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Problems in Continuum Theory in Memory of Sam B. Nadler, Jr.

by

Edited by Logan Hoehn, Piotr Minc, Murat Tuncali

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PROBLEMS IN CONTINUUM THEORY IN MEMORY OF SAM B. NADLER, JR.

EDITED BY LOGAN HOEHN, PIOTR MINC, AND MURAT TUNCALI

Abstract. This paper contains annotated problems in continuum theory. Topics include homogeneous continua, hereditarily indecomposable continua (metric or Hausdorff), embedding continua in the plane, connections of continuum theory with dynamical systems, co-existential arcs, degree of homogeneity, hereditarily equivalent continua, hereditarily self-like continua, the semi-Kelley property, products of solenoids, span 0, and chainable, subchainable, and tree-like continua.

Contributions:

2010 *Mathematics Subject Classification.* 54-06, 54F15, 54H20, 37B40, 37B45.

Key words and phrases. chainable, dynamical systems, embedding in the plane, hereditary indecomposability, homogeneity, problems in continuum theory, subchainable and tree-like continua.

Introduction

This problem set is dedicated to the memory of Sam B. Nadler, Jr. (June 3, 1939 – February 4, 2016), a towering figure in continuum theory, our dear colleague, mentor, and friend. Shortly before his death, Sam expressed his wish to have a continuum theory problem session organized for him. Sam knew very well how important new problems are for the future of continuum theory and he wanted his beloved field to flourish. The Continuum Theory Special Session of the 51st Spring Topology and Dynamical Systems Conference was dedicated to Sam's memory. The speakers were asked to honor Sam's wishes and his memory by devoting a portion of their talk to open continuum theory problems. Those problems are collected in this set. Each of the sections was written by a different author, or a set of authors. Many of the sections include questions on homogeneous continua, and many include questions on hereditarily indecomposable continua (metric or Hausdorff). Four sections are devoted to questions connected with embedding continua in the plane. Another two are on connections of continuum theory with dynamical systems. Other topics include co-existential arcs, degree of homogeneity, hereditarily equivalent continua, hereditarily self-like continua, the semi-Kelley property, products of solenoids, span 0, and chainable, subchainable, and tree-like continua.

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1. Problems on Planar Embeddings of Chainable Continua and Accessibility

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A *continuum* X is a compact connected metric space. A collection $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ of open sets in X is called a *chain* in X if $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The *mesh* of $\mathcal{C}$ is $\mathrm{mesh}(\mathcal{C}) = \max_i \mathrm{diam}\,(\ell_i)$. A continuum is *chainable* if it can be covered by a chain of arbitrary small mesh. It is a well-known fact that every chainable continuum can be embedded in the plane; see [13]. For a planar continuum $Y \subset \mathbb{R}^2$ and $y \in Y$, we say that y is *accessible* if there exists an arc $A \subset \mathbb{R}^2$ such that $A \cap Y = \{y\}$.

The following text is motivated by the long-standing open problem posed by Sam B. Nadler, Jr., and J. Quinn in 1972. (See also Question 48 and Question 50 below).

Question (Nadler and Quinn (see [75, p. 229, Question]))**.** Let X be a chainable continuum and $x \in X$. Can X be embedded in the plane such that x is accessible?

The positive answer to the above question is obtained for the unimodal inverse limit space X and an arbitrary point x in [5]. The construction from [5] is further generalized in [6]. However, the embeddings constructed in the above references are all nested intersections of planar chains with connected links. In general, links do not have to be connected sets and that is the main obstacle in solving the Nadler–Quinn problem.

For chains $\mathcal{C}' = \{\ell'_1, \dots, \ell'_m\}$ and $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$, we say that $\mathcal{C}'$ *properly refines* $\mathcal{C}$, denoted by $\mathcal{C}' \prec \mathcal{C}$, if, for every $j \in \{1, \dots, m\}$, there exists $i \in \{1, \dots, n\}$ such that $\overline{\ell}'_j \subset \ell_i$.

Let $Y \subset \mathbb{R}^2$ be a continuum. We say that Y is *C-chainable* if there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of chains in $\mathbb{R}^2$ such that $Y = \cap_{n\in\mathbb{N}} \mathcal{C}_n^*$, where $\mathcal{C}_n^*$ denotes the union of links of $\mathcal{C}_n$, $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}$, $\mathrm{mesh}(\mathcal{C}_n) \to 0$ as $n \to \infty$, and the links of $\mathcal{C}_n$ are connected sets in $\mathbb{R}^2$ (note that links are open in the topology of $\mathbb{R}^2$).

Let X be a chainable continuum. We say that an embedding $\phi\colon X \to \mathbb{R}^2$ is a *C-embedding* if $\phi(X)$ is C-chainable. Otherwise, ϕ is called a non-C-embedding.

Question 1. Which chainable continua can be non-C-embedded in the plane?

An example of non-C-embedding of an arc+ray continuum is constructed by R. H. Bing in [13]; see Figure 1. An example of non-C-embedding of the 3-Knaster continuum is given by W. Dębski and E. D. Tymchatyn in [29].

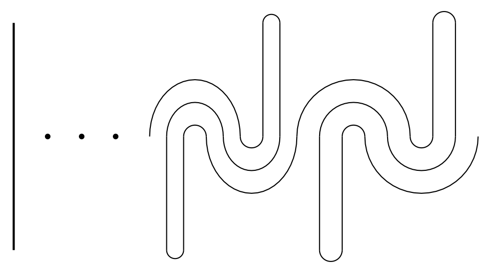

FIGURE 1. Bing's example from [13].

Piotr Minc suggested a possible counterexample to the Nadler-Quinn question: a continuum which is the inverse limit with single bonding map f given in Figure 2 (see the map $\tilde{g}$ in Section 10 and the reference therein). We conjecture that there is no C-embedding of Minc's continuum making point $p = (\frac{1}{2}, \frac{1}{2}, \dots)$ accessible.

