THE TOTAL NEGATION OF A TOPOLOGICAL PROPERTY

BY

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0. Introduction

The central theme in this paper is the uniform generation of new topological properties from old. Two of the best known properties obtained in this way are total disconnectedness (deriving from connectedness) and scatteredness (deriving from perfectness, i.e. having no isolated points). A third property, lesser known but interesting in its own right, is pseudofiniteness (the cf-spaces studied in [8], [9], [10], [12]) or the class of spaces whose compact subsets are finite. This last-mentioned property derives from compactness in the manner we will explore here.

In general, given a class $K$ of topological spaces ($K$ is closed under homeomorphism) we define the class Anti ($K$) in such a way that "totally disconnected" is co-extensive with "Anti (connected)" and so on. The "anti-property" of most interest to us here is pseudofiniteness which we henceforth relabel "anticompectness". We will also be interested in related anti-properties (Anti (sequentially compact), Anti (Lindelöf), etc.) but they will receive secondary emphasis. The general behavior of the operation Anti ($\cdot$) itself will occupy some of our attention. However at this stage there are many more questions than answers, so our general treatment will be sketchy, serving mainly to tie together ideas which otherwise may appear to be unrelated.

Our set-theoretic conventions are as follows: (i) $\omega_\alpha$ denotes the $\alpha$th infinite initial ordinal, where $\alpha$ is any ordinal. Since we assume the Axiom of Choice throughout, we identify $\omega_\alpha$ with the cardinal $\aleph_\alpha$. $\omega = \omega_0$. (ii) An ordinal $\alpha$ is the set of its predecessors. Greek letters near the beginning of the alphabet will usually denote ordinals, while the letters $K$, $A$, $T$ will be reserved for cardinals. (iii) The ordinal successor of $\alpha$ is $\alpha + 1 = \alpha \cup \{\alpha\}$, the cardinal successor of $K$ is $K^+$. (iv) If $A$ is any set $P(A)$ denotes the power set of $A$. (v) $B^A$ is the set of all maps $f: A \to B$. The cardinality of $A$ is written $|A|$. (vi) If $K$ is a cardinal then $\exp(K) = |2^K| = |P(K)|$. $\exp(\omega)$ is usually denoted by $\alpha$. (vii) The cartesian product of a family $(A_i: i \in I)$ of sets is denoted $\prod_I A_i$. If $A_i = A$ for all $i \in I$ then the set $A^I$ will also at times be denoted $\prod_I (A)$. Further notations will be introduced as they arise in the discussion. The referee's kind suggestions regarding exposition are gratefully acknowledged.

Received September 16, 1977.
1. A Description of the process

Let $K$ be a topological class. The spectrum $\text{Spec}(K)$ of $K$ is the class of cardinal numbers $\kappa$ such that any topology on a set of power $\kappa$ lies in $K$. For example, any topology on a finite set must be compact; and any infinite set supports noncompact topologies. Thus $\text{Spec}(\{\text{compact spaces}\}) = \omega$. Other spectra can be computed quite readily, such as $\text{Spec}(\{\text{connected spaces}\}) = 2(=\{0, 1\})$, and $\text{Spec}(\{\text{perfect spaces}\}) = 1$.

Now let $K$ be a class of spaces and define $\text{Anti}(K)$ to be the class of spaces $X$ such that whenever $Y \subseteq X$, $Y \in K$ iff $|Y| \in \text{Spec}(K)$. Thus $X \in \text{Anti}(K)$ iff the only subspaces of $X$ which are in $K$ are those which "have to be" on account of their cardinalities. Clearly $\text{Anti}(K)$ is always hereditary.

1.1 Proposition. $\text{Anti}(K)$ is never empty.

Proof. If $\text{Anti}(K)$ were empty for some $K$ then for every $X$ there would be a subspace $Y$ with $Y \subseteq K$ and $|Y| \notin \text{Spec}(K)$. Pick $X = \emptyset$. Then $\emptyset \in K$ but $0 \notin \text{Spec}(K)$. This is nonsense since there is only one topology on the empty set.

