.5 Fact. Let U be κ-complete and let $\mathscr{X}$ be a space of power $< \kappa$. Then $\Delta_U : \mathscr{X} \to \prod_U(\mathscr{X})$ is a homeomorphism.

Proof. The trick is in showing that Δ_U is onto. Assume otherwise. Then there is an $f: I \to X$ such that for each $x \in X$ $\{i : f(i) \neq x\} \in U$. Since $|X| < \kappa$ and U is κ-complete, we have $\{i : \text{for all } x \in X, f(i) \neq x\} \in U$; and hence some $i \in I$ has no image under f, a contradiction. □

A corollary of (3.3, 3.5) is the following:

3.6 Fact. Let U be countably complete and let $\mathfrak{X}$ be a space whose power is smaller than the first measurable cardinal. Then Δ_U is a homeomorphism.

3.7 Theorem. *The following properties are preserved under taking ultraproducts via countably complete ultrafilters: first and second countability, separability, metrizability, path disconnectedness, and ω_1-openness (i.e. G_δ sets are open).*

Proof, sketch. Re second countability: Let each $\mathfrak{X}_i$ have a countable base β_i. Then $\Pi_U \mathfrak{X}_i$ has a countable base $\Pi_U \beta_i$.

Re metrizability: Let d_i be a metric on $\mathfrak{X}_i$. Then $\Pi_U d_i$ is a $\Pi_U(\mathbf{R})$-valued "metric" on $\Pi_U \mathfrak{X}_i$ (i.e. $\Pi_U d_i([f]_U, [g]_U) = [h]_U \in \Pi_U(\mathbf{R})$, where $h(i) = d_i(f(i), g(i))$); and the ultraproduct topology is easily seen to be generated by this "metric" (which clearly satisfies the usual metric laws). But U is countably complete; so by (3.1, 3.6), $\Pi_U(\mathbf{R})$ is canonically isomorphic to $\mathbf{R}$.

Re path disconnectedness: Suppose $\Pi_U \mathfrak{X}_i$ is path connected; and for each $i \in I$ let $x_i, y_i \in X_i$. Let $[f]_U, [g]_U \in \Pi_U X_i$ be such that $f(i) = x_i$, $g(i) = y_i$, $i \in I$; and let $p : [0, 1] \to \Pi_U X_i$ connect $[f]_U$, $[g]_U$. Identify p with its graph in $[0, 1] \times \Pi_U X_i$. Then $|p| = \exp(\omega)$; whence $p \circ \Delta_U^{-1} : \Pi_U([0, 1]) \to \Pi_U \mathfrak{X}_i$ is an ultraproduct $\Pi_U p_i$ where almost every p_i is a path from x_i to y_i.

Re ω_1-openness: Countably complete ultraproducts commute with countable set operations. □

Although a property may be "countable", it may not be preserved under countably complete ultraproducts. Such a property seems to be Lindelöfness (and its negation).

In the next section we will show that the class of ω_1-open spaces (otherwise known as P-*spaces*) is closed. Here it seems to be necessary to divide the proof into two cases as we mentioned in the first paragraph of the present section.

4. Countably incomplete ultraproducts

In ZFC the existence of free countably complete ultrafilters is a knotty question; for not only can it not be shown that such ultrafilters exist (viz. Scott's theorem and the consistency of $V = L$), it cannot even be shown that the existence question is consistent (If M were a model of ZFC which had a measurable cardinal then M

would have an inaccessible cardinal and so would contain a model of ZFC. This would contradict Gödel's Incompleteness Theorem). By contrast countably incomplete ultrafilters are a dime a dozen. One way to measure the degree of countable incompleteness is to introduce the notion of "κ-regularity" for $\kappa \geq \omega$ a cardinal. We say U is *κ-regular* if there is a point-finite collection $E \subseteq U$ of power κ (i.e. each $i \in I$ is contained in only finitely many members of E). It can easily be shown that U is countably incomplete iff there is a sequence $J_0 \supseteq J_1 \supseteq \ldots$ of members of U whose intersection is empty. Thus countable incompleteness is ω-regularity. Clearly κ-regularity gets stronger as κ increases. Also if $\kappa = |I|$ then there are no κ^+-regular ultrafilters on I for if there were one, say U, and if $E \subseteq U$ were a "regularizing set" of power κ^+ then there would be a choice function $\chi : E \to I\,(\chi(J) \in J)$. Since for each $i \in I$ $|\chi^{-1}(i)| < \omega$, this would force $|I| \geq |E|$, an impossibility). However there are lots of κ-regular ultrafilters on sets of power κ. To see this we have only to regard I as the set of all finite subsets of some other set J of power κ $(I = \mathbf{P}_\omega(J))$. For each $i \in I$ let $i^\# = \{i' \in I : i \subseteq i'\}$. Then $I^\# = \{i^\# : i \in I\} \subseteq \mathbf{P}(I)$ has the finite intersection property $(i_1^\# \cap \ldots \cap i_n^\# = (i_1 \cup \ldots \cup i_n)^\#)$ and thus extends to a κ-regular ultrafilter on I, the regularizing set being $I^\#$.

A space $\mathfrak{X}$ is *κ-open* iff its topology is closed under $< \kappa$ intersections. All spaces are ω-open; and $\mathfrak{X}$ is ω_1-open iff $\mathfrak{X}$ is a P-space (see [12]) iff every $x \in X$ is a P-point. We prove the following:

4.1 Theorem (Openness Lemma). *An ultrafilter U is κ-regular iff all topological U-ultraproducts are κ^+-open.*

Proof. Model-theoretic analogues of this theorem can be found in [19]. We prove the most useful direction first. Assume U is κ-regular and that $\langle \mathfrak{X}_i : i \in I\rangle$ is a family of spaces. To check κ-openness in a space it clearly suffices to restrict attention to families of basic open sets, so let $\langle \prod_U M_{i,\xi} : \xi < \kappa\rangle$ be such a family in $\prod_U \mathfrak{X}_i$ and let $[f]_U \in \bigcap_{\xi<\kappa} \prod_U M_{i,\xi}$. By κ-regularity there is a (well-ordered) regularizing set $E = \langle J_\xi : \xi < \kappa\rangle \subseteq U$. Let $K_\xi = \{i : f(i) \in M_{i,\xi}\} \in U$; and for each $i \in I$ let $F(i) = \{\xi < \kappa : i \in J_\xi \cap K_\xi\}$. Each $F(i)$ is finite, so the set $N_i = \bigcap_{\xi \in F(i)} M_{i,\xi}$ is open. We show $[f]_U \in \prod_U N_i \subseteq \bigcap_{\xi<\kappa} \prod_U M_{i,\xi}$. Indeed $\{i : f(i) \in N_i\} = I \in U$. Now suppose $\eta < \kappa$. Then $\{i : N_i \subseteq M_{i,\eta}\} \supseteq \{i : \eta \in F(i)\} \supseteq J_\eta \cap K_\eta \in U$. We thus have the desired conclusion.

For the converse (only the first half is proved in [3]) assume all topological U-ultraproducts are κ^+-open, and let X be a set (of power $\geq \kappa$). Then X has κ-regular ultrafilters; in particular there is a set $S = \langle Y_\xi : \xi < \kappa\rangle \subseteq \mathbf{P}(X)$ with the finite intersection property such that no $x \in X$ is contained in infinitely many members of S. Let p be an ultrafilter extending S and let $X' = X \cup \{p\}$ (disjoint union). We form the space $\mathfrak{X}'$ using X' for points, by letting points of X be isolated, and by letting the sets $\langle Y_\xi \cup \{p\} : \xi < \kappa\rangle$ form a neighborhood subbase for p. Observe that $\prod_U (X)$ is a discrete subset of $\prod_U(\mathfrak{X}')$, a κ^+-open space. Now

$\bigcap_{\xi<\kappa}\Pi_U(Y_\xi \cup \{p\})$, being a neighborhood of $\Delta_U(p)$, contains a set $\Pi_U Z_i$ where $Z_i - \{p\} \in p$. Thus there is a point $[f]_U \in \bigcap_{\xi<\kappa}\Pi_U(Y_\xi)$. Let $J_\xi = \{i : f(i) \in Y_\xi\}$, $\xi < \kappa$, and let $E = \langle J_\xi : \xi < \kappa\rangle$. Then E is clearly a κ-regularizing set for U. □

An immediate corollary of (4.1) is that all countably incomplete topological ultraproducts are P-spaces. We have already observed that ω_1-openness is preserved under countably complete ultraproducts. Thus we have

4.2 Corollary. *The class of* P*-spaces is closed.* □

A little less immediate is the following:

4.3 Theorem. *The class of path-disconnected* T_1 *spaces is closed.*

Before proving (4.3) we first define a space to be *totally non-compact* if its only compact subsets are the ones that have to be, namely the finite ones (one could define "totally non-P" for any property P, e.g. connectedness, in a similar manner).

