TOPOLOGICAL MEASURES OF SYSTEMS OF SETS

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Abstract

If X is an infinite set, we may topologize the power set of X naturally so that systems of subsets of X can be described in the language of Baire category. Systems we consider include: subalgebra lattices of algebraic structures; topologies; filters; and families of almost disjoint sets.

AMS Subject Classification (1991): 05B35, 08A99, 54A05, 54E52.

Key Words and Phrases: systems of sets, closure systems, exchange property, Baire category.

0. Introduction.

By a *system of sets* we mean a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a collection of subsets of X. Our objective in this paper is to compare lattice properties of $\mathcal{S}$ (where $\mathcal{S}$ is viewed as a subset of the power set lattice $\wp(X)$) with topological properties of $\mathcal{S}$ (where $\mathcal{S}$ is viewed as a subset/subspace of the set $\wp(X)$, suitably topologized). The lattice properties we have in mind include whether $\mathcal{S}$ is closed under certain Boolean operations, whether $\mathcal{S}$ forms the subalgebra lattice of a universal algebra structure on X, and the like. The topological properties we consider largely deal with how "thick" $\mathcal{S}$ is.

A subset of a set Y is considered "thick" if its complement in Y is considered "thin." In pure set-theoretic terms, if Y has infinite cardinality λ, then subsets of cardinality $< \lambda$ may be fairly viewed as being "thin," but we seek notions that are more essentially topological. Suppose κ is an infinite cardinal and that Y is equipped with a topology satisfying the κ-Baire category property. (This means that intersections of at most κ dense open sets are dense. In case κ is countable, any space that embeds as a G_δ subset of a compact Hausdorff space is κ-Baire; in the uncountable case, examples are more exotic, but still plentiful.) In this setting, there are essentially five gradations for subsets on the "thin-thick" scale. The "thinnest" sets are the *nowhere dense* sets, those with closures that have empty interior. Unions of at most κ nowhere dense sets have empty interior, and are called κ-*meager*. (When κ is countable, these sets are also called *meager* sets, or sets of the *first category*.) κ-meager sets can be dense in the space, and are thus generally "thicker" than the nowhere dense sets. We adopt the convention that if P is an adjective describing subsets of a set, then the subsets whose complements are P are called *co-P*. However, since the term "co-κ-meager" is so awkward-sounding, and since the property it names plays such a large role in the sequel, we break the convention in this case only and use "*κ-residual*" instead. Note that, relative to a given topology, κ-meagerness and κ-residuality become weaker with increasing κ. When the κ-Baire property is present, κ-residual sets (just residual when κ is countable) are "thick" in the sense that any intersection of at most κ κ-residual sets is still κ-residual; however, κ-residual sets can have empty interior. "Thickest" of all, then, are the co-nowhere dense sets, sets with dense interiors. The middle ground on this scale comprises those sets that are neither κ-meager nor κ-residual, neither "thin" nor "thick." (In the countable case, sets that are not of the first category are called sets of the *second category*, or *nonmeager* sets. Such sets are merely "not thin.") In this paper, we use the words "thin" and "thick," without quotes, to convey, in most cases, the more precise notions of κ-meager and κ-residual with respect to a given κ-Baire topology. This somewhat vague (but less cumbersome) terminology will be used only in informal discourse, and should cause no confusion.

In the next section, we indicate ways in which one can topologize the power set to obtain a space satisfying the κ-Baire category property (much as we did in [1] and [2]). In succeeding sections we first deal with general systems of sets, then with closure systems, and finally with closure systems satisfying the exchange property.

Acknowledgement. The author would like to extend his gratitude to the referee for many helpful comments that improved both the form and the content of this paper.