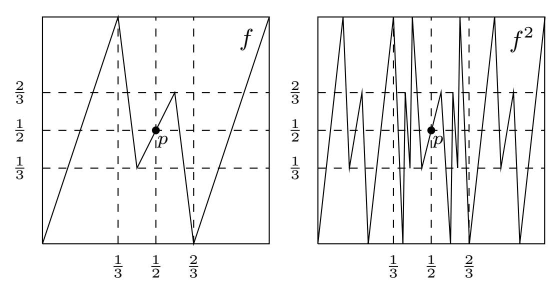

FIGURE 2. Bonding map (and its second iterate) in Minc's possible counterexample.

Question 2. Is there a chainable continuum X and a non-C-embedding ψ of X such that the set of accessible points of $\psi(X)$ is different from the set of accessible points of $\phi(X)$ for any C-embedding ϕ of X?

2. Monotone and Co-Existential Images of Ultra-Arcs: Problems

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If X is a compact Hausdorff space and $\mathcal{D}$ is an ultrafilter on an infinite discrete set I, the $\mathcal{D}$-*ultracopower* $X_{\mathcal{D}}$ is the inverse image of $\mathcal{D}$ under the Stone-Čech lift of the coordinate projection $q : X \times I \to I$. The Stone-Čech lift of the coordinate projection $p : X \times I \to X$, when restricted to $X_{\mathcal{D}}$, is called the *codiagonal map*, denoted $p_{X,\mathcal{D}} : X_{\mathcal{D}} \to X$, and is a continuous surjection.

A continuous map $f : X \to Y$ between compacta is called *co-existential* if there is an ultrafilter $\mathcal{D}$ and a continuous surjection $g : Y_{\mathcal{D}} \to X$ such that $f \circ g = p_{Y,\mathcal{D}}$.

An *ultra-arc* is a $\mathcal{D}$-ultracopower of the closed unit interval $\mathbb{I}$, where $\mathcal{D}$ is a free ultrafilter on the natural numbers ω. Alternatively, it is any of the nonmetrizable components of $\beta(\mathbb{I} \times \omega)$. Ultra-arcs are the "standard subcontinua" of the Stone-Čech remainder of the half-line.

It is well known that every continuum of weight $\leq \aleph_1$ is a continuous image of any ultra-arc; we are interested in the monotone and the co-existential ones.

Question 3. Does every nondegenerate monotone image of an ultra-arc have covering dimension one?

Question 4. Is a nondegenerate monotone metrizable image of an ultra-arc always an arc?

Question 5. Is a co-existential image of an ultra-arc necessarily irreducible? (Yes, if it is metrizable.)

Question 6. Is every nondegenerate monotone image of an ultra-arc also a co-existential one?

Question 7. Is a solenoid a co-existential image of an ultra-arc? (Every nondegenerate chainable metrizable continuum is a co-existential image, but there are nonchainable ones too. The ones we know about, however, are all hereditarily indecomposable.)

Question 8. Is a chainable continuum of weight $\aleph_1$ necessarily a co-existential image of an ultra-arc? (Yes, if the ultrafilter is the Fubini product of two free ultrafilters.)

For more information, see [8].

3. A question on hereditarily indecomposable tree-like continua

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In 1951, R. H. Bing [12] asked whether the pseudo-arc is the only non-degenerate hereditarily indecomposable plane continuum which does not separate the plane. Each non-separating plane continuum with empty interior is tree-like. Also in 1951, R. D. Anderson [3] announced a construction of uncountably many topologically distinct hereditarily indecomposable tree like plane continua. However, Anderson's construction has not appeared in print. In 1979, W. T. Ingram [47] constructed an uncountable collection of mutually exclusive hereditarily indecomposable tree-like continua in the plane such that if M is a compact metric continuum, then M cannot be mapped onto every member of the collection. Two years later, Ingram [48] constructed a hereditarily indecomposable tree-like continuum such that each non-degenerate subcontinuum of it has positive span. It is not clear if the continuum described in [48] is planar or not. In general, it would be interesting to answer the following question.

Question 9. Is every tree-like hereditarily indecomposable continuum planar?

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4. Problems on Degree of Homogeneity, Dynamics, and Mapping Properties

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Degree of homogeneity:

Given a natural number $n > 0$, a topological space X is said to be $1/n$-*homogeneous* if it has exactly $n > 1$ orbits under the action of its homeomorphism group.

Question 10. Does there exist a planar indecomposable 1/2-homogeneous circle-like continuum?

Question 11. Does there exist an indecomposable 1/2-homogeneous chainable continuum?

Note that "Yes" to Question 11 implies "Yes" to Question 10. Nonplanar indecomposable circle-like examples were constructed independently by Jan P. Boroński [17] and Pavel Pyrih and Benjamin Vejnar [86].

Question 12. Does there exist a hereditarily indecomposable continuum that is $1/n$-homogeneous, for some $n > 1$?

Question 13 (Acosta and Pacheco-Juárez [2])**.** Given a positive integer $n > 3$, what are $1/n$-homogeneous dendrites? (The answer is well known for $n = 1, 2$. In [2], Gerardo Acosta and Yaziel Pacheco-Juárez give a classification for $n = 3$.)

Dynamics:

A map : $X \to X$ is said to be *minimal* if the forward orbit $\{f^n(x) : n = 1, 2, 3, \dots\}$ is dense for every x. A *noninvertible map* is a map that is not injective.

Question 14. Does the pseudo-circle admit a minimal noninvertible map?

Note that the pseudo-circle admits minimal homeomorphisms [37] as well as the circle, but the latter does not admit minimal noninvertible maps [7]. There are continua that admit minimal noninvertible maps but not minimal homeomorphisms, continua that admit both types of minimal maps, and continua that admit neither [22].