4.4 Lemma. T_1 P*-spaces are totally non-compact.*

Proof. Assume $\mathfrak{X}$ is a T_1 P-space with an infinite compact subset C. C is also an infinite T_1 P-space, so if $A \subseteq C$ is countable then A is closed (hence compact) and discrete, an impossibility. □

Remark. The referee has produced a totally non-compact Hausdorff space which is not a P-space. Let $\langle L_n : n < \omega\rangle$ be a partition of an η_1-set L (i.e. a linearly ordered space such that to each pair of countable $A, B \subseteq L$ with $a < b$ for all $a \in A$, $b \in B$ there exists a point $c \in L$ with $a < c < b$ for all $a \in A$, $b \in B$) into dense sets and let $X = \{\langle x, n\rangle \in L \times (\omega + 1) : x \in L_n\}$ inherit the subspace topology. Then X is Hausdorff, totally non-compact, and *ultraparacompact* (i.e. open covers refine to clopen partitions); but it is not a P-space.

Proof of 4.3. Let $\langle \mathfrak{X}_i : i \in I\rangle$ be a collection of path-disconnected T_1 spaces. If U is countably complete then $\Pi_U\mathfrak{X}_i$ is path-disconnected T_1 by (1.1, 3.7). If U is countably incomplete then $\Pi_U\mathfrak{X}_i$, being a T_1 P-space by (1.1, 4.1), is totally non-compact. Thus $\Pi_U\mathfrak{X}_i$ has only constant paths, so indeed $\Pi_U\mathfrak{X}_i$ is *totally* path-disconnected. Since its cardinality must exceed 1, it is path-disconnected T_1. □

Further applications of (4.1) are the following:

4.5 Theorem. *No countably incomplete ultraproduct of non-discrete* T_1 *spaces is first countable. Thus if there are no measurable cardinals then the classes of non-(first countable), non-(second countable), and non-metrizable* T_1 *spaces are closed.*

Proof. Let the $\mathfrak{X}_i$'s be non-discrete T_1 spaces. Then $\Pi_U\mathfrak{X}_i$ is non-discrete T_1. If $\Pi_U\mathfrak{X}_i$ were first countable then every point would be the intersection of its countable nbd basis so would be isolated by (4.1). Now if there are no measurable cardinals then every free ultrafilter is countably incomplete; and if the $\mathfrak{X}_i$'s are non-(first countable) T_1 then by the above (since they must then be non-discrete) $\Pi_U\mathfrak{X}_i$ is non-(first countable) T_1. If the $\mathfrak{X}_i$'s are non-(second countable) T_1 then $\Pi_U\mathfrak{X}_i$ is a T_1 P-space so cannot be second countable unless it is discrete. But second countable discrete spaces are necessarily countable; and it is an elementary fact in the theory of ultraproducts (see [8]) that infinite ultraproducts via countably incomplete ultrafilters are uncountable. Thus $\Pi_U\mathfrak{X}_i$ is non-(second countable) T_1. Finally if the $\mathfrak{X}_i$'s are non-metrizable and T_1 then $\Pi_U\mathfrak{X}_i$ is not discrete but is a T_1 P-space. Hence it cannot be first countable, let alone metrizable. □

4.6 Theorem. *No infinite countably incomplete Hausdorff ultraproduct has the c.c.c.*

Proof. Suppose $\Pi_U\mathfrak{X}_i$ is infinite. Then for each $n<\omega\{i:|X_i|>n\}\in U$. Let $I=J_0\supseteq J_1\supseteq\dots$ be a descending sequence of members of U with empty intersection. Without loss of generality we may assume that for each $n<\omega$, $i\in J_n$, $|X_i|>n$. Now for each $n<\omega$, $i\in J_n-J_{n+1}$, find n points in X_i and separate them via n disjoint open sets, thereby creating a family $\mathfrak{M}_i=\{M_{i,1},\dots,M_{i,n}\}$ of mutually disjoint nonempty open sets. Since U is countably incomplete, $\Pi_U\mathfrak{M}_i$ is an uncountable collection of mutually disjoint nonempty open sets in $\Pi_U\mathfrak{X}_i$. □

We will see later on that topological ultraproducts, endowed with suitable separation properties, are totally disconnected in very strong ways. However, they are not often extremally disconnected, as the next theorem shows.

4.7 Theorem. *Let $\langle\mathfrak{X}_i : i\in I\rangle$ be a family of nondiscrete regular spaces such that none of the cardinals $|X_i|$ $(i\in I)$, $|I|$ are Ulam-measurable. Then if U is countably incomplete on I, $\Pi_U\mathfrak{X}_i$ cannot be extremally disconnected.*

Proof. A cardinal is *moderate* if it is not Ulam-measurable. Now $\Pi_U\mathfrak{X}_i$ is a nondiscrete regular P-space. If the cardinals $|X_i|$ $(i\in I)$, $|I|$ are moderate then, since measurable cardinals are strongly inaccessible, $|\Pi_U\mathfrak{X}_i|$ is moderate too. Let $\mathfrak{X}=\Pi_U\mathfrak{X}_i$ and assume that $\mathfrak{X}$ is extremally disconnected. Pick a non-isolated $p\in X$. Now regular P-spaces are zero dimensional (in the weak inductive sense) since if $x\in X$ and M_0 is a nbd of x we can find a nbd M_1 with $x\in M_1\subseteq\bar{M}_1\subseteq M_0$. Repeating for M_1, we obtain M_2, M_3, and so on. Thus $\bigcap_{n<\omega}M_n$ is a clopen nbd of x contained in M_0. Clearly there is a family $\mathfrak{M}$ of pairwise disjoint nonempty clopen sets not containing p; and by Zorn's Lemma, we can pick $\mathfrak{M}$ maximal with this property. p is not isolated so $\bigcup\mathfrak{M}$ is dense in X, i.e. $p\in\overline{\bigcup\mathfrak{M}}$. Let $\mathfrak{U}=\{\mathfrak{M}'\subseteq\mathfrak{M}:p\in\overline{\bigcup\mathfrak{M}'}\}$. We show $\mathfrak{U}$ is a countably complete free ultrafilter on $\mathfrak{M}$. This will force $|\mathfrak{M}|$ to be Ulam-measurable; and since $|\mathfrak{M}| \leq |X|$, this will yield a contradiction.

Clearly, $\emptyset \notin \mathfrak{U}$, $\mathfrak{M} \in \mathfrak{U}$, $\mathfrak{U}$ is closed under superset, and no singleton $\{M\}$ is in $\mathfrak{U}$ (since elements $M \in \mathfrak{M}$ are clopen). Moreover if $\bigcup_{n<\omega}\mathfrak{M}_n \in \mathfrak{U}$ then, since $\mathfrak{X}$ is a P-space, $p \in \overline{\bigcup(\bigcup_{n<\omega}\mathfrak{M}_n)} = \overline{\bigcup_{n<\omega}(\bigcup\mathfrak{M}_n)} = \bigcup_{n<\omega}\overline{(\bigcup\mathfrak{M}_n)}$. Thus for some $n < \omega$, $\mathfrak{M}_n \in \mathfrak{U}$. It remains to show that $\mathfrak{U}$ is closed under finite intersections. Let $\mathfrak{M}_1$, $\mathfrak{M}_2 \in \mathfrak{U}$ with $M_1 = \bigcup\mathfrak{M}_1$, $M_2 = \bigcup\mathfrak{M}_2$. Then $p \in \bar{M}_1 \cap \bar{M}_2 \supseteq \overline{M_1 \cap M_2}$. To show equality, suppose N is open, $N \subseteq \bar{M}_1 \cap \bar{M}_2$. Then $\bar{N} - M_1$, $\bar{N} - M_2$ are closed nowhere dense, thus so is $\bar{N} - (M_1 \cap M_2)$. We then conclude that $N \subseteq \overline{M_1 \cap M_2}$. Now $\mathfrak{X}$ is extremally disconnected. Therefore $\bar{M}_1 \cap \bar{M}_2 = \mathrm{Int}(\bar{M}_1) \cap \mathrm{Int}(\bar{M}_2) = \mathrm{Int}(\bar{M}_1 \cap \bar{M}_2) \subseteq \mathrm{Int}(\overline{M_1 \cap M_2}) = \overline{M_1 \cap M_2}$ so $\mathfrak{M}_1 \cap \mathfrak{M}_2 \in \mathfrak{U}$. □

Remark. The above proof derives its inspiration from [12] problem 12H. R. Button (in a personal communication) also noted the same problem and independently obtained a similar result for Q-topologies.

4.8 Corollary. *The class of extremally disconnected spaces is not closed.* □

5. Strong disconnected properties

Since proving that the class of T_0 spaces is both open and closed, we have been assuming all spaces to obey the T_1 axiom. We will continue to do so. We define the disconnectedness classes $\langle D_n : n = 0, 1, 2, 3, 3.5, 4\rangle$ as follows:

D_0 = totally path-disconnected spaces.

D_1 = totally disconnected spaces.

D_2 = *ultra-Hausdorff* spaces = those spaces in which two distinct points are separable via disjoint clopen sets.

D_3 = *ultra-regular* spaces = those spaces in which a point and a disjoint closed set are separable via disjoint clopen sets = the weak inductive zero-dimensional spaces.

$D_{3.5}$ = *strongly zero-dimensional* spaces = those Tichonov spaces in which two disjoint zero sets are separable via disjoint clopen sets.

D_4 = *ultranormal* spaces = those spaces in which two disjoint closed sets are separable via disjoint clopen sets = the strong inductive zero-dimensional spaces.