1. Topological Preliminaries.

We first fix some standard notation: The cardinality of a set X is denoted $|X|$; an ordinal α is the set of its ordinal predecessors ($\alpha = \{\beta : \beta < \alpha\}$); a cardinal is an initial ordinal; ω is the first infinite cardinal; the cardinal successor of a cardinal κ is denoted κ^+; $2^\kappa := |\wp(\kappa)|$; and $2^{<\kappa} := Sup\,\{2^\lambda : \lambda < \kappa\}$. Let X be an infinite set. For each pair F, G of subsets of X, let $[F, G] := \{Y \subseteq X : F \subseteq Y \subseteq X \setminus G\}$. If κ is any infinite cardinal, we let $\mathcal{T}_\kappa$ be the topology that is basically generated by those sets $[F, G]$ where $|F|, |G| < \kappa$. (Such sets are called *κ-intervals*.) Then, by identifying subsets with characteristic functions, we know that $(\wp(X), \mathcal{T}_\omega)$ is homeomorphic to the $|X|$-fold topological power of the two-point discrete space. We refer to $\mathcal{T}_\omega$ as the *coarse* topology (closed (resp. open, discrete, etc.) sets in this topology being termed *coarse-closed* (resp. *coarse-open,* *coarse-discrete*, etc.)). This space is totally disconnected compact Hausdorff, which is also self-dense (i.e., no isolated points) and homogeneous (i.e., any point may be carried to any other point via an autohomeomorphism). This space also satisfies the Baire category property (where "Baire" means "ω-Baire"), so we may place systems $\mathcal{S} \subseteq \wp(X)$ on the coarse thickness scale.

A subset $F \subseteq X$ is *small* if $|F| < |X|$. When we take κ to be $|X|$, $\mathcal{T}_\kappa$ is referred to as the *fine* topology, and is basically generated by $|X|$-intervals. The Hung-Negrepontis category theorem (Theorem 15.8 in [3]) tells us that when $\kappa = |X|$ is a regular cardinal (i.e., not the union of $< \kappa$ small subsets), any intersection of at most κ dense open subsets of $(\wp(X), \mathcal{T}_\kappa)$ is dense (even homeomorphic to $(\wp(X), \mathcal{T}_\kappa)$, provided κ is not weakly compact). Thus $(\wp(X), \mathcal{T}_\kappa)$ is a κ-Baire space, and we may also place systems $\mathcal{S} \subseteq \wp(X)$ on the fine thickness scale.

Of course, as long as a space is not a thin subset of itself, and unions of finitely many thin sets are still thin, no subset can be both thin and thick on the same scale. The question naturally arises as to whether a subset can be both thin on one given scale and thick on another. A reassuring answer is given in the following.

1.1 Proposition. Let X be an infinite set, $\mathcal{S} \subseteq \wp(X)$.

(*i*) If X is uncountable, then $\mathcal{S}$ cannot be both coarse-meager and fine-dense.

(*ii*) If $|X|$ is regular, then $\mathcal{S}$ cannot be both coarse-meager and fine-$|X|$-residual (or both coarse-residual and fine-$|X|$-meager).

Proof. Ad (*i*). Suppose $\mathcal{S}$ is fine-dense, X an uncountable set. Then $\mathcal{S}$ is $\mathcal{T}_{\omega^+}$-dense; hence $\mathcal{S}$ intersects every coarse-G_δ set. Consequently any coarse-F_σ set containing $\mathcal{S}$ must be all of $\wp(X)$. Since the coarse topology satisfies the Baire category property, $\mathcal{S}$ cannot be coarse-meager.

Ad (*ii*). If X is countable, the conclusion is immediate; so assume X is uncountable and regular. If $\mathcal{S}$ is fine-$|X|$-residual, then $\mathcal{S}$ is fine-dense by the Hung-Negrepontis theorem cited above. Thus $\mathcal{S}$ is not coarse-meager by (*i*). If $\mathcal{S}$ were both coarse-residual and fine-$|X|$-meager, then $\wp(X) \setminus \mathcal{S}$ would be both coarse-meager and fine-$|X|$-residual, an impossibility. □

Under certain circumstances, a system may itself satisfy a category property when viewed as a subspace of $\wp(X)$, suitably topologized. In particular, if $\mathcal{S}$ is coarse-closed, then $\mathcal{S}$ is compact Hausdorff; hence a Baire space. We may then place subsystems

$\mathcal{T} \subseteq \mathcal{S}$ on the relativized thickness scale in the obvious way. When we try to repeat this approach using the fine topology (assuming $|X|$ is regular), we encounter difficulties. A closed subspace in the fine topology is not *a priori* a $|X|$-Baire space. The following result is of some help with this problem, and is an easy consequence of Theorem 15.9 in [3].