Remark. (1.1) shows that not every hereditary class need be of the form $\text{Anti}(K)$. In a private communication, B. Scott has proved the following (the proof to appear elsewhere) Theorem: Let $L$ be a hereditary class. Then $L$ is not of the form $\text{Anti}(K)$ iff there is an $n < \omega$ with $n \in \text{Spec}(L)$ and $X \notin L$ for all spaces $X$ of power at least $n + 1$. Moreover if $L$ is of the form $\text{Anti}(K)$ then $K$ can be chosen to have empty spectrum; and if $L$ contains spaces of all cardinalities then $K$ can be taken to be the complement of $L$.

The reader can easily check that $\text{Anti}(\{\text{connected spaces}\}) = \{\text{totally disconnected spaces}\}$, $\text{Anti}(\{\text{perfect spaces}\}) = \{\text{scattered spaces}\}$, and $\text{Anti}(\{\text{compact spaces}\}) = \{\text{pseudofinite spaces}\}$. Other "anti-classes" are easy to compute as well.

We mention some of the general properties of the operation $\text{Anti}(\cdot)$. The proofs are straightforward.

1.2 Proposition. (i) If $K \subseteq L$ then $\text{Spec}(K) \subseteq \text{Spec}(L)$.
(ii) If $K \subseteq L$ and $\text{Spec}(K) = \text{Spec}(L)$ then $\text{Anti}(K) \supseteq \text{Anti}(L)$. $\text{Anti}(K)$ and $\text{Anti}(L)$ can be unrelated, however.
(iii) $\text{Anti}(\cdot)$ is not idempotent.
(iv) If $K$ is hereditary then $K \subseteq \text{Anti}(\text{Anti}(K))$.
(v) $\text{Anti}(K) \subseteq \text{Anti}(\text{Anti}(\text{Anti}(K)))$ for all $K$.

Proof. (i) This is obvious.
(ii) Let $K$, $L$ be as in the hypothesis, with $X \notin \text{Anti}(K)$. Let $Y \subseteq X$ be such that $Y \subseteq K$ but $|Y| \notin \text{Spec}(K)$. Then $Y \subseteq L$ but $|Y| \notin \text{Spec}(L)$; whence $X \notin \text{Anti}(L)$.

Let $K = \{\text{compact spaces}\}$, $L = \{\text{Lindelöf spaces}\}$. Then $K \subseteq L$ but $\text{Anti}(K) \notin \text{Anti}(L)$ and $\text{Anti}(L) \notin \text{Anti}(K)$ by (1.3(iii)).
(iii) Let $K = \{\text{compact spaces}\}$ again. Then $X \in \text{Anti}(\text{Anti}(K))$ iff every infinite subset of $X$ contains an infinite compact set. The ordinal space $[0, \omega] (= \omega + 1$ with the order topology) is in $K$ but not in $\text{Anti}(\text{Anti}(K))$.

(iv) Let $K$ be hereditary, with $X \notin \text{Anti}(\text{Anti}(K))$. Then for some $Y \subset X$, $|Y| \notin \text{Spec}(\text{Anti}(K))$ but $Y \in \text{Anti}(K)$. Since $K$ is hereditary, we have $\text{Spec}(K) \subset \text{Spec}(\text{Anti}(K))$. Thus $|Y| \notin \text{Spec}(K)$. But $Y \in \text{Anti}(K)$; whence $Y \notin K$, so $X \notin K$.

(v) $\text{Anti}(K)$ is always hereditary, so use (iv). □

In the remainder of this paper we will concentrate on $\text{Anti}(K)$ where $K$ is one of the properties “compact”, “sequentially compact”, “Lindelöf”.

1.3 Proposition. (i) If $X$ is anti-Lindelöf, anti-(sequentially compact), and $T_2$ then $X$ is anticompact.

(ii) If $X$ is anticompact and $T_1$ then $X$ is anti-(path connected).

(iii) Anticompactness and anti-Lindelöfness are implicationally unrelated.

(iv) $\text{[12]}$ Anticompact spaces are anti-(sequentially compact). The converse is false.

(v) A $T_2$ space is discrete iff it is an anticompact $k$-space.

Proof. (i) A $T_2$ space is anti-(sequentially compact) iff it contains no embedded copies of the compact ordinal space $[0, \omega]$. To see this, suppose first that $[0, \omega]$ is embedded in $X$. Then $X$ would contain an infinite sequentially compact subspace. On the other hand suppose $X$ fails to be anti-(sequentially compact). Let $Y \subset X$ be infinite and sequentially compact. Since $Y$ is $T_2$, it contains a discrete sequence $(y_0, y_1, \ldots)$. By sequential compactness there is a convergent subsequence, hence an embedded copy of $[0, \omega]$.