A continuous map $\varphi\colon X \to X$ acting on a compact metric space (X, ρ) is *Li–Yorke chaotic* (or *chaotic in the sense of Li–Yorke*) if there is an uncountable set $S \subset X$ such that $\liminf_{n\to\infty} \rho(\varphi^n(x), \varphi^n(y)) = 0$ and $\limsup_{n\to\infty} \rho(\varphi^n(x), \varphi^n(y)) > 0$ for any distinct points $x, y \in S$. This notion of chaos comes from [62] and is weaker than the one of positive entropy, as well as Devaney chaos. In [18], Boroński and Piotr Oprocha construct a chainable continuum that admits a Li–Yorke chaotic zero entropy homeomorphism.

Question 15. Is there a Li–Yorke chaotic zero entropy homeomorphism of the pseudo-arc?

Note that the homeomorphism in question cannot be obtained as the shift homeomorphism on an inverse limit of arcs; see [16].

Question 16. Suppose f and g are two Li–Yorke chaotic weakly unimodal interval maps of type 2^∞. Must the inverse limits I_f and I_g be homeomorphic? See [19].

A self-map f of the unit interval is said to be *weakly unimodal* if there exists a c, $0 < c < 1$, such that $f|[0, c]$ is nondecreasing and $f|[c, 1]$ is nonincreasing. It is of type 2^∞ if it admits a periodic point of period 2^n for all $n = 0, 1, 2, \dots$, and no other periods.

Question 17. Let f be a self-map of the unit interval I and let $\hat{f}$ be the induced map on the hyperspace of subcontinua $C(I)$. Suppose $\hat{K}$ is a subcontinuum of $C(I)$ invariant under $\hat{f}$ with the property that if $\hat{L} = \hat{f}(\hat{L})$ is another subcontinuum of $C(I)$ containing $\hat{K}$, then $\hat{L} = C(I)$. Could $\hat{K}$ be (a) non-locally connected? (b) a Sierpinski triangle? (c) a non-trivial dendrite? (d) not a finite graph or 2-cell?

Question 18. Given a map f on an arc-like continuum X, does the map induced on the n-fold hyperspace $F^n(X)$ fix a point in every subcontinuum of $F^n(X)$ invariant under $\hat{f}$?

The answer is "Yes" for the hyperspace of subcontinua for I [87] and more generally for all chainable continua [20].

Question 19 (Nadler [76])**.** For what circle-like X does the hyperspace suspension $C(X)/F^1(X)$ have the fixed point property?

Question 20. What are circle-like continua with the fixed point property?

Krystyna Kuperberg [55] asks under what assumptions must an orientation reversing planar homeomorphism have a Cantor set of fixed points in an invariant continuum. We suggest the following variant of her question.

Question 21. Suppose h is an orientation reversing homeomorphism of the plane with an invariant lakes of Wada continuum X for which all complementary domains are invariant. Must X contain a Cantor set of fixed points?

Other Mapping Properties:

A *pseudo-solenoid* is a hereditarily indecomposable nonplanar circle-like continuum. Given a sequence of prime numbers $P = (p_i : i = 1, 2, ...)$, the pseudo-solenoid S_P is defined as the inverse limit of the inverse system $\{C_i, \tau_i : i = 1, 2, \ldots\}$, where C_i is a pseudo-circle and $\tau_i : C_{i+1} \to C_i$ is a p_i-fold covering map. The questions below come from [21].

Question 22. For a given pseudo-solenoid S, can each subcontinuum of $S \times S$ that projects onto both coordinate spaces be approximated (in the hyperspace of $S \times S$) by a local homeomorphism?

Michel Smith [89] shows that each subcontinuum of the Cartesian square of the pseudo-arc that projects onto both coordinate spaces can be approximated by a homeomorphism. Jan M. Aarts and Robbert J. Fokkink [1] show that if f is a k-to-1 self-map of the 2-adic solenoid, then k is odd. It would be interesting to know the answer to the following.

Question 23. If S is a P-adic pseudo-solenoid, when is there a k-to-1 self-map of S? Must k be coprime with all but finitely many $p_i \in P$? What if S is a P-adic solenoid?

Question 24. If S is a P-adic pseudo-solenoid, is every exactly k-to-1 map on S a k-fold covering map?

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5. Continua That Are Subcontinuum-like or Hereditarily Self-like

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Definition. Let X be a non-degenerate continuum.

- X is *tree-like* if, for every $\varepsilon > 0$, there is an ε-map from X to a tree; i.e., for every $\varepsilon > 0$, there exists a tree T and a continuous function $f : X \to T$ such that for each $t \in T$, the set $f^{-1}(t)$ has diameter less than ε. Note that the tree T may depend on ε.
- If K is a continuum, then X is K*-like* if, for every $\varepsilon > 0$, there is an ε-map from X onto K.
- We say that X is *hereditarily self-like* provided that each of its non-degenerate subcontinua is X-like.
- We say that X is *subcontinuum-like* provided that, for every non-degenerate subcontinuum K of X, X is K-like.
- X is *hereditarily equivalent* if X is homeomorphic to each of its non-degenerate subcontinua.

It is easy to see that any (non-degenerate) hereditarily equivalent continuum is both hereditarily self-like and subcontinuum-like. Thus, these two new classes extend the class of hereditarily equivalent continua, but to what degree?

Question 25. Is there a continuum that is both hereditarily self-like and subcontinuum-like but not hereditarily equivalent?

In [48], W. T. Ingram gives an example of a continuum X such that every non-degenerate subcontinuum of X has positive span.

Question 26. Is the Ingram continuum with hereditarily positive span from [48] subcontinuum-like?

Question 27. Is the Ingram continuum with hereditarily positive span from [48] hereditarily self-like?

As shown by H. Cook [27], (non-degenerate) hereditarily equivalent continua are tree-like.

Question 28. Is every hereditarily self-like continuum also tree-like?

Question 29. Is every subcontinuum-like continuum also tree-like?

A property $\mathcal{P}$ of continua is called a *hereditary property* if, whenever a non-degenerate continuum has property $\mathcal{P}$, each of its non-degenerate subcontinua also has property $\mathcal{P}$.