1.2 Lemma. Let X be a set of infinite regular cardinality κ, and $\mathcal{S} \subseteq \wp(X)$ a coarse-closed fine-self-dense system of sets. Then $\mathcal{S}$, as a subspace of $(\wp(X), \mathcal{T}_\kappa)$, is homeomorphic to $(\wp(X), \mathcal{T}_\kappa)$; and is hence a κ-Baire space. □

The topologies $\mathcal{T}_\kappa$ on $\wp(X)$ are always homogeneous (and self-dense when $\kappa \leq |X|$), hence the space $(\wp(X), \mathcal{T}_\kappa)$ admits many autohomeomorphisms. The most useful seem to arise from lattice considerations. For each $R \subseteq X$, let $\Delta_R : \wp(X) \to \wp(X)$ be "symmetric difference with R;" i.e., $\Delta_R(Y) := (R \setminus Y) \cup (Y \setminus R)$.

1.3 Lemma. Let X be a set, $R \subseteq X$, and κ an infinite cardinal. Then Δ_R is an involutive autohomeomorphism on $(\wp(X), \mathcal{T}_\kappa)$.

Proof. Clearly Δ_R is an involution. Let $[F, G]$ be a κ-interval of subsets of X, and set $F' := (F \setminus R) \cup (R \cap G)$ and $G' := (G \setminus R) \cup (R \cap F)$. Then F' and G' have cardinality $< \kappa$, and we claim $\Delta_R([F, G]) = [F', G']$. For suppose $F \subseteq Y \subseteq X \setminus G$. Then $R \setminus (X \setminus G) \subseteq R \setminus Y \subseteq R \setminus F$ and $F \setminus R \subseteq Y \setminus R \subseteq (X \setminus G) \setminus R$. Thus $(R \setminus (X \setminus G)) \cup (F \setminus R) \subseteq \Delta_R(Y) \subseteq (R \setminus F) \cup ((X \setminus G) \setminus R)$. The left side is clearly F', and the right (after a brief calculation, noting that F and G are disjoint) is $X \setminus G'$. Conversely, suppose $F' \subseteq Y \subseteq X \setminus G'$. Then, as above, $(F' \setminus R) \cup (R \cap G') \subseteq \Delta_R(Y) \subseteq X \setminus ((G' \setminus R) \cup (R \cap F'))$. After another simple calculation, the left side is F, and the right is $X \setminus G$. □

Thus, no matter what thickness scale we use, the positions of $\mathcal{S} \subseteq \wp(X)$ and $\Delta_R(\mathcal{S})$ are the same for each $R \subseteq X$.

2. Systems of Sets.

Let $(X, \mathcal{S})$ be a system of sets. The elements of $\mathcal{S}$ are called $\mathcal{S}$-*distinguished* (so the elements of the dual system $\Delta_X(\mathcal{S})$ are $\mathcal{S}$-co-distinguished).

2.1 Theorem. Suppose X is an infinite set, and $(X, \mathcal{S})$ is a coarse-residual (resp. fine-$|X|$-residual, provided $|X|$ is regular,) system of sets. Then every finite (resp. small) subset of X is the intersection of two distinct members of $\mathcal{S}$, whose complements are also in $\mathcal{S}$. Thus every co-finite (resp. co-small) subset of X is the union of two distinct $\mathcal{S}$-distinguished sets.

Proof. Assume $\mathcal{S}$ is coarse-residual (resp. fine-$|X|$-residual, where $|X|$ is regular,), let F be any finite (resp. small) subset of X, and set $R := X \setminus F$. By Lemma 1.3, $\mathcal{S} \cap \Delta_R(\mathcal{S}) \cap \Delta_X(\mathcal{S}) \cap \Delta_R(\Delta_X(\mathcal{S}))$ is coarse-residual (resp. fine-$|X|$-residual), and is therefore coarse-dense (resp fine-dense). It consequently intersects $[F, \emptyset]$. Since Δ_R and Δ_X are involutive, there is some $Y \in \mathcal{S}$, containing F, such that $\Delta_R(Y), X \setminus Y$, and $X \setminus \Delta_R(Y)$ are in $\mathcal{S}$ also. But then $F = Y \cap ((X \setminus Y) \cup F)) = Y \cap (((X \setminus F) \setminus Y) \cup (Y \setminus (X \setminus F))) = Y \cap ((R \setminus Y) \cup (Y \setminus R)) = Y \cap \Delta_R(Y)$. Since $R \neq \emptyset$, we have $Y \neq \Delta_R(Y)$. Also $X \setminus F = (X \setminus Y) \cup (X \setminus \Delta_R(Y))$. □

Theorem 2.1 is useful for showing that certain well-known systems of sets are not thick.