Now suppose $X$ satisfies the hypotheses of (i). If $Y \subset X$ is compact then $Y$ is also Lindelöf, hence countable. Thus, if $Y$ isn’t finite, $Y$ is infinite compact metric and must contain a copy of $[0, \omega]$, an impossibility.

(ii) Let $f: [0, 1] \rightarrow X$ be a path in $X$. Since $X$ is anticompact, range $(f) \subset X$ is a connected finite set. Since $X$ is also $T_1$, range $(f)$ must be a singleton.

(iii) The ordinal space $[0, \omega]$ is anti-Lindelöf but not anticompact. To get a space which is anticompact but not anti-Lindelöf, let $R^#$ denote the real numbers with the topology basically generated by sets of the form (open interval)-(countable set). This is the classical example of a hereditarily Lindelöf nonseparable $T_2$ space. Since $|R^#| = c$, this space cannot be anti-Lindelöf. However let $K \subset R^#$ be compact. If $K$ is infinite then $K$ contains a countable subset. But such sets are closed discrete in $R^#$. Thus $K$ is finite and $R^#$ is anticompact.

(iv) This is proved in [12] (c.f. the observation after their Theorem 5).

(v) Let $X$ be $T_2$. If $X$ is discrete it is clearly anti-compact as well as compactly generated. Conversely if $X$ is an anticompact $k$-space and $Y \subset X$
then \( Y \cap K \) is closed in \( K \) whenever \( K \) is compact (hence finite discrete) in \( X \). Thus \( Y \) is closed in \( X \), whence \( X \) is discrete. 

**Remark.** The class of \( k \)-spaces is quite large, including the first countable spaces as well as the \( p \)-spaces of Archangel'skiï. Thus it doesn't take much to force an anticom pact space to be discrete. We will see in \( \S 2 \), however, that nondiscrete anticom pact spaces abound.

### 2. Anticom pactness and connectedness

In view of the fact that anticom pact \( T_1 \) spaces have trivial path components, one might suspect that connectedness and anticom pactness are mutually inconsistent properties for reasonably nice (with regard to separation axioms) spaces. However our only nontrivial example, up to this point, of an anticom pact space turns out also to be connected \( T_2 \) (but not regular). The space \( R^# \), originally suggested to me by F. Galvin, is connected for the following reason. Let \( \mathcal{U} = \{ U_\alpha - A_\alpha : \alpha < \kappa \} \) be a basic open cover of \( R^# \). That is, \( U_\alpha \) is a nonempty open interval and \( A_\alpha \) is countable for each \( \alpha < \kappa \). It will suffice to show that \( \mathcal{U} \) is "connected" in the sense that whenever \( V, W \in \mathcal{U} \) there is a "simple chain" \( M_1, \ldots, M_n \in \mathcal{U} \) with \( V \cap M_1 \neq \emptyset, M_n \cap W \neq \emptyset \), and \( M_i \cap M_{i+1} \neq \emptyset \) for \( 1 \leq i \leq n - 1 \). But this is clearly true for \( \mathcal{U} \) since \( R \) is connected in its usual topology and the overlap of two open intervals is either empty or uncountable.

Let us now look at two important sources of anticom pact examples, the \( P \)-spaces and the \( MI \)-spaces. \( X \) is a \( P \)-space if intersections of countably many open sets are open. A \( P \)-space which is also \( T_1 \) must be anticom pact since countable subsets are always closed discrete. As far as existence is concerned, these spaces are quite common and there are many ways of systematically constructing them (see [2], [3], [4], [11], [13], [14], also \( \S 3 \)). In [11] Misra constructs a connected \( T_2 \) \( P \)-space, so again connectedness and anticom pactness co-occur in the presence of the Hausdorff axiom. The inevitable question then is whether there are any regular connected anticom pact spaces. \( R^# \) is well-known to be nonregular. Moreover no regular \( P \)-space with more than one point is connected since, as can be seen in [2], [11], such spaces are always strongly zero-dimensional.

This brings us to our second source, namely the \( MI \)-spaces of Hewitt [7]. \( X \) is an \( MI \)-space if it is perfect, Hausdorff, and "sub-maximal" in the sense of [5], i.e. every dense subset is open. There are several ways of constructing these spaces (see [1], [5], [7], [9], [10]); and in [1] Anderson gives a uniform way of constructing connected examples. To complete the picture, Kirch [9] shows that \( MI \)-spaces are anticom pact. To the best of our knowledge, however, it is an open question whether a connected \( MI \)-space can be regular.