Clearly, being hereditarily equivalent is a hereditary property.

Question 30. Is being hereditarily self-like a hereditary property?

Question 31. Is being subcontinuum-like a hereditary property?

It is known that the property of being *both* hereditarily self-like and subcontinuum-like is a hereditary property.

George W. Henderson [38] proves that the only decomposable hereditarily equivalent continuum is the arc.

Question 32. Is there a hereditarily decomposable continuum other than an arc that is both subcontinuum-like and hereditarily self-like?

Question 33. Is every hereditarily self-like continuum one-dimensional?

Question 34. Is every subcontinuum-like continuum one-dimensional?

The answer to this last question is affirmative for finite-dimensional subcontinuum-like continua.

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6. A Question on Semi-Kelley Continua

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A *continuum* is a nonempty compact connected metric space. A continuum X has the *property of Kelley* provided that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every pair of points x and y of X such that $d(x, y) < \delta$, and each subcontinuum A of X for which $x \in A$, there exists a subcontinuum B of X such that $y \in B$ and $\mathcal{H}(A, B) < \varepsilon$, where $\mathcal{H}(A, B)$ is the Hausdorff distance in 2^X. In order to define when a continuum has the property of semi-Kelley, we need the following concept.

Definition 1. Let K be a subcontinuum of a continuum X. A continuum $M \subseteq K$ is called a *maximal limit continuum in* K provided that there is a sequence of subcontinua M_n of X converging to M such that for each convergent sequence of subcontinua M'_n of X with $M_n \subseteq M'_n$ for each $n \in \mathbb{N}$, and $\lim M'_n = M' \subseteq K$, we have $M' = M$.

Definition 2. A continuum X is said to be *semi-Kelley* provided that, for each subcontinuum K of X and for every two maximal limit continua M_1 and M_2 in K, either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$.

Remark. If a continuum has the property of Kelley, then it is semi-Kelley.

Theorem 1 (Kelley [53, Theorem 3.3])**.** *If a continuum X has the property of Kelley, then 2^X and $C(X)$ are contractible.*

Several properties of semi-Kelley continua are studied in [23], and, due to the previous theorem, a natural question is that if a continuum X has the property of semi-Kelley, is it true that 2^X or $C(X)$ is contractible? We post here the question as it appears in [23].

Question 35 (J. J. Charatonik and W. J. Charatonik [23, Question 5.16])**.** Is it true that if a continuum X is semi-Kelley, then the hyperspace 2^X and/or $C(X)$ is contractible?

The following results aim in the direction of this question and their proofs may be found in [25, Corollary 4] and [53, Lemma 3.1], respectively.

Theorem 2. *If a continuum X contains an R^i-continuum for some $i \in \{1, 2, 3\}$, then 2^X and $C(X)$ are not contractible.*

Theorem 3. *For a compact metric continuum, the following properties are equivalent:*

(1) *$F_1(X)$ is contractible in 2^X;*
(2) *2^X is contractible;*
(3) *$C(X)$ is contractible (in itself).*

The following result may be found in [32].

Theorem 4. *Let X be a continuum with the property of semi-Kelley. Then X does not contain R^2-continua.*

7. Characterizations of the Pseudo-Arc

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R. H. Bing [14] shows that every hereditarily indecomposable arc-like continuum is homeomorphic to the pseudo-arc. A continuum X has *span zero* [58] if, for any continuum C and any two maps $f, g : C \to X$ such that $f(C) \subseteq g(C)$, there exists $p \in C$ with $f(p) = g(p)$. It is well known that every arc-like continuum has span zero, but L. C. Hoehn [39] shows that there exists a continuum of span zero which is not arc-like. On the other hand, he and Lex G. Oversteegen [41] show that all hereditarily indecomposable continua of span zero are arc-like, hence homeomorphic to the pseudo-arc by Bing's characterization.

A continuum is weakly-chainable [30] and [57] if it is the continuous image of the pseudo-arc. Since it is known that every continuum of span zero is weakly chainable, atriodic, and tree-like, not all continua with those properties are arc-like. A positive solution to the following problem would provide a useful characterization of the pseudo-arc.

Question 36. Is every hereditarily indecomposable weakly chainable continuum homeomorphic to the pseudo-arc?

It is known [64] that such a continuum must be tree-like.

F. B. Jones [49] asks: What are all the homogeneous hereditarily indecomposable continua? Since every homogeneous planar tree-like continuum is arc-like, a complete classification of all planar homogeneous continua is known [41]. By results of Paweł Krupski and Janusz R. Prajs [54] and James T. Rogers, Jr. [88], a homogeneous continuum is hereditarily indecomposable if and only if it is tree-like. Thus far, the pseudo-arc is the only known example of a non-degenerate homogeneous tree-like continuum.

Question 37. If X is a homogeneous tree-like (equivalently, hereditarily indecomposable) continuum, must X be homeomorphic to the pseudo-arc?

An affirmative answer would follow if one could prove that every homogeneous tree-like continuum has span zero. The question of whether every homogeneous tree-like continuum has span zero is raised by W. T. Ingram in [28, Problem 93].

In light of Question 36, it would be nice to know the answer to the following question.

Question 38. Is every tree-like homogeneous continuum weakly chainable?

A continuum X is *hereditarily equivalent* if X is homeomorphic to each of its non-degenerate subcontinua. Hoehn and Oversteegen have announced that the only non-degenerate hereditarily equivalent plane continua are the arc and the pseudo-arc. It is known that every decomposable hereditarily equivalent continuum is an arc [38]. The following problem remains open.

Question 39. Is every hereditarily equivalent and hereditarily indecomposable continuum homeomorphic to the pseudo-arc?

Again, in light of Question 36, the following question is of interest.

Question 40. Is every hereditarily equivalent and hereditarily indecomposable continuum weakly chainable?