2.2 Corollary. Suppose X is infinite, and $\mathcal{S}$ is a proper filter of subsets of X. Then $\mathcal{S}$ cannot be coarse-residual. Nor can it be fine-$|X|$-residual, if $|X|$ is regular. If $\mathcal{S}$ is an ultrafilter, then, in addition, it cannot be coarse-meager (or fine-$|X|$-meager, in the case $|X|$ is regular).

Proof. Otherwise, by Theorem 2.1, $\mathcal{S}$ would have to contain the empty set. This takes care of the first assertion. If $\mathcal{S}$ were a proper ultrafilter that was coarse-meager (or fine-$|X|$-meager, in the case $|X|$ is regular),then its dual $\Delta_X(\mathcal{S})$ would have the same status. But the dual is just $\wp(X) \setminus \mathcal{S}$ in this case, and we obtain a contradiction. □

So proper ultrafilters occupy the middle of whichever thickness scale one chooses. There are other systems that can easily be placed, e.g., the infinite co-infinite subsets; and, with a little more work, the "moieties" (i.e., those subsets that are neither small nor co-small.)

2.3 Theorem. Let X be a set of infinite cardinality κ.

(*i*) $\mathcal{S}_{ic} := \{Y \subseteq X : Y \text{ is infinite and co-infinite}\}$ is coarse-residual; and, if κ is regular, fine-residual as well.

(*ii*) $\mathcal{S}_m := \{Y \subseteq X : Y \text{ is a moiety}\}$ is fine-κ-residual, provided κ is regular, and either a limit cardinal (i.e., weakly inaccessible) or equal to $2^{<\kappa}$.

(*iii*) If κ is uncountable, then $\mathcal{S}_m$ is neither coarse-meager nor coarse-residual.

Proof. Ad (i). For each $\alpha < \omega$, let $\mathcal{F}_\alpha := \{Y \subseteq X : |Y| \geq |\alpha|\}$. One easily shows that each $\mathcal{F}_\alpha$ is coarse-open and fine-dense, and that $\mathcal{S}_{ic} = \mathcal{F} \cap \Delta_X(\mathcal{F})$, where $\mathcal{F} := \bigcap_{\alpha<\omega} \mathcal{F}_\alpha$. (The coarse topology always satisfies the Baire category property; we assume the regularity of κ to ensure the same for the fine topology.)

Ad (ii). Assuming κ is a regular limit cardinal, repeat the argument in (i), replacing ω with κ. (Each $\mathcal{F}_\alpha$ is fine-open and fine-dense.) That $\mathcal{S}_m$ contains $\mathcal{F} \cap \Delta_X(\mathcal{F})$ follows from the limit assumption. In the case $\kappa = 2^{<\kappa}$, simply note that then we have $|\wp(X) \setminus \mathcal{S}_m| = \kappa$.

Ad (iii). It is easy to see that $\mathcal{S}_m$ is fine-dense. If κ is uncountable, then, by Proposition 1.1(i), $\mathcal{S}_m$ cannot be coarse-meager.

To show $\mathcal{S}_m$ cannot be coarse-residual when κ is uncountable, let $\mathcal{F} := \{Y \subseteq X : |Y| < |X|\}$. We note that $\mathcal{F}$ is clearly fine-dense (coarse-dense is all we need), and claim that it is a Baire subspace of $(\wp(X), \mathcal{T}_\omega)$. Assuming for the moment that the claim is true, it is easy to see that $\mathcal{F}$ cannot be coarse-meager. Indeed, if $\mathcal{U}_n$ is dense open in the coarse topology, $n < \omega$, and if $\mathcal{U}$ is any nonempty coarse-open set, then $(\mathcal{U} \cap \mathcal{F}) \cap (\mathcal{U}_n \cap \mathcal{F}) = \mathcal{U} \cap \mathcal{U}_n \cap \mathcal{F} \neq \emptyset$, so $\mathcal{U}_n \cap \mathcal{F}$ is dense open in the Baire space $\mathcal{F}$. Consequently $\mathcal{F} \cap \bigcap_{n<\omega} \mathcal{U}_n \neq \emptyset$. Now since $\mathcal{S}_m$ is disjoint from $\mathcal{F}$, $\mathcal{S}_m$ cannot be coarse-residual.