As a side remark, the space \( R^# \) is neither a \( P \)-space nor an \( MI \)-space. For
on the one hand each of its points is a $G_6$; and on the other hand, $R^\#$ is "resolvable" into a disjoint union of two dense subsets (see [7]).

With the above ample introduction aside, we now answer our "inevitable" question in the affirmative with the following offering. This example owes its beginnings to an enlightening conversation with E. K. van Douwen and M. E. Rudin.

2.1 Example. A connected anticom pact Tikhonov space which is resolvable, hence neither an MI-space nor a $P$-Space.

Construction. Let $A = [0, 1)$ be the half-open unit interval with $A^* = \beta A - A$ its Stone-Cech remainder. We construct a subspace $\Sigma$ of $A^*$ and show that $\Sigma$, automatically a Tikhonov space, satisfies the remaining conditions. The basic facts we use are the following:

(i) $A^*$ is an indecomposable continuum (see Walker [15]).

(ii) There are $\exp(c)$ infinite closed subsets of $A^*$, and each has $\exp(c)$ points (again, see [15]).

(iii) If $X$ is any connected $T_1$ space and $p \in X$ is a cutpoint of $X$ then there are disjoint nonempty open sets $U, V$ in $X$ with $U \cup V = X - \{p\}$ and $U \cup \{p\}$, $V \cup \{p\}$ connected (see Ward [16]).

We first prove the claim that if $X$ is a nondegenerate indecomposable continuum and $F \subset X$ is finite then $X - F$ is connected. Induct on $|F|$. If $|F| = 0$ there's nothing to prove, so suppose $X - F$ is connected with $p \in X - F$. We show $X - (F \cup \{p\})$ is connected by proving that $p$ isn't a cutpoint of $X - F$. For if it were then (since finite sets are closed) there would be disjoint nonempty open sets $U, V$ of $X$ with $U \cap V = X - (F \cup \{p\})$ and $U \cup \{p\}$, $V \cup \{p\}$ connected. Now $X - (F \cup \{p\})$ is dense in $X$. Thus

$$\text{Cl}(U \cup \{p\}) \cup \text{Cl}(V \cup \{p\}) = X.$$ 

But neither subcontinuum is all of $X$. This contradicts indecomposability.

Now, using (ii) above, let $(F_\alpha : \alpha < \exp(c))$ be a well ordering in type $\exp(c)$ of the infinite closed subsets of $A^*$. By induction we can pick distinct points

$$x^{(i)}_\alpha \in F_\alpha - \bigcup_{\beta < \alpha} \{x^{(1)}_\beta, x^{(2)}_\beta, x^{(3)}_\beta\}, \quad i = 1, 2, 3, \alpha < \exp(c).$$

Let $X^{(i)} = \{x^{(i)}_\alpha : \alpha < \exp(c)\}, i = 1, 2, 3$, and set $\Sigma = X^{(1)} \cup X^{(2)}$. We check that $\Sigma$ has the properties we want.

Let $U$ be nonempty open in $A^*$, $x \in U$. Let $V$ be open in $A^*$ with $x \in V \subset \text{Cl}(V) \subset U$. Then $\text{Cl}(V)$ is infinite closed in $A^*$ so hits each $X^{(i)}$, $i = 1, 2, 3$; whence each $X^{(i)}$ is dense in $A^*$. Thus $\Sigma$ is resolvable into the disjoint union of the dense subsets $X^{(1)}$, $X^{(2)}$, so isn't an $MI$-space. $\Sigma$ is anticom pact since if $K$ is a compact subset then $K$ is closed in $A^*$. Since infinite closed sets share points with $X^{(3)} \supset A^* - \Sigma$, $K$ must be finite. To see
that $\Sigma$ is connected let $\mathcal{U}$ be a collection of open subsets of $A^*$ which covers $\Sigma$. We show the relativized cover $\mathcal{U} \upharpoonright \Sigma$ to be connected. Since $\Sigma$ is dense in $A^*$, it will suffice to show that $\mathcal{U}$ itself is connected. But $A^* - \bigcup \mathcal{U}$ is finite; and hence by (i), (iii) above, $\bigcup \mathcal{U}$ is a connected set. The argument is completed by noting that $\Sigma$ is not a $P$-space since it is regular and not zero-dimensional.