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8. A Question on Solenoids

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In [11], David P. Bellamy and Janusz M. Lysko consider the following property for the product of two nondegenerate continua X and Y:

> $(*)$ If M is a subcontinuum of $X \times Y$ such that $\pi_X(M) = X$ and $\pi_Y(M) = Y$, then M has arbitrarily small connected neighborhoods in $X \times Y$ (that is, for each open subset U of $X \times Y$ such that $M \subset U$, there exists an open connected subset V of $X \times Y$ such that $M \subset V \subset U$).

They also pose a series of problems, most of which are answered in [42], [43], and [45]. The following problem remains unsolved.

Question 41 (Bellamy and Lysko [11, p. 230, (4)])**.** Suppose that S_a and S_b are the respective a and b solenoids; where $(a, b) = 1$, does $S_a \times S_b$ have property $(*)$?

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9. Questions on Subchainable, Chainable, and Span 0 Continua

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A continuum X is *subchainable* if every proper subcontinuum of X is chainable.

Question 42. Is every nondegenerate homogeneous subchainable continuum circle-like?

- If every nondegenerate proper subcontinuum is an arc, then yes. [34]
- If every nondegenerate proper subcontinuum is a pseudo-arc, then yes. [59]
- If every nondegenerate proper subcontinuum is either a pseudo-arc or an arc of pseudo-arcs, then yes.

(In the last case, by [59], the continuum has a continuous decomposition into maximal pseudo-arcs. If the continuum is not itself a pseudo-arc, then every nondegenerate proper subcontinuum of the decomposition is an arc and, by [34], the decomposition space is either a circle or a solenoid. By [60], the original continuum is then either a circle of pseudo-arcs or a solenoid of pseudo-arcs.)

A complete classification of homogeneous circle-like continua is known (see [36] and [60]). Each nondegenerate circle-like continuum is one of the following:

- the pseudo-arc, or
- the simple closed curve, or the circle of pseudo-arcs, or
- a solenoid, or a solenoid of pseudo-arcs.

Thus, if the answer to Question 42 is yes, then the classification of subchainable homogeneous continua is complete.

J. J. Charatonik and T. Maćkowiak [24] show that the pseudo-arc is the only nondegenerate chainable continuum which is homogeneous with respect to confluent mappings. The arc is homogeneous with respect to weakly confluent mappings, as well as with respect to other classes of mappings, so this does not generalize to homogeneity with respect to continuous functions.

Question 43. Is the pseudo-arc the only nondegenerate subchainable hereditarily indecomposable continuum which is homogeneous with respect to confluent mappings?

Frank Sturm [90] shows that no pseudo-solenoid (including the pseudo-circle) is homogeneous with respect to continuous functions. By [26] and [63], a continuum X is hereditarily indecomposable if and only if every continuous function from a continuum to X is confluent. Thus, for hereditarily indecomposable continua, being homogeneous with respect to confluent mappings is the same as being homogeneous with respect to continuous functions.

Every subchainable hereditarily indecomposable continuum is locally planar, since every such continuum is locally homeomorphic to the pseudo-arc [61]. Thus, it would be global, not local, properties which would prevent such a continuum from being planar.

Question 44. Is every tree-like subchainable hereditarily indecomposable continuum embeddable in the plane?

R. H. Bing [15] shows that every circle-like continuum, such as the solenoids or pseudo-solenoids, can be embedded in the 3-book, the product of a simple triod and an interval.

Question 45. Is every subchainable hereditarily indecomposable continuum embeddable in the 3-book?

The following questions are not about subchainable continua, but are still of interest in consideration of L. C. Hoehn's [39] example of a continuum with span 0 which is not chainable.

Question 46. Does there exist a nonchainable continuum of span 0 with every nondegenerate subcontinuum nonchainable?

Question 47. Is every continuum of span 0 embeddable in the plane?

In 1972 Nadler and Quinn asked the following question, which is still open.

Question 48 (Nadler [75, p. 229])**.** If C is a chainable continuum and p is any given point of C, then can C be embedded in the plane so that p is arcwise accessible from the complement (of the embedding)?

If p is an endpoint of the chainable continuum C, then there is an embedding of C in the plane with p arcwise accessible from the complement of C. The union of C and arc A joined at p with an endpoint of A is chainable and, hence, by a result of Bing [13], embeddable in the plane.

A point p of a chainable continuum is a *pseudo-endpoint* of the chainable continuum C if, for every $\epsilon > 0$, there exists an $\epsilon-$chain $U = \{U_1, U_2, \ldots, U_n\}$ covering C such that $\mathrm{dist}(p, U_1) < \epsilon$.

Question 49. If p is a pseudo-endpoint of the chainable continuum C, then is there an embedding of C in the plane with p arcwise accessible from the complement of C?

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10. Embedding Tree-Like Continua in the Plane

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Let I denote the interval $[0,1]$. Recall that a continuum is chainable if and only if it can be expressed as the inverse limit $\varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ where $X_i = I$ and $f_i : X_{i+1} \to X_i$ is continuous. A *tree* is a finite, connected and simply connected, one-dimensional polyhedron. A continuum is *tree-like* if and only if it can be expressed as an inverse limit of trees.

By a *simplicial tree* (or a *simplicial arc*), we understand a tree (or an arc) T with a finite subset $\mathcal{V}(T)$ such that each component of $T \setminus \mathcal{V}(T)$ is an open arc. Points of $\mathcal{V}(T)$ are called *vertices* of T and the closures of components of $T \setminus \mathcal{V}(T)$ are called *edges*. Given two simplicial trees T and T', we say that a continuous mapping $\varphi : T' \to T$ is *simplicial* if $\varphi(\mathcal{V}(T')) \subset \mathcal{V}(T)$, and φ restricted to each edge of T' is either constant or a homeomorphism onto an edge of T.