To prove the claim, it suffices, by the Baire category theorem, to show that $\mathcal{F}$ is a countably compact subspace of $(\wp(X), \mathcal{T}_\omega)$. Now ω-intervals are easily seen to be coarse-closed; hence complements of ω-intervals are finite unions of ω-intervals, since the coarse topology is compact. Thus the collection of all finite unions of ω-intervals furnishes a basis for the coarse-closed sets. Let $\{\mathcal{C}_n : n < \omega\}$ be basic coarse-closed sets such that $\{\mathcal{C}_n \cap \mathcal{F} : n < \omega\}$ satisfies the finite intersection property. Then $\{\mathcal{C}_n : n < \omega\}$ satisfies this property too, hence $\bigcap_{n<\omega} \mathcal{C}_n \neq \emptyset$. This set is a countable intersection of finite unions of ω-intervals; by infinite distributivity, it may be written as a union of countable intersections of ω-intervals. At least one of these intersections, say $\bigcap_{n<\omega}[F_n, G_n]$, must therefore be nonempty. This says that $(\bigcup_{n<\omega} F_n) \cap (\bigcup_{n<\omega} G_n) = \emptyset$. Thus $\bigcup_{n<\omega} F_n$, a countable set and hence in $\mathcal{F}$, must be a member of $\bigcap_{n<\omega} \mathcal{C}_n$. This proves the claim and hence the result. □

As an application of Theorems 2.1 and 2.3, recall that a system $\mathcal{S}$ of subsets of a set X of infinite cardinality κ is an *almost disjoint family* (resp. a *strongly almost disjoint family*) if the intersection of any two distinct members of $\mathcal{S}$ is small (resp. finite). It is a simple, but elegant, result (see [7]) that if $\kappa = 2^{<\kappa}$, then $\wp(X)$ contains almost disjoint families of cardinality 2^κ. However, such families are generally not thick.

2.4 Corollary. Let X be a set of infinite cardinality κ, and $\mathcal{S}$ be a system of subsets of X.

(i) If $\mathcal{S}$ is a strongly almost disjoint family, then $\mathcal{S}$ is not coarse-residual (or fine-κ-residual, when κ is regular).

(ii) If $\mathcal{S}$ is an almost disjoint family, κ is regular, and either κ is a limit cardinal or $\kappa = 2^{<\kappa}$, then $\mathcal{S}$ is not fine-κ-residual.

Proof. Ad (i). Assume $\mathcal{S}$ is a strongly almost disjoint family that is coarse-residual. Using Theorems 2.1 and 2.3(i), there exist two distinct sets $Y, Z \in \mathcal{S} \cap \mathcal{S}_{ic}$, whose complements are also in $\mathcal{S}$. But then $Y \cap Z$ and $Y \cap (X \setminus Z)$ are both finite; hence their union, which is Y, is also finite, a contradiction.

The proof that $\mathcal{S}$ cannot be fine-κ-residual is a repetition of the argument above, making use of the parts of Theorems 2.1 and 2.3(i) that deal with the fine topology.

Ad (ii). Repeat the argument in (i), making use of Theorems 2.1 and 2.3(ii). The sets Y and Z must then be moieties. □

The question naturally arises as to the thickness of chains and antichains in $\wp(X)$. Of course chains can be neither coarse-residual nor fine-$|X|$-residual by Theorem 2.1. (Another way to see this is to note that since the cardinality of any chain in $\wp(X)$ is bounded by $|X|$, we know that chains are automatically fine-$|X|$-meager. We may then cite Proposition 1.1(ii).) The issue for antichains is only slightly less clear.

2.5 Proposition. Let X be an infinite set. Then no antichain in $\wp(X)$ can be coarse-residual (or fine-$|X|$-residual, when $|X|$ is regular).