We end this section with some open questions.

2.2 Problems. (i) Are there nontrivial examples of connected anticom- pact spaces which are normal? paracompact? We don't know whether $\Sigma$ above is normal.

(ii) Are there regular connected anticom pact spaces of power $c$? Our space $\Sigma$ has power $\exp(c)$.

(iii) Are regular anticom pact spaces always Tichonov? We know an- ticom pactness fails to collapse any of the other pairs of well-known separa- tion axioms.

(iv) Find an interesting class of spaces (not contained in the class of $k$-spaces) whose intersection with the anticom pact spaces is contained within the totally disconnected spaces.

3. Preservation of anticom pactness

In this section we consider questions involving the preservation of anticom pactness under topological operations. For instance anticom pactness is trivially preserved by open bijections (e.g. expansion of topologies). Also anticom pactness is preserved by "compact covering maps" (i.e. continuous maps such that compact sets in the range are images of compact sets in the domain).

We next turn our attention to the preservation of anticom pactness under various topological product formations. The following is stated in [12].

3.1 Proposition. The Tichonov product of topological spaces is anticom- pact iff all of the factors are anticom pact and all but finitely many of them are singletons.

A generalization of the Tichonov product is the "$\lambda$-box product" where $\lambda$ is an infinite cardinal. Specifically let $\langle X_i : i \in I \rangle$ be a collection of spaces with $\prod_i X_i$ denoting the cartesian product of the underlying sets of the $X_i$'s. An open $\lambda$-box is a product $\prod_i U_i$ where $U_i$ is open in $X_i$ and $|\{i : U_i \neq X_i\}| < \lambda$. The $\lambda$-box product, denoted by $\prod_\lambda X_i$, is the space with underlying set $\prod_i X_i$ and topology generated by the open $\lambda$-boxes. The Tichonov product is then $\prod_\omega X_i$; and the full box product is $\prod_\nu X_i$, denoted $\prod_\nu X_i$.

3.2 Theorem. Let $\langle X_i : i \in I \rangle$ be a collection of $T_1$ spaces with $\omega_1 \leq \lambda \leq \infty$. Then $\prod_\lambda X_i$ is anticom pact iff each $X_i$ is anticom pact.
Proof. Suppose $\prod_i X_i$ is anticom pact, $j \in I$. Then $X_j$ embeds in $\prod_i X_i$ and is thus anticom pact.

Conversely suppose each $X_i$ is anticom pact. Since $\lambda \geq \omega_1$ and anticom pactness is preserved under expansion of topologies, we need only prove the assertion for $\lambda = \omega_1$. So let $K$ be compact in $\prod_i X_i$ with $K_j$ the $j$th projection of $K$. Then each $K_i$ is compact, hence finite; and is therefore discrete since $X_i$ is $T_1$. So $K \subseteq \prod_i K_i$, an $\omega_1$-box product of discrete spaces. But such spaces are clearly $P$-spaces. Since they are also $T_1$, they are therefore anticom pact. Thus $K$ is finite.

Another generalized product (generalizing not the Tichonov product but the box product) is the topological reduced product. This construction, borrowed from model theory, is studied in [2], [3], [4] and is defined as follows. Let $(X_i : i \in I)$ be a collection of spaces with $D$ a filter of subsets of $I$. In $\prod X_i$ define the equivalence relation $x \sim_D y$ if $\{i : x_i = y_i\} \in D$. Let $\prod_D X_i$ be the resulting quotient space. This space is the $D$-reduced product of the $X_i$'s. Clearly $\prod_D X_i = \prod_i X_i$; and if $D$ is an ultrafilter on $I$ then $\prod_D X_i$ is called the $D$-ultraproduct of the $X_i$'s. We consider the preservation of anticom pactness under reduced products. The reader is assumed to have a nodding acquaintance with some of the lore of measurable cardinals and of the set-theoretic properties of filters. In particular a filter $D$ is $\lambda$-complete if $D$ is closed under $< \lambda$ intersections. $D$ is $\lambda$-regular if there is a set $E \subseteq D$ of power $\lambda$ such that each $i \in I$ is contained in only finitely many members of $E$. An ultrafilter $D$ is $\omega$-regular iff it is $\omega_1$-incomplete (i.e. countably incomplete). This is an important property of ultrafilters.