Clearly $D_0 \supseteq D_1 \supseteq D_2 \supseteq D_{3.5} \supseteq D_4$; and if $\mathfrak{X}$ is compact T_2 then $\mathfrak{X} \in D_1$ iff $\mathfrak{X} \in D_4$ iff $\mathfrak{X}$ is Boolean.

Three more disconnectedness classes, listed in decreasing order with respect to inclusion (see [1, 14]) are ultraparacompactness, non-Archimedeanness, and ultrametrizability. The following facts are well-known:

5.1 Fact. $D_4 = T_4 \cap D_{3.5}$ (by Tietze's extension theorem).

5.2 Fact. $\mathfrak{X} \in D_{3.5}$ iff $\beta(\mathfrak{X})$, the Stone-Čech compactification of $\mathfrak{X}$, is Boolean.

5.3 Fact. $\mathfrak{X}$ is ultraparacompact iff $\mathfrak{X}$ is paracompact and D_4 (see [11]).

5.4 Fact. $\mathfrak{X}$ is D_4 iff $\mathfrak{X}$ has covering dimension zero (i.e. finite open covers refine to (finite) clopen partitions).

5.5 Fact. If $\mathscr{X}$ is ultraparacompact, $C \subseteq \mathscr{X}$ is closed, and $f: C \to \mathscr{Y}$ is (bounded) continuous to a complete metric space, then f extends to a (bounded) continuous $F: \mathscr{X} \to \mathscr{Y}$ (see [11]).
5.6 Fact. Boolean spaces are ultraparacompact (we will see later that they needn't be non-Archimedean).
5.7 Fact. Separable metric D_3 spaces are ultraparacompact, since they are D_4 and paracompact (use (5.3) plus standard dimension theory [20]).

P-spaces enter the picture as follows:

5.8 Theorem. (i) T_1 *P-spaces are* D_0.
(ii) *Functionally Hausdorff* P-*spaces are* D_2.
(iii) *Regular* P-*spaces are* $D_{3,5}$.
(iv) *Normal* P-*spaces are* D_4.
(v) *Paracompact* P-*spaces are ultraparacompact.*

Proof. Let $\mathscr{X}$ be a P-space.
Re (i): If $\mathscr{X}$ is T_1 then by (4.4) $\mathscr{X}$ is totally non-compact, hence D_0.
Re (ii): Let $\mathscr{X}$ be functionally Hausdorff with $x, y \in X$ distinct, let $f: \mathscr{X} \to [0,1]$ be continuous taking x to 0, y to 1, and let $M_n = f^{-1}([0, 1/n+1))$, $n < \omega$. Then $M_0 \supseteq \bar{M}_1 \supseteq M_1 \supseteq \bar{M}_2 \supseteq \ldots$, and $M = \bigcap_{n<\omega} M_n$ is a clopen nbd of x missing y.
Re (iii): Let $\mathscr{X}$ be regular with Z_1, Z_2 disjoint zero sets, say $Z_1 = f_1^{-1}(0)$, $Z_2 = f_2^{-1}(0)$ for $f_1, f_2: \mathscr{X} \to [0,1]$ continuous. For each $x \in X$, $f_1(x) + f_2(x) > 0$ so $f = f_1/(f_1+f_2): \mathscr{X} \to [0,1]$ is continuous, takes Z_1 to 0, and takes Z_2 to 1. For $n < \omega$ let $M_n = f^{-1}([0, 1/n+1))$. Then the M_n's are open; and for $m < n < \omega$ we have $Z_1 \subseteq M_n \subseteq \overline{M_n} \subseteq M_m \subseteq X - Z_2$. Let $M = \bigcap_{n<\omega} M_n$. Then M is a clopen separation of Z_1, Z_2. We need $\mathscr{X}$ to be completely regular (= Tichonov). But we saw in the proof of (4.7) that $\mathscr{X}$ is actually ultraregular.
Re (iv): Similar to (iii).
Re (v): If $\mathscr{X}$ is paracompact then $\mathscr{X}$ is normal, hence D_4. Then use (5.3). □

6. Normal and paracompact ultraproducts

Since regularity is preserved under ultraproducts, any countably incomplete ultraproduct of regular spaces is $D_{3.5}$. However, there are examples to be seen later to show that ultrapowers of compact Hausdorff spaces needn't even be normal. How then might we hope to ensure normality and paracompactness in ultraproducts? One way is to condition the spaces $\mathscr{X}_i$ by letting them have closed properties which are stronger than the properties we want $\prod_U \mathscr{X}_i$ to enjoy. Such closed properties are at our disposal, namely linear orderability (implying collectionwise normality), linear uniformizability (implying paracompactness (see [13])), and non-Archimedeanness (also implying paracompactness (see [1])). In fact, if $\mathscr{X}$ is either linearly uniformizable or non-Archimedean then $\mathscr{X}$ is hereditarily paracompact. Thus we have the following corollary of (1.8), (1.11), (1.12):

6.1 Theorem. *Let U be countably incomplete, $\langle \mathscr{X}_i : i \in I\rangle$ a family of spaces. If the $\mathscr{X}_i$'s are linearly orderable then $\Pi_U \mathscr{X}_i$ is ultranormal; and if the $\mathscr{X}_i$'s are either non-Archimedean or linearly uniformizable then $\Pi_U \mathscr{X}_i$ is hereditarily ultraparacompact.* □

A second way to obtain paracompact ultraproducts is to condition the ultrafilter. We will show (using the Generalized Continuum Hypothesis) that to every collection $\langle \mathscr{X}_i : i \in I\rangle$ of regular spaces (where I is sufficiently large) there is an ultrafilter U so that $\Pi_U \mathscr{X}_i$ is hereditarily ultraparacompact (in an appendix we will give a more model-theoretic proof which avoids the GCH and which shows that every regular space has an ultrametrizable ultrapower). An immediate consequence of this result is that *the classes of Tichonov, normal, hereditarily normal, collectionwise normal and paracompact (as well as lots of other classes, e.g. linearly orderable spaces etc.) spaces are not open.*

6.2 Theorem. *Assume $\kappa^+ = \exp(\kappa)$ and suppose for each $i \in I$, $\mathscr{X}_i$ is a regular space with weight $\leq \kappa = |I|$. If U is regular on I then $\Pi_U \mathscr{X}_i$ is hereditarily ultraparacompact.*

Proof. $\Pi_U \mathscr{X}_i$ is regular and κ^+-open, hence zero-dimensional. Also since each $\mathscr{X}_i$ has a basis β_i of power $\leq \kappa$ we know that $\Pi_U \mathscr{X}_i$ has a basis $\Pi_U \beta_i$ of power $\leq |\kappa^\kappa| = \exp(\kappa) = \kappa^+$. Thus the ultraproduct is hereditarily κ^+-Lindelöf (i.e. open covers have subcovers of power $\leq \kappa^+$). Now regular α-open α-Lindelöf spaces are paracompact for $\alpha = \omega$ and ultraparacompact for $\alpha > \omega$. The countable case is well-known; the uncountable case even easier. For let $\mathscr{X}$ be given with open cover $\mathfrak{M}$. Since $\mathscr{X}$ is zero-dimensional and α-Lindelöf ($\alpha > \omega$) we can assume $\mathfrak{M}$ can be well-ordered in type α, say $\langle M_\xi : \xi < \alpha\rangle$ where each M_ξ is clopen. Use α-openness to refine $\mathfrak{M}$ to a clopen partition by letting $N_\xi = M_\xi - \bigcup_{\lambda<\xi} M_\lambda$ for $\xi < \alpha$. The result follows immediately. □

6.3 Corollary (GCH). *Every regular space has a hereditarily ultraparacompact ultrapower.* □

7. Ultrapowers and diagonals

We concentrate for much of the remainder of this paper on ultrapowers. In this special case there is the additional structure accorded us by the presence of the diagonal $\Delta_U : \mathscr{X} \to \Pi_U(\mathscr{X})$. When context permits we drop the U-subscript so that Δ will denote the diagonal map with $\Delta(\mathscr{X})$ its image in $\Pi_U(\mathscr{X})$. Now for box powers and cartesian powers the diagonal is always a retract of the power, for we can simply fix $i_0 \in I$ and map f to $f(i_0)$. This mapping is no longer well-defined in the ultrapower case so a natural question to ask is what conditions suffice to force the diagonal to be a retract of the ultrapower. As a dividend, the techniques we develop here will yield a normal space with a non-normal ultrapower.

We assume all spaces to be Hausdorff. Let $\mathfrak{X}$ and U be given. The *coarse topology* on $\Pi_U(\mathfrak{X})$ is that topology on the set $\Pi_U(X)$ which is basically generated by the open ultracubes $\Pi_U(M)$, $M \subseteq \mathfrak{X}$ open. If $Y \subseteq \Pi_U(X)$, Y^0 denotes the underlying set of Y topologized via the coarse topology (thus $\Delta(\mathfrak{X})^0$ is homeomorphic, via Δ, to $\mathfrak{X}$). For $x \in X$ define $\mu(x) = \mu_U(x)$, the *monad* of x, to be $\bigcap\{\Pi_U(M) : x \in M$ open$\}$. By Hausdorffness, $x \neq y$ implies $\mu(x) \cap \mu(y) = \emptyset$. The *near-diagonal* $N(\mathfrak{X}) = N_U(\mathfrak{X})$ is the union of the monads, and the *U-limit map* $\lim = \lim_U$ takes $[f] \in \mu(x)$ to x. Clearly lim is a left-inverse for Δ. lim is *diagonal continuous* if it is continuous as a map from $N(\mathfrak{X})$ to $\Delta(\mathfrak{X})$ (i.e. if $\Delta \circ \lim$ is continuous). lim is *coarse continuous* if it is continuous as a map from $N(\mathfrak{X})^0$ to $\mathfrak{X}$.