Proof. Let $\mathcal{S} \subseteq \wp(X)$ be coarse-residual (resp. fine-$|X|$-residual, when $|X|$ is regular), and let $F \subseteq X$ be any nonempty finite set. By Lemma 1.3, we have that $\mathcal{S} \cap \Delta_F(\mathcal{S})$ is coarse-residual (resp. fine-$|X|$-residual), so there is some $Y \in \mathcal{S} \cap \Delta_F(\mathcal{S}) \cap [\emptyset, F]$. Thus both Y and $Y \cup F$ (a proper superset) are in $\mathcal{S}$, and $\mathcal{S}$ is therefore not an antichain. □

3. Closure Systems of Sets.

A system $(X, \mathcal{S})$ is a *closure system* if $\mathcal{S}$ is closed under arbitrary intersections. For the remainder of the paper, we concentrate on closure systems, as well as associated systems that naturally arise. When $\mathcal{S}$ is a closure system, there is the associated closure operator $\langle\ \rangle := \langle\ \rangle_{\mathcal{S}}$ mapping $\wp(X)$ to itself: for any $Y \subseteq X$, $\langle Y \rangle$ is the intersection of all members of $\mathcal{S}$ containing Y.

It is easy to verify that $\langle\ \rangle$ is expansive, monotone and idempotent (see, e.g., [6]). A closure system is *topological* if it is closed under finite unions. For any infinite cardinal κ, define the closure system $(X, \mathcal{S})$ to be *κ-algebraic* (simply *algebraic* if $\kappa = \omega$) if whenever $Y \subseteq X$ and $x \in \langle Y \rangle$, then $x \in \langle Z \rangle$ for some $Z \subseteq Y$ of cardinality $< \kappa$. ($\mathcal{S}$ is algebraic if and only if it is closed under unions of nonempty up-directed subsystems [6], so Zorn's lemma arguments come into play in this context.) $(X, \mathcal{S})$ is *monotonically κ-additive* if whenever $\alpha < \kappa$ and $\{F_\beta : \beta < \alpha\}$ is an increasing sequence of subsets of X of cardinality $< \kappa$, then $\langle \bigcup_{\beta<\alpha} F_\beta \rangle = \bigcup_{\beta<\alpha} \langle F_\beta \rangle$. (Clearly when $\kappa = \omega$, this is no restriction.) For any closure system $(X, \mathcal{S})$, a subset Y is *$\mathcal{S}$-dense* (resp. *$\mathcal{S}$-discrete*) if $\langle Y \rangle = X$ (resp. $x \in X \setminus \langle Y \setminus \{x\} \rangle$ for each $x \in Y$). The words "dense" and "discrete" come from topology; nevertheless we have decided on this terminology to apply in a more general context. Of course, when $\mathcal{S}$ is algebraic, and can thus be viewed as the subalgebra lattice of some finitary universal algebra structure on X [6], dense sets are "generating" sets (or "spanning" sets, in the vector space context); discrete sets are "independent" sets in a weak sense (this sense being strong enough in the context of vector spaces). We denote the associated systems of $\mathcal{S}$-dense sets and $\mathcal{S}$-discrete sets by $Den(\mathcal{S})$ and $Dis(\mathcal{S})$ respectively. (Clearly $Den(\mathcal{S})$ (resp. $Dis(\mathcal{S})$) is closed under supersets (resp. subsets).) Finally, a closure system $(X, \mathcal{S})$ satisfies the *exchange property* (EP) if whenever $Y \subseteq X$ and $x, y \in X$ are such that $y \in \langle Y \cup \{x\} \rangle \setminus \langle Y \rangle$, then $x \in \langle Y \cup \{y\} \rangle$.

3.1 Theorem. Let $(X, \mathcal{S})$ be a closure system, κ an infinite cardinal.

(i) If $\mathcal{S}$ is κ-algebraic, then $\mathcal{S}$ is $\mathcal{T}_\kappa$-closed. The converse is true when κ is regular.

(ii) If $(X, \mathcal{S})$ satisfies the EP and $Den(\mathcal{S})$ contains no subsets of cardinality $< \kappa$, then $\mathcal{S}$ is $\mathcal{T}_\kappa$-self-dense. If $Den(\mathcal{S})$ does contain a subset of cardinality $< \kappa$, then X is a $\mathcal{T}_\kappa$-isolated point of $\mathcal{S}$.