3.3 Lemma. Let $(X_i : i \in I)$ be a collection of spaces with $D$ a $\lambda$-regular filter on $I$. Then $\prod_D X_i$ is a $P_\lambda$-space (i.e. intersections of $\leq \lambda$ open sets are open).

Proof. This is proved in [2] for $D$ a $\lambda$-regular ultrafilter. The proof for arbitrary $\lambda$-regular $D$ is identical.

3.4 Theorem. Let $(X_i : i \in I)$ be a collection of $T_1$ spaces, with $D$ an $\omega$-regular filter on $I$. Then $\prod_D X_i$ is anticom pact.

Proof. Clearly $\prod_D X_i$ is $T_1$. By (3.3) it is also a $P_{\omega_1}$-space (i.e. $P$-space); and is hence anticom pact. Thus the $\omega$-regular reduced product formation not only preserves anticom pactness of $T_1$ spaces, it confers the property for free. We next consider the behavior of countably complete filters.

3.5 Theorem. Let $(X_i : i \in I)$ be a collection of anticom pact $T_2$ spaces with $D$ a countably complete ultrafilter on $I$. Then $\prod_D X_i$ is anticom pact.

Proof. $D$ is either fixed, in which case we're done, or free and $\mu$-complete where $\mu$ is a measurable cardinal (see [6]). Let $C$ be compact in
\[ \prod_D X_i \] and assume \( C \) is infinite. If \( K \subset C \) is countable then of course \( \text{Cl}(K) \subset C \) is compact. We show first that \( |\text{Cl}(K)| < \mu \). Indeed reduced products preserve the Hausdorff axiom so \( \text{Cl}(K) \) is a separable \( T_2 \) space; and its cardinal therefore is \( \leq \exp (c) \). This follows from a well-known general fact about \( T_2 \) spaces; for let \( X \) be \( T_2 \), let \( S \subset X \) be dense, and let

\[ \Pi_x = \{ U \cap S: U \text{ is an open neighborhood of } x \in X \} \]

Then \( \Pi_x \neq \Pi_y \), whenever \( x \neq y \), so we obtain a one-one map of \( X \) into

\[ P(\{ U \cap S: U \text{ open in } X \}) \]

whence \( |X| \leq \exp (\exp (d(X))) \), where \( d(X) \) is the density character of \( X \). Since measurable cardinals are inaccessible, we have \( |\text{Cl}(K)| < \mu \). Now let

\[ K_i = \{ x_i: \text{there is a } [f]_D \in \text{Cl}(K) \text{ with } f_i = x_i \} \]

We show \( \text{Cl}(K) = \prod_D K_i \). First it is clear that \( \text{Cl}(K) \subset \prod_D K_i \) so suppose that \( [f]_D \in \prod_D K_i - \text{Cl}(K) \). Then \( \{ i: f_i \in K_i \} \in D \); and for each \( [g]_D \in \text{Cl}(K) \), \( \{ i: f_i = g_i \} \in D \). Since \( |\text{Cl}(K)| < \mu \) and \( D \) is \( \mu \)-complete we know

\[ \{ i: f_i \in K_i \text{ and for all } [g]_D \in \text{Cl}(K), f_i \neq g_i \} \in D. \]

In particular there is an \( i \in I \) with \( f_i \in K_i \) and \( f_i \neq g_i \) for any \( [g]_D \in \text{Cl}(K) \), contradicting the definition of \( K_i \).

Now since \( \text{Cl}(K) = \prod_D K_i \) is compact, it is a basic fact about topological ultraproducts (see [2]) that \( \{ i: K_i \text{ is compact} \} \in D \). Since each \( X_i \) is anticom pact, \( \{ i: K_i \text{ is finite} \} \in D \). Finally, since \( D \) is countably complete, it follows that \( \prod_D K_i \) is finite, a contradiction. Thus \( C \) must have been finite to begin with.

Now ultrafilters are either countably complete or \( \omega \)-regular. Thus combining (3.4) and (3.5) gives:

3.6 Corollary. Topological ultraproducts of anticom pact \( T_2 \) spaces are anticom pact. ■

Remark. The argument in (3.5) clearly requires that the countably complete filter \( D \) be an ultrafilter and that the spaces \( X_i \) be Hausdorff (as opposed to \( T_1 \) for \( \lambda \)-box products and \( \omega \)-regular reduced products). The inequality \( |X| \leq \exp (\exp (d(X))) \) fails for \( T_1 \) spaces since the cofinite topology on any set is separable \( T_1 \).