Exercise. Let g and $\tilde{g}$ be the functions whose graphs are depicted in Figure 3. Suppose that T and T' are simplicial trees, each of which is a copy of I with a certain choice of vertices. Let $\mathcal{V}(T)$ be a finite subset of $T = I$ containing $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Show that there is a unique finite set $\mathcal{V}(T') \subset T' = I$ such that $\tilde{g} : T' \to T$ is simplicial. Observe also that $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\} \subset \mathcal{V}(T')$. Repeat the exercise with $\frac{1}{3}, \frac{2}{3}$, and $\tilde{g}$ replaced by $\frac{1}{5}, \frac{4}{5}$, and g, respectively.

In this section, we will be interested in the class of tree-like continua and its subclass of inverse limits of simplicial trees with simplicial bonding maps. When considering such inverse limit $\varprojlim\{T_i, \varphi_i\}_{i=0}^{\infty}$, it is important to understand that $\mathcal{V}(T_i)$ is defined only once for each i. In particular, the set $\mathcal{V}(T_{i+1})$ is the same for both simplicial maps $\varphi_i : T_{i+1} \to T_i$ and $\varphi_{i+1} : T_{i+2} \to T_{i+1}$.

We would like to return now to the very important problem by Sam B. Nadler, Jr., and J. Quinn (see [75, p. 229]) already recalled above by Ana Anušić, Henk Bruin, and Jernej Činč in section 1, and by Wayne Lewis as Question 48. Another version of this question was asked by John Mayer.

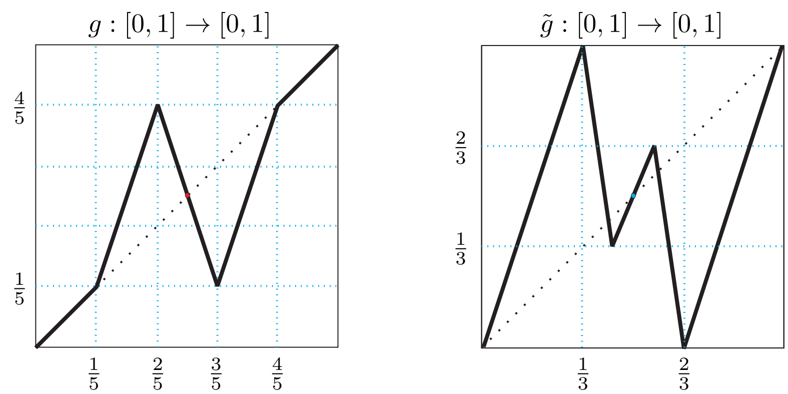

FIGURE 3. Graphs of g and $\tilde{g}$.

Question 50 (Mayer, see [44, p. 335, Question 16])**.** Let C be an *indecomposable* chainable continuum and p a point of C. Is there an embedding of C in the plane such that p is arcwise accessible from the complement?

Mayer gives full credit for the question to Nadler [75, p. 229]; however, the two versions are not the same. Mayer additionally requires C to be indecomposable. Both questions are still unanswered.

If $f : I \to I$ is continuous, let I_f denote the inverse limit of the inverse sequence with every coordinate space equal to I and each bonding map equal to f. Consider I_g and $I_{\tilde{g}}$ where g and $\tilde{g}$ are the functions whose graphs are depicted in Figure 3; see also Figure 2 in section 1 by Anušić, Bruin, and Činč. Notice that $\frac{1}{2}$ is a fixed point of both g and $\tilde{g}$. So $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ belongs to both I_g and $I_{\tilde{g}}$. Observe that I_g consists of two arcs attached to endpoints of the three-fold Knaster continuum depicted in [56, p. 205, Figure 5]. Consequently, I_g is decomposable. It can be easily observed that $I_{\tilde{g}}$ is indecomposable. An example homeomorphic to I_g is described by Nadler [75, p. 229] in connection with the question. Nadler remarks that there are "sandwiched" points in the continuum C, (I_g in our notation), that are not arcwise accessible from the complement of $\mathbb{R}^2 \setminus C'$ under any embedding C' of C in the plane which can be chained by connected open subsets of $\mathbb{R}^2$. $I_{\tilde{g}}$ was constructed by the author of this section (see [44, p. 335, Question 19]).

Continua in the form I_f, where $f : I \to I$ is continuous, play an important role in topological dynamics. For instance, Marcy Barge and Joe Martin [9] prove that each such I_f can be realized as a global attractor for a homeomorphism of the plane. Therefore, it would be especially interesting to answer questions 48 and 50 in the case where $C = I_f$ for some continuous $f : I \to I$. In particular, it would be interesting to finally settle the case for $C = I_g, I_{\tilde{g}}$, and $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, where g and $\tilde{g}$ are described above. By repeatedly using the exercise, one can show that each of I_g and $I_{\tilde{g}}$ can be expressed as an inverse limit of simplicial arcs with simplicial bonding maps. So, questions 48 and 50 are both still open in the simpler case where C is an inverse limit of simplicial arcs with simplicial bonding maps.

Note Added in Proof. It follows from a theorem recently announced by Anušić, Bruin, and Činč [6] that I_g is not a counter-example for Question 48.

There is another well-known way of looking at questions 48 and 50. If C is a chainable continuum and p is an arbitrary point of C, let $C(p)$ denote $C \cup A_p$, where A_p is an arc such that p is an endpoint of A_p and $C \cap A_p = \{p\}$. Questions 48 and 50 may be rephrased by asking whether $C(p)$ can be embedded in the plane. Observe that $C(p)$ is a tree-like continuum. We may expand $C(p)$ into an inverse limit in the following special way. Since C is chainable, it can be expressed as the inverse limit $\varprojlim \{X_i, f_i\}_{i=0}^{\infty}$, where $X_i = I$ and $f_i : X_{i+1} \to X_i$ is continuous. Denote by π_i the projection of C into X_i. Let Y_i be a copy of X_i with an arc A_i attached such that $X_i \cap A_i$ is a single point which is $\pi_i(p)$ in X_i and, simultaneously, an endpoint of A_i. Notice that Y_i is either an arc or a simple triod. Finally, let $f_i^* : Y_{i+1} \to Y_i$ be a mapping such that $f_i^*|X_{i+1} = f_i$ and $f_i^*|A_{i+1}$ is a homeomorphism onto A_i. (Observe that if the inverse sequence $\{X_i, f_i\}_{i=0}^{\infty}$ can be made simplicial for some choice of vertices, then the same is possible for $\{Y_i, f_i^*\}_{i=0}^{\infty}$.) Clearly, $\varprojlim \{Y_i, f_i^*\}_{i=0}^{\infty}$ is homeomorphic to $C(p)$. So, answering questions 48 and 50 can be viewed as a special part of the following much more general project stated here in two versions.