(iii) If κ is a regular cardinal, and $(X, \mathcal{S})$ is κ-algebraic, monotonically κ-additive and $\mathcal{T}_\kappa$-self-dense, then $|\mathcal{S}| \geq 2^\kappa$.

(iv) $\mathcal{S}$ contains all subsets of cardinality $< \kappa$ if and only if $\mathcal{S}$ is $\mathcal{T}_\kappa$-dense.

Proof. Ad (i). With very little difficulty, one can translate the assertion that $\mathcal{S}$ is κ-algebraic (resp. $\mathcal{T}_\kappa$-closed) to the following: for every $Y \notin \mathcal{S}$, for every $x \in \langle Y \rangle \setminus Y$ (resp. there is some $x \in \langle Y \rangle \setminus Y$), there is some $F \subseteq Y$ of cardinality $< \kappa$, such that $x \in \langle F \rangle$.

To see the converse, assume κ is a regular cardinal, and let $Y \notin \mathcal{S}$ be fixed. Set $Z := \bigcup\{\langle F \rangle : F \subseteq Y$ has cardinality $< \kappa\}$. Then $Z \subseteq \langle Y \rangle$ and $Y \subseteq Z$; hence $\langle Y \rangle = \langle Z \rangle$. We show $Z \in \mathcal{S}$. Indeed, suppose otherwise. Then there is some $G \subseteq Z$

of cardinality $< \kappa$ such that $\langle G \rangle \not\subseteq Z$, since $\mathcal{S}$ is $\mathcal{T}_\kappa$-closed. But κ is a regular cardinal, so $G \subseteq \langle F \rangle$ for some $F \subseteq Y$ of cardinality $< \kappa$. Thus $\langle G \rangle \subseteq Z$ after all. Therefore, since $\langle Y \rangle = Z$, we know that every $x \in \langle Y \rangle$ is in $\langle F \rangle$ for some $F \subseteq Y$ of cardinality $< \kappa$.

Ad (ii). Suppose no $\mathcal{S}$-dense subset of X has cardinality $< \kappa$, and that the EP holds. Let $[F, G]$ be a κ-interval that intersects $\mathcal{S}$. Then $\langle F \rangle \cap G = \emptyset$. We need to find another $\mathcal{S}$-distinguished set in $[F, G]$, so it suffices to find some $x \in X \setminus \langle F \rangle$ such that $\langle F \cup \{x\} \rangle \cap G = \emptyset$. Assuming this cannot be done, we have for each $x \in X \setminus \langle F \rangle$, some $g(x) \in G \cap \langle F \cup \{x\} \rangle$. Since $g(x) \notin \langle F \rangle$, the EP tells us that $x \in \langle F \cup \{g(x)\} \rangle$; whence $X \setminus \langle F \rangle \subseteq \langle F \cup G \rangle$. This implies that $F \cup G$ is $\mathcal{S}$-dense, contradicting our hypothesis.

Now assume there is some $F \in Den(\mathcal{S})$ of cardinality $< \kappa$. Then $[F, \emptyset]$ is a κ-interval whose intersection with $\mathcal{S}$ is $\{X\}$.

Ad (iii). Suppose $\lambda \leq \kappa$, and $\{F_\alpha : \alpha < \lambda\}$ and $\{G_\alpha : \alpha < \lambda\}$ are two increasing λ-sequences of subsets of X, such that for each $\alpha < \lambda$, F_α and G_α are disjoint and of cardinality $< \kappa$. Then $\{[F_\alpha, G_\alpha] : \alpha < \lambda\}$ is a decreasing λ-sequence of κ-intervals. Assume each $[F_\alpha,\ G_\alpha]$ intersects $\mathcal{S}$, a κ-algebraic, monotonically κ-additive closure system. Let $F := \bigcup_{\alpha<\lambda} F_\alpha$, and $G := \bigcup_{\alpha<\lambda} G_\alpha$. Then we claim that $[F, G]$ intersects $\mathcal{S}$. If not, then $\langle F \rangle \cap G \neq \emptyset$. Assume first that $\lambda = \kappa$. Since $\mathcal{S}$ is κ-algebraic, there is some $H \subseteq F$ of cardinality $< \kappa$ such that $\langle H \rangle \cap G \neq \emptyset$. Because κ is regular, there is some $\alpha < \kappa$ with $H \subseteq F_\alpha$. Thus $\langle F_\alpha \rangle$ intersects G, so there is some $\beta < \kappa$ such that $\langle F_\alpha \rangle$ intersects G_β. Let $\gamma := max\{\alpha, \beta\}$. Then $\langle F_\gamma \rangle \cap G_\gamma \neq \emptyset$, a contradiction. If $\lambda < \kappa$, we invoke monotonic κ-additivity to assert that $\langle F \rangle = \bigcup_{\alpha<\lambda} \langle F_\alpha \rangle$, and argue much as above.