3.7 Problem. Do all reduced products preserve anticom pactness? An interesting special case to consider might be whether the countably complete filter generated by the closed unbounded subsets of \([0, \omega_1]\) preserves anticom pactness. Judging by the special properties of countably complete ultrafilters which we had to use in proving (3.5), it seems likely that a counterexample awaits discovery here.
4. The anti-Lindelöf property

Let $\kappa$ be a cardinal, $X$ a space. $X$ is $\kappa$-compact if every open cover of $X$ has a subcover of power $<\kappa$. Thus $X$ is anti-$\kappa$-compact iff the only subsets of $X$ which are $\kappa$-compact are those of power $<\kappa$. Of course compact $=\omega$-compact and Lindelöf $=\omega_1$-compact. We concern ourselves here with generalizing the results of Sections 2 and 3. Unfortunately many of these results have proved highly resistant to generalization, so there is no shortage of open questions in this connection. Since many of the difficulties which arise for general $\kappa$ evidence themselves already in the case $\kappa=\omega_1$, this is the case which will receive most of our attention.

Let us first examine the existence question. The space consisting of $I \cup \{\infty\}$, where $I$ is uncountable discrete, $\infty \notin I$, and the neighborhoods of $\infty$ are of the form $J \cup \{\infty\}$ where $I-J$ is countable, is Lindelöf and not anti-Lindelöf, as well as a paracompact $P$-space. So, not surprisingly, $P$-spaces do not provide us automatically with anti-Lindelöf examples. As regards the $MI$-spaces, less is known.

4.1 Problem. Can uncountable $MI$-spaces be Lindelöf?

4.2 Proposition. If $X$ is a $T_\lambda P_{\lambda^+}$-space then $X$ is anti-$\kappa$-compact for $\kappa \leq \lambda$.

Proof. Every subset of $X$ of power $\leq \lambda$ is closed discrete. ■

As we saw in (3.3), $\lambda$-regular reduced products of $T_1$ spaces provide an excellent source of anti-$\lambda$-compact examples.

4.3 Example. For each infinite cardinal $\lambda$, a space of power $\lambda$ which is nondiscrete, paracompact, and anti-$\kappa$-compact for each $\kappa \leq \lambda$.

Construction. Let $I$ be discrete of power $\lambda$ and let $p \in \beta I - I$. Let $X = I \cup \{p\} \subset \beta I$. This space clearly has the desired properties. ■

The space $R^\#$ introduced in the proof of (1.3iii) motivates the next example.

4.4 Example. For each infinite cardinal $\lambda$, a connected $\exp(\lambda)$-compact $T_2$ space of power $\exp(\lambda)$ which is anti-$\kappa$-compact for $\kappa < \exp(\lambda)$.

Construction. Let $X$ be the cube $[0, 1]^\lambda$ where we allow sets of the form (open Tichonov box)−(set of power $< \exp(\lambda)$) to form a topological basis. Thus sets of power $< \exp(\lambda)$ are automatically closed. Clearly $X$ is $T_2$ and of power $\exp(\lambda)$. Also since sets of power $< \exp(\lambda)$ are discrete in $X$, this space is clearly anti-$\kappa$-compact for $\kappa < \exp(\lambda)$. $X$ is connected for almost exactly the same reason that $R^\#$ is connected. $X$ is $\exp(\lambda)$-compact by a straightforward argument using the fact that $[0, 1]^\lambda$ with the Tichonov topology is compact. ■
4.5 Problem. Are there any regular (Tichonov) connected anti-Lindelöf spaces?

We next look for analogues of the theorems in §3 regarding preservation of anti-Lindelöfness. For example it is easy to see that this property is preserved under open bijections as well as "Lindelöf covering maps". Nonetheless it turns out that anti-Lindelöfness is generally a more difficult property to work with than is anticompactness.

The proof of the following is identical to that of (3.1) [12].

4.6 Proposition. The Tichonov product of topological spaces is anti-Lindelöf iff all of the factors are anti-Lindelöf and all but finitely many of them are singletons. ■

In analogy with (3.2) we have:

4.7 Theorem. Let \( \langle X_i : i \in I \rangle \) be a collection of \( T_1 \) P-spaces with \( \omega_2 \leq \lambda \leq \infty \). Then \( \prod_i X_i \) is anti-Lindelöf iff each \( X_i \) is anti-Lindelöf.