Project 51 (General Version)**.** Develop a general theory allowing to determine whether the inverse limit $\varprojlim \{T_i, g_i\}_{i=0}^{\infty}$ of an arbitrary given inverse sequence of trees can be embedded in the plane.

Project 51′ (Simplicial Version)**.** Restrict Project 51 to inverse sequences of simplicial trees with simplicial bonding maps.

The project (regardless of the version) must go in two directions. First, we need to be able to embed $\varprojlim \{T_i, g_i\}_{i=0}^{\infty}$ in the plane if, in fact, that can be done. There is a considerable body of research allowing us to do so in many cases. For instance, R. H. Bing [13, Theorem 4] proves that every chainable continuum (i.e., an inverse limit of arcs) can be embedded in the plane. Another very useful result in the same direction is the Anderson–Choquet Embedding Theorem [4, Theorem I]; see also [77, Theorem 2.10]. However, not much is known on how to prove that a given tree-like continuum X cannot be embedded in the plane, especially if X is atriodic. Developing new results in this direction will probably require discovering some combinatorial type of obstructions that prevent an embedding in the plane. That may be considerably easier to do in the simplicial case. Developing a comprehensive theory of embedding in this case would be very useful since many examples of tree-like continua known in the literature can be expressed as inverse limits of simplicial trees with simplicial bonding maps.

Some interesting questions concerning embedding of tree-like continua have already been asked in this problem set by Anušić, Bruin, and Činč (questions 1 and 2), David P. Bellamy (Question 9), and Lewis (questions 44, 47, and 49). We would like to add a few more in the remainder of this section.

One of the most important open problems in continuum theory asks whether every non-separating plane continuum has the fixed point property (fpp). Every one-dimensional non-separating plane continuum is tree-like. In 1980, Bellamy [10] constructed his spectacular example of a tree-like continuum without the fpp. Subsequently, several other examples, all of them based on Bellamy's idea, have been constructed; see [31], [35], [65], [66], [67], [68], [69], [78], [79], and [80].

Bellamy [10, Epiloque] proves that his example is not embeddable in the plane. The proof relies on the cone over a certain 0-dimensional set included in the example. Bellamy also notes that the Fugate–Mohler [33] technique used on his example produces an atriodic example of a tree-like continuum without the fpp. It is not known if this atriodic Bellamy's continuum is embeddable in the plane; see [10, Epiloque], [35, p. 3660], and [80, p. 85].

Most of the other known examples of tree-like continua without the fpp are either already atriotic or they become atriotic after applying the Fugate–Mohler technique. It should be mentioned that all of those examples, except for the hereditarily indecomposable continuum described in [69], can be expressed as an inverse limit of simplicial trees with simplicial bonding maps.

Question 52. Are any of the known atriodic tree-like continua without the fpp embeddable in the plane?

The expected answer should be negative. It is not very likely that any of the already known atriodic tree-like continua without the fpp can be embedded in the plane. But, showing that any of those continua cannot be embedded would be a major step toward the completion of Project 51.

We say that Y is a *compactification of the half-line* $[0, \infty)$ with X as the *remainder* provided that Y and X are continua, $X \subset Y$, and $Y \setminus X$ is dense in Y and homeomorphic to $[0, \infty)$. It would be interesting to know when a compactification Y of the half-line with a planar continuum X as the remainder is also planar. If X is chainable, then Y is also chainable and, therefore, can be embedded in the plane by Bing's theorem [13, Theorem 4]. On the other hand, if X contains a (not necessarily simple) triod, one can construct a nonplanar compactification of the half-line with a planar continuum X as the remainder.

Question 53. Let X be an arbitrary atriodic, nonchainable, tree-like continuum contained in the plane. Is there a nonplanar compactification of the half-line with an X as the remainder? What if X has span 0?

It would be interesting to answer the above question in the case of any specific atriodic, nonchainable, tree-like continuum X contained in the plane. In particular, what if X is the atriodic continuum constructed by W. T. Ingram in [46]? What if X is one of the span 0 continua constructed by L. C. Hoehn in [39] and [40]?

Mayer (see [44, Question 17]) asks the following very interesting question (that he attributes to Ingram).

Question 54. Let X be Ingram's atriodic tree-like nonchainable continuum [46]. Is there more than one inequivalent embedding of X in the plane? An embedding with the endpoint not arcwise accessible from the complement?

We would like to extend this question to the following one.

Question 55. Is there an atriodic tree-like nonchainable continuum with more than one inequivalent embedding of in the plane?

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11. Questions on Expansive Homeomorphisms

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A homeomorphism $h : X \to X$ is called *expansive* provided that there exists a constant $c > 0$ such that, for every $x, y \in X$, there exists an integer n such that $\mathrm{d}(h^n(x), h^n(y)) > c$. Here, c is called the expansive constant. A *continuum* is a compact, connected metric space. Expansive homeomorphisms on continua are some of the most beautifully chaotic functions because they have so many chaotic properties:

(1) They have positive entropy. [50]
(2) They exhibit sensitive dependence on initial conditions in the strongest sense in that no matter how close any two points are, either their images or their pre-images will, at some point, be a certain distance apart.
(3) They are chaotic in the sense of Li and Yorke. [52]

(See the definition of a Li–Yorke chaotic map on page 7 in the paragraph preceding Question 15.)