Note that Δ is rarely continuous. In fact if $\mathfrak{X}$ is first countable and U is countably incomplete then $\Delta(\mathfrak{X})$ is discrete.

The *character* $\operatorname{ch}(\mathfrak{X})$ of a space $\mathfrak{X}$ is the least cardinal κ such that each $x \in X$ has a nbd basis of power κ. The *regularity* $\operatorname{reg}(U)$ of an ultrafilter U is the least κ such that U is not κ-regular. Observe that $\mathfrak{X}$ is first countable iff $\operatorname{ch}(\mathfrak{X}) = \omega$ and that U is countably complete iff $\operatorname{reg}(U) = \omega$ (its least possible value). U is regular iff $\operatorname{reg}(U) = |I|^+$ (its greatest possible value).

We now look to the continuity of lim.

7.1 Theorem. *If either $\mathfrak{X}$ is regular or* $\operatorname{ch}(\mathfrak{X}) < \operatorname{reg}(U)$ *then* lim *is continuous. If* $\operatorname{ch}(\mathfrak{X}) < \operatorname{reg}(U)$ *then each monad is open so $N(\mathfrak{X})$ is open and $\Delta(\mathfrak{X})$ is discrete.*

Proof. Suppose first that $\mathfrak{X}$ is regular. We show lim to be coarse continuous, a result which will be useful later on. If $M \subseteq \mathfrak{X}$ is open we show that $\lim^{-1}(M) = N(\mathfrak{X}) \cap \bigcup\{\Pi_U(M') : M'$ open, $\overline{M'} \subseteq M\}$. Indeed suppose $[f] \in \lim^{-1}(M)$, say $\lim([f]) = x$. By regularity there is an open M' with $x \in M' \subseteq \overline{M'} \subseteq M$, so $[f] \in \Pi_U(M')$. For the reverse inclusion suppose $[f] \in N(\mathfrak{X}) \cap \Pi_U(M')$, M' open, $\overline{M'} \subseteq M$. If M'' is any nbd of $x = \lim([f])$ then $[f] \in \Pi_U(M'')$ so that in particular $\Pi_U(M') \cap \Pi_U(M'') = \Pi_U(M' \cap M'') \neq \emptyset$. This gives $x \in \overline{M'}$, so $[f] \in \lim^{-1}(M)$.

Now lift the regularity condition but impose $\operatorname{ch}(\mathfrak{X}) < \operatorname{reg}(U)$. Let $x \in X$ with $\mathfrak{M}$ a nbd basis of power $\operatorname{ch}(\mathfrak{X})$. Then $\mu(x) = \bigcap\{\Pi_U(M) : M \in \mathfrak{M}\}$. Now U is $\operatorname{ch}(\mathfrak{X})$-regular so by (4.1) $\Pi_U(\mathfrak{X})$ is $(\operatorname{ch}\mathfrak{X}))^+$-open, whence $\mu(x)$ is open. It then follows immediately that lim is continuous, $N(\mathfrak{X})$ is open, and $\Delta(\mathfrak{X})$ is discrete. □

To answer the retraction question there are two broad avenues which present themselves: Firstly we could find out when $\Delta(\mathfrak{X})$ is a retract (via lim, say) of $N(\mathfrak{X})$ and try to retract $\Pi_U(\mathfrak{X})$ onto $N(\mathfrak{X})$; and secondly we could determine when $\Delta(\mathfrak{X})$ is closed discrete in $\Pi_U(\mathfrak{X})$ with $\Pi_U(\mathfrak{X})$ ultraparacompact. For then we could apply Ellis' theorem [11] directly since discrete spaces are complete metrizable.

Let $\mathscr{X}$ be a space, κ a cardinal. We form the *κ-modification* $(\mathscr{X})_\kappa$ of $\mathscr{X}$ by using the closure of the topology of $\mathscr{X}$ under $<\kappa$ intersections as a new basis. The operation $(\mathscr{X})_\kappa$ can easily be made functorial for the benefit of category theorists; and the result is a coreflection of the category Top of spaces and continuous maps onto its full subcategory of κ-open spaces. We abuse notation and write $(f)_\kappa$ again as f since the two functions are point-wise identical.

7.2 Theorem. *Let U be regular on a set of power κ. Then $\Delta : (\mathscr{X})_{\kappa^+} \to \Pi_U(\mathscr{X})$ is a homeomorphism onto $\Delta(\mathscr{X})$.*

Proof. It is a triviality to show that Δ is relatively open (i.e. for $M \subseteq X$ open, $\Delta(M) = \Delta(\mathscr{X}) \cap \Pi_U(M)$, so is open in $\Delta(\mathscr{X})$). To show that $(\Delta)_{\kappa^+}$ is also relatively open we note first that since Δ is one-one it preserves intersections. Also since $\Pi_U(\mathscr{X})$ is κ^+-open it follows that $\Delta(\bigcap_{\xi<\kappa} M_\xi) = \bigcap_{\xi<\kappa} \Delta(M_\xi)$ is open in $\Delta(\mathscr{X})$. As for continuity let $\Pi_U M_i$ be open in $\Pi_U(\mathscr{X})$. Then $\Delta^{-1}(\Pi_U M_i) = \bigcup_{J\in U} \bigcap_{i\in J} M_i$ which is open in $(\mathscr{X})_{\kappa^+}$ since $|J| \leqslant \kappa$. □

7.3 Corollary. *Let U be regular on a set of power κ. If either $\mathscr{X}$ is regular or* $\mathrm{ch}(\mathscr{X}) \leqslant \kappa$ *then* lim *is diagonal continuous, so that $\Delta(\mathscr{X})$ is a retract of $N(\mathscr{X})$.* □

To get $N(\mathscr{X})$ to be a retract of $\Pi_U(\mathscr{X})$ we will try to ensure that $N(\mathscr{X})$ is clopen in $\Pi_U(\mathscr{X})$; for clopen sets are always retracts. One straightforward approach is to ask when $N(\mathscr{X}) = \Pi_U(\mathscr{X})$. The answer goes back to Robinson; the following is a slight modification of a result in [21].

7.4 Theorem. $N(\mathscr{X}) = \Pi_U(\mathscr{X})$ *for all ultrafilters U iff $\mathscr{X}$ is compact.*

Proof. Let $\mathscr{X}$ be compact with $[f] \in \Pi_U(\mathscr{X}) - N(\mathscr{X})$. Then for $x \in X$ there is a nbd M_x of x with $[f] \notin \Pi_U(M_x)$. Take a finite subcover of the M_x's. We can then conclude that $\{i : f(i) \notin X\} \in U$ which is nonsense.

Conversely if $\mathscr{X}$ is noncompact and $\mathfrak{M}$ is an open cover with no finite subcover, we let $I = \mathbf{P}_\omega(\mathfrak{M})$, $I^\# = \{i^\# : i \in I\}$, where $i^\# = \{j \in I : i \subseteq j\}$, and let $U \supseteq I^\#$ be an ultrafilter on I. Then, letting $f(i) \in X - \cup\, i$, we show $[f] \in \Pi_U(\mathscr{X}) - N(\mathscr{X})$. For if $M \in \mathfrak{M}$ then, supposing $i \in \{\mathfrak{M}\}^\#$ (i.e. $M \in i$), we have $f(i) \notin M$. But $\{M\}^\# \in U$, so $[f] \notin \Pi_U(M)$. □

More generally we have the following:

7.5 Theorem. *If $\mathscr{X}$ is locally compact and* $\mathrm{reg}(U)$*-compact (where α-compact means that every open cover has a subcover of power $<\alpha$.* N.B. *α-Lindelöf = α^+-compact) then $N(\mathscr{X})$ is clopen in $\Pi_U(\mathscr{X})$.*

Proof. Assume $\mathscr{X}$ is locally compact and pick for each $x \in X$ a nbd M_x with $\overline{M_x}$ compact. Then $N(\mathscr{X}) = \cup\{\Pi_U(M_x) : x \in X\}$. For on the one hand if $[f] \in \mu(x)$ for some $x \in X$ then by definition $[f] \in \Pi_U(M_x)$; and on the other hand, since each $\overline{M_x}$ is compact, the elements of $\Pi_U(M_x)$ must be near-diagonal by (7.4). Now $\mathfrak{X}$ is also reg (U)-compact so that we may take a subcover $\langle M_\xi : \xi < \kappa \rangle$ of the M_x's where $\kappa < \text{reg}(U)$. Then $N(\mathfrak{X}) = \bigcup_{\xi<\kappa} \overline{\Pi_U(M_\xi)}$, a union of $\leqslant \kappa$ closed sets in a κ^+-open space. Hence $N(\mathfrak{X})$ is also closed. □

Collecting what we know so far, we can state the following:

7.6 Theorem. *Let $\mathfrak{X}$ be a Hausdorff space, U an ultrafilter on a set I of power κ. Then $\Delta(\mathfrak{X})$ is a retract of $\Pi_U(\mathfrak{X})$ if any of the following holds:*
(i) $|X| + \text{ch}(\mathfrak{X}) < \text{reg}(U)$;
(ii) $\mathfrak{X}$ *compact, U regular; or*
(iii) $\mathfrak{X}$ *locally compact κ-Lindelöf, U regular.*

Proof. Re (i): By (7.1) $N(\mathfrak{X})$ is open. Also, since $\Pi_U(\mathfrak{X})$ is $|X|^+$-open and each monad is closed, $N(\mathfrak{X})$ is closed as well. By (7.3) $\Delta(\mathfrak{X})$ is a retract of $N(\mathfrak{X})$.