Proof. Mimic the proof of (3.2). At the stage where \( K \subset \prod_i K_i \) (i.e. \( K \) is Lindelöf, \( K_i \) is the \( i \)-th projection of \( K \), and each \( K_i \) is discrete, owing to the fact that \( X_i \) is a \( T_1 \) P-space with \( K_i \) countable), we use the fact that \( \omega_2 \)-box products of discrete spaces are \( P_{\omega_2} \)-spaces which in turn are anti-Lindelöf. Thus \( K \) is countable. ■

Concerning \( \omega_1 \)-box products of anti-Lindelöf spaces, the "real" analogue of (3.1) is:

4.8 Theorem. Suppose \( \langle X_i : i \in I \rangle \) is a collection of \( T_1 \) spaces.

(i) If \( \inf \{ i : |X_i| \geq 2 \} \geq \omega \) then \( \prod_i X_i \) is not anti-Lindelöf.

(ii) If \( \inf \{ i : |X_i| \geq 2 \} \leq \omega \) and each \( X_i \) is an anti-Lindelöf P-space then \( \prod_i X_i \) is anti-Lindelöf.

Proof. (i) If each \( X_i \) is \( T_1 \) and uncountably many \( X_i \)'s have more than one point then \( \prod_i X_i \) contains a copy of \( \prod_{\omega} (2) \) which, while not Lindelöf itself (it further contains a copy of \( \prod_{\omega} (2) \), an uncountable closed discrete subset), fails to be anti-Lindelöf. To see this, let \( (X)_\lambda \) denote the space obtained from \( X \) by closing up the topology of \( X \) under intersections of \( <\lambda \) open sets. It is a straightforward matter to prove that if \( X \) is discrete and \( \lambda \) is a regular cardinal then for any index set \( J \), \( \prod_j (X)_\lambda \) is homeomorphic to \( (\prod_j (X))_\lambda \). Thus \( \prod_{\omega_1} (2) = (\prod_{\omega_1} (2))_{\omega_1} \). Let \( F: \omega_1 \to 2^{\omega_1} \) be defined by

\[
F(\alpha)(\beta) = \begin{cases} 0 & \text{if } \beta < \alpha \\ 1 & \text{if } \beta \geq \alpha \end{cases}
\]

Then range \((F)\) is a discrete subset of \( \prod_{\omega_1} (2) \) (i.e. \( F \) is an "\( \omega_1 \)-Cauchy sequence" in the sense of Sikorski (see [13], [14])), and it has precisely one limit point, namely the zero sequence. Thus range \((F) \cup \{ \text{zero sequence} \} \) is a copy of the modified ordinal space \( ([0, \omega_1])_{\omega_1} \), which itself is uncountable Lindelöf.
(ii) A box product of countably many anti-Lindelöf $P$-spaces is anti-Lindelöf. For again look at the proof of (3.2). When we get to the stage $K \in \prod_{i \in I} K_i$ (where $I$ is countable), each $K_i$ is discrete and box products of discrete spaces are discrete. Thus $K$ is countable.

4.9 Problem. Is the box product of countably many copies of $[0, \omega]$ anti-Lindelöf?

We now move to consider reduced products of anti-Lindelöf spaces. By (3.3) and (4.2), an $\omega_1$-regular reduced product of $T_1$ spaces is anti-Lindelöf as well as anticompact. Moreover the analogy of (3.5) goes through intact. The proof is virtually the same (with the obvious minor adjustments).

4.10 Theorem. Let $\langle X_i : i \in I \rangle$ be a collection of anti-Lindelöf $T_2$ spaces with $D$ a countably complete ultrafilter on $I$. Then $\prod_D X_i$ is anti-Lindelöf.

4.11 Corollary. Topological ultraproducts of anti-Lindelöf $T_2$ spaces are anti-Lindelöf, provided the ultrafilters are either countably complete or $\omega_1$-regular.

We leave the subject with the obvious question implied by (4.11), namely:

4.12 Problem. Decide whether free ultrafilters on the integers (as examples of $\omega$-regular but not $\omega_1$-regular ultrafilters) yield preservation of anti-Lindelöfness for $T_2$ spaces.

References