One of the more interesting problems in dynamical continuum theory is determining necessary and sufficient conditions for a continuum to admit an expansive homeomorphism. Many of the results have been on k-cyclic continua. A connected graph G is *0-cyclic* if it contains no cycles, i.e., is a tree. A connected graph G is *k-cyclic* if k is the smallest number edges $\{e_1, ..., e_k\}$ that must be removed from G so that there are no longer any cycles; that is, $G - \{e_1, ..., e_k\}$ is a *spanning tree* for G. A continuum is *k-cyclic* if it can be constructed from the inverse limit of k-cyclic graphs (k is fixed). If a one-dimensional continuum is not k-cyclic for some k, then it is said to be *infinitely-cyclic*.

Question 56. Does there exist a one-dimensional Peano continuum that admits an expansive homeomorphism?

If so, it cannot be k-cyclic [70]. In particular, the following question is unknown.

Question 57. Does the Menger curve admit an expansive homeomorphism?

A continuum is *indecomposable* if every proper subcontinuum has empty interior. Every known continuum that admits an expansive homeomorphism is known to contain a non-degenerate indecomposable subcontinuum. However, the following question remains open.

Question 58. If X is a one-dimensional continuum that admits an expansive homeomorphism, must X contain an indecomposable subcontinuum?

We need only to consider infinite-cyclic continua [70].

Question 59. Does there exist a two-dimensional non-separating plane continuum that admits an expansive homeomorphism?

There do not exist one-dimensional non-separating or 2-separating plane continua that admit expansive homeomorphisms [71] and [72]. For $4 \leq$

$n < \infty$, there do exist n-separating plane continua that admit expansive homeomorphisms, for example, Plykin attractors [81].

Question 60. Does there exist a one-dimensional 3-separating plane continuum that admits an expansive homeomorphism?

Question 61. If X is a continuum that admits an expansive homeomorphism, must X be the union of arcs.

Question 62. Suppose that $h : X \longrightarrow X$ is an expansive homeomorphism of a k-cyclic continuum (or a G-like continuum). Do a graph H and a map $f : H \longrightarrow H$ exist such that

(1) X is homeomorphic to $Y = \varprojlim(H, f)$,

(2) the shift homeomorphism $\widehat{f}$ of $\varprojlim(H, f)$ is expansive, and

(3) there exists a map $\phi : X \longrightarrow Y$ such that $\widehat{f} \circ \phi = \phi \circ h$?

A homeomorphism is *positively continuum-wise fully expansive* if, for every pair $\epsilon, \delta > 0$, there is an $N(\epsilon, \delta) > 0$ such that if Y is a subcontinuum of X with $\mathrm{diam}(Y) \geq \delta$, then $\mathrm{d}_H(h^n(Y), X) < \epsilon$ for all $n \geq N(\epsilon, \delta)$. Hisao Kato [51] shows that if continuum X admits a positively continuum-wise fully expansive homeomorphism, then X must be indecomposable.

A homeomorphism is *positively fully expansive* if it is expansive and either h or h^{-1} is positively continuum-wise fully expansive. Consider the following result.

Theorem ([73]). *If X is a circle-like continuum that admits an expansive homeomorphism, then X is a solenoid formed by the same bonding map. Furthermore, any expansive homeomorphism must also be positively fully expansive, and all shift homeomorphisms are expansive.*

This result creates even more questions.

Question 63. If $h : X \longrightarrow X$ is a positively fully expansive homeomorphism of k-cyclic continuum, then is X homeomorphic to the inverse limit of a bouquet of k circles?

Question 64. If so, must all the bonding maps be the same and must the shift be positively fully expansive?

Question 65. If X is a one-dimensional continuum that admits an expansive homeomorphism, must X contain a k-cyclic continuum that admits a fully expansive homeomorphism?

Again, we need only to consider infinite-cyclic continua by a new result [74].

12. Questions on Semi-Indecomposable Homogeneous Continua

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Definition. A continuum X is semi-indecomposable if for every two disjoint subcontinua of X at least one has empty interior.

Question 66. Is every one-dimensional semi-indecomposable homogeneous continuum indecomposable?

Question 67. Is every aposyndetic semi-indecomposable homogeneous continuum two-dimensional?

Notice that the Cartesian square of the pseudo-arc is a two-dimensional aposyndetic semi-indecomposable homogeneous continuum. See [85] for more examples of such continua. See also [83] and [84] for the background and more information on the above two questions.

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13. Questions on Products of Hereditarily Indecomposable Hausdorff Continua

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Definitions:

A *continuum* is a compact connected Hausdorff space. The continuum X is *decomposable* if X contains two proper subcontinua H and K so that $X = H \cup K$; otherwise, it is *indecomposable*. The continuum X is *hereditarily indecomposable* if every subcontinuum of X is indecomposable.

Background:

Regina Jackson, Michel Smith, and Jennifer Stone have an example of a non-metric indecomposable continuum X of size $\mathfrak{c}$ which contains no non-degenerate hereditarily indecomposable subcontinuum, and is such that $X \times X$ contains a non-metric hereditarily indecomposable continuum. (Paper in preparation.)

Note that by a result presented at the conference, by Smith, hereditarily indecomposable subcontinua of the product of two Hausdorff arcs must be metric.

Note that **the author presented a result at the conference showing that** hereditarily indecomposable subcontinua of the product of two Hausdorff arcs must be metric.

Question 68. Is there a continuum X of cardinality greater than $\mathfrak{c}$ which contains no non-degenerate hereditarily indecomposable subcontinua so that $X \times X$ contains a nonmetric hereditarily indecomposable continuum?

Question 69. Is there a continuum X so that $X \times X$ contains no nonmetric hereditarily indecomposable continuum, but $X \times X \times X$ contains such a continuum?

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