Re (ii): This is a special case of (iii).

Re (iii): By (7.3) (locally compact spaces are regular) $\Delta(\mathfrak{X})$ is a retract of $N_U(\mathfrak{X})$. Invoke (7.5). □

Let us now return to our earlier-mentioned second tack, namely to insure that $\Pi_U(\mathfrak{X})$ is ultraparacompact and that $\Delta(\mathfrak{X})$ is both discrete and closed. We already know how to obtain ultraparacompactness and discreteness, so the trick is to get $\Delta(\mathfrak{X})$ closed. *Until further notice assume I is countable, say $I = \omega$.* An ultrafilter U on ω is *preselective* (*selective*) if whenever we partition ω into countably many blocks each of which fails to be in U there is a set $J \in U$ which intersects each member of the partition in at most finitely many ($\leqslant 1$) points. There is quite a literature on preselective and selective ultrafilters (as points of the space $\beta(\omega) - \omega$), see [7] or [22] for details. An important property of preselective ultrafilters (the P-points of $\beta(\omega) - \omega$) U is that for any countable sequence $\langle J_n : n < \omega \rangle$ of members of U there is an element $J \in U$ "contained in each J_n modulo a finite set" (i.e. $|J - J_n| < \omega$ for each $n < \omega$). The existence of such ultrafilters is ensured by Martin's Axiom (MA) (= Every c.c.c. compact T_2 space is $c = \exp(\omega)$ – Baire). It is unknown whether MA is necessary.

7.7 Theorem. *Let $\mathfrak{X}$ be regular, U preselective. Then $\Delta(\mathfrak{X})$ is closed in $\Pi_U(\mathfrak{X})$.*

Proof. Let $[f] \in \Pi_U(\mathfrak{X}) - \Delta(\mathfrak{X})$. We show that $[f]$ has a representative g such that there are nbds N_n, $n < \omega$, pairwise disjoint, such that $g(n) \in N_n$ for all $n < \omega$. Then $[f] \in \Pi_U N_n$ which is disjoint from $\Delta(\mathfrak{X})$.

Let $J = \{m < \omega : \lim([f]) \neq f(m)\}$. Then if either $[f] \notin N(\mathfrak{X})$ or $\lim([f]) \neq f(m)$ for any $m < \omega$, we have $J = \omega$. Otherwise if $\lim([f]) = f(m_0)$ then since $\mathfrak{X}$ is Hausdorff and $[f]$ is not U-constant, we have that $J \supseteq \{m : f(m) \neq f(m_0)\} \in U$. In any event, $J \in U$. Now for each $m \in J$ let M_m be a nbd of $f(m)$ such that $K_m = \{n < \omega : f(n) \notin M_m\} \in U$, and by preselectivity let $K \in U$ be such that $K - K_m$ is finite for each $m \in J$, with $L = J \cap K \in U$. Then $\{n \in K : f(n) \in M_m\}$ is finite for each $m \in J$ so shrink M_m to a nbd M'_m of $f(m)$ so that $\{n \in K : f(n) \in M'_m\} = \{m\}$. Now use regularity plus the countability of our index set. By regularity there is an open M''_m with $f(m) \in M''_m \subseteq \overline{M''_m} \subseteq M'_m$. For each $m \in L$ $\{n \in L : f(n) \in \overline{M''_m}\} = \{m\}$. So let $N_m = M'_m - \bigcup_{n<m} \overline{M''_n}$, $m \in L$. Then the N_m's are pairwise disjoint so $\Pi_U N_n$ is a nbd of $[f]$ missing $\Delta(\mathfrak{X})$ as promised. □

We can now state another theorem like (7.6).

7.8 Theorem. *Let $\mathfrak{X}$ be a first countable regular space, U a preselective ultrafilter. Then $\Delta(\mathfrak{X})$ is a retract of $\Pi_U(\mathfrak{X})$ if any of the following holds:*

(i) *$\mathfrak{X}$ is non-Archimedean*;

(ii) *$\mathfrak{X}$ is linearly uniformizable*; *or*

(iii) *(CH) $\mathfrak{X}$ has weight $\leq c$.*

Proof. We want $\Delta(\mathfrak{X})$ to be closed discrete and for $\Pi_U(\mathfrak{X})$ to be ultraparacompact. Then Ellis' Theorem will come into play. If $\mathfrak{X}$ is first countable regular then $\Delta(\mathfrak{X})$ is discrete closed by (4.1), (7.7). If either (i) or (ii) holds, then $\Pi_U(\mathfrak{X})$ is ultraparacompact by (6.1). If CH and $\mathfrak{X}$ has weight $\leq c$ then $\Pi_U(\mathfrak{X})$ has weight $\leq |c^\omega| = c = \omega_1$. We then proceed as in the proof of (6.2) to show that $\Pi_U(\mathfrak{X})$ is ultraparacompact. □

Remark. The operation $\Pi_U(\cdot)$ acts on functions between spaces as well as on spaces themselves; and it is a triviality to show that $f : \mathfrak{X} \to \mathfrak{Y}$ is continuous iff $\Pi_U(f)$ is continuous. However, $\Pi_U(f)$ needn't be the only $g : \Pi_U(\mathfrak{X}) \to \Pi_U(\mathfrak{Y})$ extending f (i.e. $\Delta \circ f = g \circ \Delta$). In fact under a variety of conditions (i.e. where $\Delta(\mathfrak{X})$ is a discrete retract of $\Pi_U(\mathfrak{X})$, (see (7.6), (7.8))) any function $f : \mathfrak{X} \to \mathfrak{Y}$ has a continuous extension $g : \Pi_U(\mathfrak{X}) \to \Pi_U(\mathfrak{Y})$. This approach can also be used to find continuous fixed point free maps on ultrapowers (cf. the end of Section 2) with diagonals that are discrete retracts. Merely take a fixed point free permutation of the diagonal and compose with inclusion and the retraction.

8. Non-normal ultraproducts

In this section we produce a normal space (in fact a perfect Boolean space) which has a non-normal ultrapower. Not only will this show that the classes of normal, collection-wise normal, and paracompact spaces fail to be closed, but also that a Boolean space needn't be linearly orderable, linearly uniformizable, or non-Archimedean. There are two proofs known to us: the first (in order of discovery) uses MA plus (7.2), (7.7); the second uses merely (7.2), (7.4). We are indebted to Kunen [16] for the following:

8.1 Theorem. *Let* $\kappa = c^+$ $(c = \exp(\omega))$ *and let* $\mathscr{X} = 2^\kappa$ *with the cartesian product topology. Then* $\mathscr{X}$ *is a perfect Boolean space whose* ω_1*-modification fails to be normal.* □

Remark. Via personal communication from E. K. van Douwen, we have learned that (8.1) holds for $\kappa = \omega_2$ (van Douwen and Borges). Now if $\kappa < \lambda$ then 2^κ canonically imbeds within 2^λ as a closed subset. Thus $(2^\kappa)_{\omega_1}$ sits as a closed subset of $(2^\lambda)_{\omega_1}$. Kunen's theorem can then be strengthened to read: *For* $\kappa \geqslant \omega_2$, $(2^\kappa)_{\omega_1}$ *fails to be normal.*

8.2 Theorem. *Let* $\mathscr{X} = 2^\kappa$ *for* $\kappa \geqslant \omega_2$. *Then there is an ultrafilter* U *on* ω *such that* $\Pi_U(\mathscr{X})$ *is not normal.*

First proof (MA). Let U be preselective on ω. By (7.2), (7.7), $(\mathscr{X})_{\omega_1}$ sits as a closed subset of $\Pi_U(\mathscr{X})$. Since $(\mathscr{X})_{\omega_1}$ is non-normal so also is $\Pi_U(\mathscr{X})$. □

Second proof. Let U be any free ultrafilter on ω. By (7.2), (7.4), $(\mathscr{X})_{\omega_1}$ sits as a retract of $\Pi_U(\mathscr{X})$. But retracts are always closed. □

8.3 Corollary. *The classes of normal, collectionwise normal, and paracompact spaces are not closed.* □

8.4 Corollary. *The space* 2^ω *is a perfect Boolean space which is linearly orderable, linearly uniformizable (indeed metrizable), and non-Archimedean (indeed non-Archimedean metric). However, for* $\kappa > \omega_1$ *the spaces* 2^κ *fail to have these properties.*

Proof. The properties in question are closed, and each implies normality. If 2^κ had any of these properties for $\kappa > \omega_1$ then each of its ultrapowers would have to be normal. This contradicts (8.2). □

Remark. (8.4) can be deduced directly from the old result of A. H. Stone that ω^κ is not normal for $\kappa > \omega$. For since $2^\kappa \cong (2^\omega)^\kappa$, ω^κ embeds within 2^κ; hence 2^κ is not hereditarily normal for $\kappa > \omega$. The properties in question imply hereditary normality however.

9. Ultraproducts and box products

Box products comprise an important source of counterexamples and open problems in contemporary point-set topology (see Rudin's lecture notes [22] for an excellent summary). Our original motivation in studying ultraproducts was to gain insight into box products; however most of the insights gained related to the ultraproducts themselves and the box products remained as enigmatic as ever.

However oddly-behaved ultraproducts may be, they are generally not as pathological as box products. Following [22] we list some properties of countable box products of regular non-discrete spaces and comment upon their ultraproduct analogues (where the ultrafilter is assumed to be free on ω).

In what follows, $\langle \mathscr{X}_n : n < \omega \rangle$ is a sequence of nondiscrete regular spaces, $\mathscr{X} = \square_\omega \mathscr{X}_n$ (the box product) U is a free ultrafilter on ω, and $\mathscr{X}^* = \prod_U \mathscr{X}_n$.

9.1 Theorem. *$\mathscr{X}$ is neither compact, connected nor first countable.* □

9.1* Theorem. *$\mathscr{X}^*$ is totally non-compact, zero dimensional, and non-(first countable).* □

Remark. Since $\Gamma_U : \mathscr{X} \to \mathscr{X}^*$ is continuous open, (9.1) follows from (9.1*).

9.2 Theorem. *$\mathscr{X}$ needn't be normal, even if the $\mathscr{X}_n$'s are separable metric or linearly orderable.* □

Remark. (9.2) is due to van Douwen [26]. His counterexample crosses the irrationals with countably many copies of the compact ordinal space $[0, \omega]$.

9.2* Theorem. *$\mathscr{X}^*$ is always normal if the $\mathscr{X}_n$'s are metric or linearly orderable, since the classes of linearly uniformizable and linearly orderable spaces are closed.* □

9.3 Theorem. *$\mathscr{X}$ needn't be normal, even if the $\mathscr{X}_n$'s are compact.* □

Remark. Kunen, in proving 9.3, uses $\mathscr{X}_n = 2^{(c^+)}$ [16].

9.3* Theorem. *$\mathscr{X}^*$ needn't be normal, even if the $\mathscr{X}_n$'s are compact* (8.2). □

The following is due to Kunen [16].

9.4 Theorem (CH). *$\mathscr{X}$ is paracompact if the $\mathscr{X}_n$'s are compact and either of weight $\leqslant c$ or scattered.* □

9.4* Theorem (CH). *$\mathscr{X}^*$ is hereditarily ultraparacompact if the $\mathscr{X}_n$'s are of weight $\leqslant c$. $\mathscr{X}^*$ is ultraparacompact if the $\mathscr{X}_n$'s are compact scattered.*

Proof. Mimic the proof of (6.2). $\mathscr{X}^*$ will be a regular P-space which is hereditarily c-Lindelöf. Now assume the $\mathscr{X}_n$'s are compact scattered. To prove (9.4) for the scattered case, Kunen first proves that $\mathscr{X}$ is c-Lindelöf. Thus, since $\mathscr{X}^*$ is a continuous image of $\mathscr{X}$, $\mathscr{X}^*$ is c-Lindelöf as well. We thus proceed as in (6.2). □

Before we leave the topic of box products, we state one nontrivial result about ultraproducts which does transfer to box products, namely (4.7).

9.5 Theorem. *Suppose $\langle \mathfrak{X}_i : i \in I \rangle$ is an infinite family of nondiscrete regular spaces such that all the cardinals $|I|$ and $\{|X_i| : i \in I\}$ are moderate. Then $\Pi_I \mathfrak{X}_i$ cannot be extremally disconnected.*

Proof. I is infinite so there is a countably incomplete ultrafilter U on I. By (4.7) $\Pi_U \mathfrak{X}_i$ cannot be extremally disconnected. Now the natural map Γ_U is a continuous open surjection, so we are done once we prove the

Claim. *Extremal disconnectedness is preserved by continuous open surjections.*

Proof of Claim. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a continuous open surjection with $\mathfrak{X}$ extremally disconnected, $M \subseteq Y$ open. To show $\bar{M}$ is open we need to show $\phi^{-1}(\bar{M})$ is open. But $\phi^{-1}(\bar{M})$ is closed and contains $\phi^{-1}(M)$, whence $\overline{\phi^{-1}(M)} \subseteq \phi^{-1}(\bar{M})$. On the other hand if $x \in \phi^{-1}(\bar{M})$ and N is an open nbd of x then $\phi(N)$ is an open nbd of $\phi(x)$ so there is an element $x' \in N$ with $\phi(x') \in M$. Thus $x' \in N \cap \phi^{-1}(M)$ so $x \in \overline{\phi^{-1}(M)}$; and we have $\phi^{-1}(\bar{M}) = \overline{\phi^{-1}(M)}$ which is open in $\mathfrak{X}$. $\square$

Parting Remark. In (9.4), *CH is actually equivalent to the statement that countable box products of compact, weight $\leqslant c$ Hausdorff spaces are paracompact.* To see the converse, use the van Douwen–Borges result that $(2^{\omega_2})_{\omega_1}$ is not normal (vide 8.2); and assume $c \geqslant \omega_2$, $\mathfrak{X}_n = 2^{\omega_2}$ each $n < \omega$, with $\Delta \subseteq \mathfrak{X}$ the diagonal. Now $\mathfrak{X}$ is a countable box product of compact weight $\leqslant c$ Hausdorff spaces. But clearly $\Delta \simeq (2^{\omega_2})_{\omega_1}$ and Δ is a retract of $\mathfrak{X}$; and is hence closed therein. Thus $\mathfrak{X}$ cannot be paracompact.

10. Some open problems

We list a few problems which came to our attention during the course of this investigation and which remain unsolved.

10.1 Problem. Can an ultrapower of a paracompact space be normal without being paracompact?

10.2 Problem (Button). Are ultraproducts of scattered Hausdorff spaces scattered?

Comment. The class of scattered spaces is easily seen to be open. Button showed that the class of scattered spaces is not closed under Q-extension; however his counterexample is not Hausdorff.

10.3 Problem. The quotient maps $\Gamma_U : \Pi_I \mathfrak{X}_i \to \Pi_U \mathfrak{X}_i$ are always open. When are they closed?

10.4 Problem. Find properties of ultraproducts (especially for uncountable index sets) which transfer in a nontrivial fashion to corresponding box products.

Appendix 1. Table of selected preservation results

Property	Closed	Open
discrete	yes	yes
T_n $(0 \leqslant n \leqslant 3)$	yes	yes
Tichonov	yes	no
normal	no	no
paracompact	no	no
homogeneous	yes	no*
fixed point property	no	yes
compact	no	yes
connected	no	yes
zero-dimensional	yes	no
non-Archimedean	yes	no*
linearly uniformizable	yes	no*
linearly orderable	yes	no*
separable	no	yes
ω_1-open	yes	no
path-disconnected T_1	yes	no
first countable	no	yes (if relativized to T_1)
second countable	no	yes (if relativized to T_1)
Lindelöf	no	yes
c.c.c.	no	yes
metrizable	no	yes (if relativized to T_1)
extremally disconnected	no	yes
scattered	no (Button)	yes

* To be proved in Appendix 2.

Appendix 2. The "space" of all spaces

We "topologize" the class of all topological spaces by using the closed classes as our "closed sets". The program which we undertake is very much like what is done in model theory in the study of elementary classes, and our techniques in this section are very much inspired by classical model theory (hence the relegation to an appendix). The reasons for including such an appendix will become plain as we proceed. One obvious dividend is a proof of (6.3) without recourse to any set theory beyond ZFC. We will in fact be able to improve on (6.3), thereby filling in the starred items in our table of preservation results.

Let $\mathfrak{K}$ denote a class of spaces (closed under homeomorphic images). $\mathfrak{K}$ is a Σ-*class* if $\mathfrak{K}$ is a (possibly proper class) union of closed classes. $\mathfrak{K}$ is a Π-*class* if $\mathfrak{K}^c$, the complement of $\mathfrak{K}$, is Σ. $\mathfrak{K}$ is a Δ-*class* if $\mathfrak{K}$ is both Σ and Π. A union of Π-classes is a $\Sigma\Pi$-*class*, a union of Δ-classes is a $\Sigma\Delta$-*class* and so on.

A 2.1 Theorem. *The open classes form a proper-class topology on the class of topological spaces. Moreover this topology is "small-compact" in the sense that open covers indexable by sets have finite subcovers.*

Proof. Any intersection of closed classes is closed, $\emptyset$ is closed, and $\mathfrak{U}$, the universal class, is closed. Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be closed. We show $\mathfrak{K}_1 \cup \mathfrak{K}_2$ is also closed. If $\langle \mathfrak{X}_i : i \in I\rangle$ is a collection of spaces in $\mathfrak{K}_1 \cup \mathfrak{K}_2$ with U an ultrafilter on I, let $J = \{i : \mathfrak{X}_i \in \mathfrak{K}_1\}$. If $J \in U$ then $\prod_U \mathfrak{X}_i \in \mathfrak{K}_1$. Otherwise $I - J \in U$, in which case $\prod_U \mathfrak{X}_i \in \mathfrak{K}_2$.

The proof of small compactness lifts right out of the ultraproduct proof of the Compactness Theorem of first order logic. Assume that for each $j \in J$, $\mathfrak{K}_j$ is closed and that for each finite $i \subseteq J$, $\bigcap_{j \in i} \mathfrak{K}_j \neq \emptyset$. We show $\bigcap_{j \in J} \mathfrak{K}_j \neq \emptyset$. Let $I = \mathbf{P}_\omega(J)$, for each $i \in I$ let $\mathfrak{X}_i \in \bigcap_{j \in i} \mathfrak{K}_j$, and let $i^* = \{i' \in I : i \subseteq i'\}$, $I^* = \{i^* : i \in I\}$. Then $I^* \subseteq \mathbf{P}(I)$ has the f.i.p. so extends to an ultrafilter U on I. Let $\mathfrak{X} = \prod_U \mathfrak{X}_i$. Then for each $j \in J$, $\{i : \mathfrak{X}_i \in \mathfrak{K}_j\} \supseteq \{j\}^* \in U$, so $\mathfrak{X} \in \mathfrak{K}_j$. $\square$

We wish to characterize Σ and $\Sigma\Delta$-classes vis a vis ultraproducts.

A 2.2 Theorem. *$\mathfrak{K}$ is a Σ-class iff $\mathfrak{K}$ is closed under ultrapowers.*

Proof. Clearly Σ-classes are closed under ultrapowers. Suppose $\mathfrak{K}$ is closed under ultrapowers. We show $\mathfrak{K} = \bigcup\{\mathfrak{K}' : \mathfrak{K}' \subseteq \mathfrak{K}, \mathfrak{K}' \text{ closed}\}$. For let $\mathfrak{X} \in \mathfrak{K}$. Then $\mathfrak{K}(\mathfrak{X})$, the closure of $\mathfrak{X}$ under ultrapowers, is in $\mathfrak{K}$. We show $\mathfrak{K}(\mathfrak{X})$ is closed. Let $\langle \mathfrak{X}_i : i \in I\rangle$ be a family of ultrapowers of $\mathfrak{X}$, say $\mathfrak{X}_i = \prod_{V_i}(\mathfrak{X})$ where V_i is an ultrafilter on J_i. If U is an ultrafilter on I let $W = \sum_U V_i =$ the *U-sum* of the V_i's = that ultrafilter on $\bigcup_{i\in I}\{i\} \times J_i$ consisting of all R such that $\{i : \{j : \langle i, j\rangle \in R\} \in V_i\} \in U$. Then by standard model-theoretic arguments $\prod_U \mathfrak{X}_i$ is canonically homeomorphic to $\prod_W(\mathfrak{X})$. $\square$

Remark. The class of finite spaces is Σ but not closed. However, for every class which we showed not to be closed we provided an ultrapower counterexample. Thus these classes (e.g. normal, paracompact) are not Σ-classes either.

Two spaces $\mathfrak{X}$, $\mathfrak{Y}$ which have the property that there are ultrafilters U, V with $\prod_U(\mathfrak{X})$ homeomorphic (though not necessarily in an ultrabox-preserving way) to $\prod_V(\mathfrak{Y})$ are called *power equivalent* and we write $\mathfrak{X} \sim \mathfrak{Y}$. Clearly this relation is symmetric and reflexive. We will show that it is indeed an equivalence relation, but we need some machinery.

Notation. Let V_i be an ultrafilter on J_i with U an ultrafilter on I, $i \in I$. When each V_i is one V and each $J_i = J$ we write $\Sigma_U V_i$ as $U \cdot V$.

It is here that we must start resorting to model theory in earnest. A pair $\mathfrak{X} = \langle X, \beta \rangle$ where β is a basis for a topology on X is called a *basoid*. If β is a basis we let β^* denote its generated topology and let $\mathfrak{X}^* = \langle X, \beta^* \rangle$. If $\mathfrak{X}$, $\mathfrak{Y}$ are basoids and there is a bijection which preserves bases then $\mathfrak{X}$, $\mathfrak{Y}$ are *isomorphic* and we write $\mathfrak{X} \cong \mathfrak{Y}$. If there is a bijection such that $\mathfrak{X}^* \cong \mathfrak{Y}^*$ then we write $\mathfrak{X} \simeq \mathfrak{Y}$ (i.e. $\mathfrak{X}$, $\mathfrak{Y}$ are *homeomorphic*). Because confusion can easily arise, we will distinguish ultraproducts of basoids (which are basoids that are hardly ever topological spaces) from their topological completions. Thus $\Pi_U \mathfrak{X}_i = \Pi_U \langle X_i, \beta_i \rangle = \langle \Pi_U X_i, \Pi_U \beta_i \rangle$, and $(\Pi_U \mathfrak{X}_i)^*$ is the associated topological space. This notation is at variance with our earlier notation, but we could afford the abuse then but not now.

The *language of basoids* consists of all first order formulae built using the usual quantifiers and connectives plus the following primitive symbols: equality ($\doteq$), membership ($\dot\in$), and two sorts of variables ($x, y, z, x_1, y_1, \ldots$ etc. for "points" and $b, c, d, b_1, c_1, \ldots$ etc. for "basic open sets"). The atomic formulae look like $x \doteq y$, $b \doteq c$, and $x \dot\in b$. If we denote this language by L, an L-*structure* is a triple $\mathfrak{A} = \langle P, B, E \rangle$ where P is a set of "points", B is a set of "basic open sets", and $E \subseteq P \times B$. Equality is interpreted by the diagonal of $(P \cup B)^2$.

We assume the reader to be on fairly amiable terms with first order logic; and he/she should have no trouble verifying that *there is an* L-*sentence* ϕ *such that for any* L-*structure* $\mathfrak{A}$, $\mathfrak{A} \models \phi$ *iff* $\mathfrak{A}$ *is isomorphic to a basoid*, where the basoid $\mathfrak{X} = \langle X, \beta \rangle$ is interpreted as an L-structure in the obvious way: $P = X$, $B = \beta$, $E = \varepsilon \restriction X \times \beta$.

We now state the principal tool of our study, the Keisler–Shelah Ultrapower Theorem (see [8] or [23]) which states that *two models* $\mathfrak{A}$, $\mathfrak{B}$ *are elementarily equivalent* (*in symbols* $\mathfrak{A} \equiv \mathfrak{B}$) *iff there are ultrafilters* U, V (*which can be taken to be equal*) *such that* $\Pi_U(\mathfrak{A}) \cong \Pi_V(\mathfrak{B})$.

A 2.3 Theorem. "*Power equivalence*" *is an equivalence relation.*

Proof. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be spaces with $\mathfrak{X} \sim \mathfrak{Y}$, $\mathfrak{Y} \sim \mathfrak{Z}$, say $\Pi_U(\mathfrak{X}) \simeq \Pi_{V_1}(\mathfrak{Y})$, $\Pi_{V_2}(\mathfrak{Y}) \simeq \Pi_W(\mathfrak{Z})$. We recall the canonical isomorphism for iterating ultrapowers of structures, namely $\Pi_U(\Pi_V(\mathfrak{A})) \cong \Pi_{U \cdot V}(\mathfrak{A})$. We also recall the notion of "elementary substructure" ($\mathfrak{A} \preccurlyeq \mathfrak{B}$). Now to continue, we know that $\mathfrak{Y} \preccurlyeq \Pi_{V_i}(\mathfrak{Y})$ $(i = 1, 2)$ so by the Keisler–Shelah Theorem there is an ultrafilter V with $\Pi_V(\Pi_{V_1}(\mathfrak{Y})) \cong \Pi_V(\Pi_{V_2}(\mathfrak{Y}))$. Thus

$$\prod_{V \cdot U}(\mathfrak{X}) \cong \prod_V\Big(\prod_U(\mathfrak{X})\Big) \simeq \prod_V\Big(\prod_{V_1}(\mathfrak{Y})\Big) \simeq \prod_V\Big(\prod_{V_2}(\mathfrak{Y})\Big) \simeq \prod_V\Big(\prod_W(\mathfrak{Z})\Big) \cong \prod_{V \cdot W}(\mathfrak{Z}). \qquad \square$$

A 2.4 Theorem. *A class $\mathfrak{K}$ of spaces is a $\Sigma\Delta$-class iff $\mathfrak{K}$ is closed under power equivalence.*

Proof. Suppose $\mathfrak{K}$ is a union of Δ-classes, and let $\mathfrak{X} \in \mathfrak{K}$ with $\mathfrak{Y} \sim \mathfrak{X}$. Pick a Δ-class $\mathfrak{K}' \subseteq \mathfrak{K}$ so that $\mathfrak{X} \in \mathfrak{K}'$, and pick ultrafilters U, V with $\prod_U(\mathfrak{X}) \simeq \prod_V(\mathfrak{Y})$. Then $\prod_U(\mathfrak{X}) \in \mathfrak{K}'$, hence $\prod_V(\mathfrak{Y}) \in \mathfrak{K}'$. Since $\mathfrak{K}'$ is a Π-class, we have $\mathfrak{Y} \in \mathfrak{K}'$.

Conversely we show $\mathfrak{K} = \bigcup\{\mathfrak{K}' \subseteq \mathfrak{K} : \mathfrak{K}' \text{ is } \Delta\}$. For if $\mathfrak{X} \in \mathfrak{K}$, let $\mathfrak{K}[\mathfrak{X}]$ denote the class of all $\mathfrak{Y} \sim \mathfrak{X}$. Then $\mathfrak{K}[\mathfrak{X}] \subseteq \mathfrak{K}$. We show this class is Δ. $\mathfrak{K}[\mathfrak{X}]$ is a Σ-class since $\mathfrak{X} \sim \prod_U(\mathfrak{X})$ always. Also if $\mathfrak{Y} \not\sim \mathfrak{X}$ then, by transitivity of $\sim$, no ultrapower of $\mathfrak{Y}$ can be power equivalent to $\mathfrak{X}$. Thus $\mathfrak{K}[\mathfrak{X}]$ is Π, and we are done. □

A 2.5 Corollary. *Every $\Sigma\Delta$-class is Σ.* □

A class $\mathfrak{K}$ is *complete* if for each $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{K}$, $\mathfrak{X} \sim \mathfrak{Y}$ holds. A very useful result is the following:

A 2.6 Theorem. *The class of perfect regular spaces is complete.*

Proof. Let $\mathfrak{X}$ be a perfect regular space. We show $\mathfrak{X} \sim \mathbf{Q}$, the ordered space of rationals, and then apply the transitivity of $\sim$. First it is a triviality to show $\mathfrak{X}$ is infinite and that the property of perfect regularity is first order, so that by the Löwenheim-Skolem Downward Theorem there is a countable basoid $\mathfrak{X}_0 \preccurlyeq \mathfrak{X}$. N.B. the universe of $\mathfrak{X}_0 = \langle X_0, \beta_0\rangle$ is $X_0 \cup \beta_0$. Then $\mathfrak{X}_0^*$ is a countable, second countable, perfect regular space. But all such spaces are homeomorphic to $\mathbf{Q}$ by a standard topological argument. Thus by the Keisler-Shelah Theorem there is an ultrafilter U such that $\prod_U(\mathfrak{X}) \cong \prod_U(\mathfrak{X}_0)$, whence $\prod_U(\mathfrak{X}) \simeq \prod_U(\mathfrak{X}_0^*) \simeq \prod_U(\mathbf{Q})$, i.e. $\mathfrak{X} \sim \mathbf{Q}$. □

And now our long-promised corollary:

A 2.7 Corollary. *Every regular space has an ultrapower which is ultrametrizable. Hence the following properties are not open (or Π): Tichonov, normal, hereditarily normal, collectionwise normal, paracompact, linearly uniformizable.*

Proof. Let $\mathfrak{X}$ be a regular space. Then $\mathfrak{X} \times \mathbf{R}$ is perfect regular, whence $\mathfrak{X} \times \mathbf{R} \sim \mathbf{R}$ by (A 2.6). Let $\prod_U(\mathfrak{X} \times \mathbf{R}) \simeq \prod_V(\mathbf{R})$. V can be chosen to be countably incomplete so that $\prod_U(\mathfrak{X} \times \mathbf{R})$ is ultrametrizable (a hereditary property). Now $\mathfrak{X}$ embeds within $\mathfrak{X} \times \mathbf{R}$, so $\prod_U(\mathfrak{X})$ embeds within $\prod_U(\mathfrak{X} \times \mathbf{R})$. Thus $\prod_U(\mathfrak{X})$ is ultrametrizable. □

Remark. There are perfect regular spaces $\mathfrak{X}$ which are not linearly orderable, homogeneous, or supportive of a topological group structure. Since $\mathbf{R}$ has all of these properties we know that they too fail to be open.

Power equivalence is not a model-theoretic notion. That is, one might ask the question whether $\mathfrak{X} \sim \mathfrak{Y}$ does in fact imply that there are ultrafilters U, V for which

$\Pi_U(\mathscr{X}) \cong \Pi_V(\mathscr{X})$ (i.e. open ultraboxes map to open ultraboxes). The question is answered in the negative by the following:

A 2.8 Theorem. *Let $\mathfrak{X}, \mathfrak{Y}$ be spaces, let β, γ be bases for $\mathfrak{X}, \mathfrak{Y}$ respectively with* $\mathfrak{X}_0 = \langle X, \beta \rangle$, $\mathfrak{Y}_0 = \langle Y, \gamma \rangle$, and $\mathfrak{X}_0 \equiv \mathfrak{Y}_0$. Then $\mathfrak{X} \sim \mathfrak{Y}$. The converse does *not* hold.

Proof. The first part of the theorem is a straightforward application of the Keisler-Shelah Theorem. As for the failure of the converse, we find two regular perfect spaces $\mathfrak{X}, \mathfrak{Y}$ such that for no generating basoids $\mathfrak{X}_0, \mathfrak{Y}_0$ it is true that $\mathfrak{X}_0 \equiv \mathfrak{Y}_0$. Pick $\mathfrak{X} = [0, 1]$. Then there is no way to pick a clopen basis for $\mathfrak{X}$. Let ϕ be the first order sentence which says that each basis set has open closure (this is straightforward to produce). Then $\mathfrak{X}_0 \models \neg\phi$. Now let $\mathfrak{Y}$ be any extremally disconnected perfect regular space (e.g. the Stone space of an atomless complete Boolean algebra). Then for any generating basoid $\mathfrak{Y}_0, \mathfrak{Y}_0 \models \phi$. □

We end this report with a few brief comments regarding the facts (2.1), (2.2). Assume CH. If U, V are free ultrafilters on ω and if $\mathbf{R}$ denotes the reals, treat $\mathbf{R}$ as an order structure. Then $\Pi_U(\mathbf{R})$, $\Pi_V(\mathbf{R})$ are c-saturated elementarily equivalent structures of power c and are hence order isomorphic. This essentially proves (2.1). To prove (2.2), consider $\mathbf{R}$ and $\mathbf{Q}$ again as basoids, with $\mathfrak{X}_0 \preccurlyeq \mathbf{R}$ a countable second countable perfect regular basoid. If U is free on ω then $\Pi_U(\mathbf{R}) \cong \Pi_U(\mathscr{X}_0)$ since both basoids are c-saturated elementarily equivalent of power c. Now $\Pi_U(\mathscr{X}_0) \simeq \Pi_U(\mathscr{X}_0^*) \cong \Pi_U(\mathbf{Q})$. We can make the same statement for $\mathbf{R}^2$ so that, in the notation of Section 2, $^*\mathbf{R} \simeq (^*\mathbf{R})^2$.

References

[1] Archangel'skii and Filippov, Spaces with bases of finite rank, Mat. Sbornik USSR 16 (1972).
[2] Alò and Shapiro, Normal Topological Spaces (Cambridge Univ. Press, 1974).
[3] P. Bankston, Topological ultraproducts, Thesis, Univ. of Wisc. (1974).
[4] A. Bernstein and F. Wattenberg, Cardinality-dependent properties of topological spaces, Victoria Symp. on Nonstandard Anal., Springer-Verlag Lecture Notes #369 (1974).
[5] R. Button, A note on the Q-topology (to appear).
[6] J. Bell and A. Slomson, Models and Ultraproducts (North-Holland, 1968).
[7] D. Booth, Ultrafilters on a countable set, Ann. Math. Logic 2 (1970).
[8] C.C. Chang and H.J. Keisler, Model Theory (North-Holland, 1973).
[9] W. Comfort and S. Negrepontis, The Theory of Ultrafilters (Springer-Verlag, 1974).
[10] J. Dugundji, Topology (Allyn and Bacon, 1966).
[11] R.L. Ellis, Extending continuous functions on zero dimensional spaces, Math. Ann. 186 (1970).
[12] L. Gillman and M. Jerison, Rings of Continuous Functions (van Nostrand Reinhold, 1960).
[13] A. Hayes, Uniformities with totally ordered bases have paracompact topologies, Proc. Cam. Phil. Soc. 74 (1973).
[14] I. Juhász, Untersuchungen über ω_μ-metrisierbare Räume, Annales Univ. Sci., Budapest Sectio Math. 8 (1965).
[15] J. Kelley, General Topology (van Nostrand, 1950).
[16] K. Kunen, Box products of compact spaces (to appear).

[17] K. Kunen, Ultrafilters and independent sets, Trans. A.M.S. 172 (1972).
[18] K. Kunen, Some comments on box products, Coll. Math. Soc. János Bolyai (10), (Keszthely, Hungary, 1973).
[19] H.J. Keisler, Ultraproducts which are not saturated, J.S.L. (32), no. 1 (1967).
[20] J. Nagata, Modern Dimension Theory (Interscience, 1965).
[21] A. Robinson, Nonstandard Analysis (North-Holland, 1966).
[22] M.E. Rudin, Lectures on Set Theoretic Topology, A.M.S. (1975).
[23] S. Shelah, Every two elementarily equivalent models have isomorphic ultrapowers, Israel J. Math. 10 (1971).
[24] J.R. Shoenfield, Mathematical Logic (Addison-Wesley, 1967).
[25] S. Willard, General Topology (Addison-Wesley, 1970).
[26] E.K. van Douwen, The box product of countably many metric spaces need not be normal, Fund. Math. (to appear).