Now assume $\mathcal{S}$ is also $\mathcal{T}_\kappa$-self-dense. We build a κ-level tree $\mathbf{T}$ by induction satisfying:

- (a) $\mathbf{T}$ is a binary tree whose nodes are (basic) $\mathcal{T}_\kappa$-neighborhoods, ordered by reverse inclusion;
- (b) each node of $\mathbf{T}$ intersects $\mathcal{S}$; and
- (c) the nodes at each level of $\mathbf{T}$ are pairwise disjoint.

It is clear, by the last paragraph, that once we establish the existence of such a tree, we are done. The construction of $\mathbf{T}$ is routine: The root node consists of any $\mathcal{T}_\kappa$-neighborhood that intersects $\mathcal{S}$; level $\alpha + 1$ is obtained by taking each node at level α and assigning two disjoint $\mathcal{T}_\kappa$-subneighborhoods, each intersecting $\mathcal{S}$ (possible because $\mathcal{S}$ is $\mathcal{T}_\kappa$-self-dense); limit levels are obtained by taking intersections.

Ad (iv). Suppose $\mathcal{S}$ contains all subsets of cardinality $< \kappa$, and let $[F, G]$ be a nonempty κ-interval. Then F and G have cardinality $< \kappa$, so $F \in [F, G] \cap \mathcal{S}$. Conversely, suppose $\mathcal{S}$ is $\mathcal{T}_\kappa$-dense, with F an arbitrary subset of cardinality $< \kappa$. If $F \notin \mathcal{S}$, then there is some $x \in \langle F \rangle \setminus F$. But $[F, \{x\}]$ intersects $\mathcal{S}$, so must contain $\langle F \rangle$, a contradiction. Thus $F \in \mathcal{S}$. $\square$

For topological closure systems, being ω^+-algebraic means having countable tightness. This property bears the same formal relationship to being $\mathcal{T}_{\omega^+}$-closed as the Fréchet-Urysohn property bears to being sequential. Since sequential spaces are well known not to be Fréchet-Urysohn in general, it came as a surprise to us that the simple (but not entirely straightforward) converse in Theorem 3.1(i) is actually true. We would like to thank Alan Dow for indicating to us the folklore result that sequential spaces have countable tightness. Our proof is an inessential generalization of that result (which appears in [8]).

The EP is easily seen to be too strong for the conclusion in the first assertion in Theorem 3.1(ii): Take X to be the (ordered) set of rational numbers, and $\mathcal{S}$ to be the closed left-unbounded intervals (in the classic sense). Then $\mathcal{S}$ is $\mathcal{T}_\omega$-self-dense, but the EP clearly fails.

An immediate consequence of Theorem 3.1(i) is the following.

3.2 Corollary. Let $\mathcal{S}$ be a proper closure system on X. If $\mathcal{S}$ is algebraic, then it is not coarse-residual; if $\mathcal{S}$ is $|X|$-algebraic, where $|X|$ is regular, then it is not fine-$|X|$-residual. $\square$

3.3 Theorem. Let $\mathcal{S}$ be a proper topological closure system on X (i.e., not discrete). Then $\mathcal{S}$ is not coarse-residual (or fine-$|X|$-residual, when $|X|$ is regular).

Proof. If $\mathcal{S}$ is thick in either sense, then each co-singleton subset of X is a union of two members of $\mathcal{S}$, by Theorem 2.1. But then every co-singleton subset of X is already in $\mathcal{S}$; hence every subset of X is in $\mathcal{S}$. $\square$

An ultrafilter on a set may be viewed as the dual of a topological closure system, once we throw in the empty set. By Corollary 2.2, then, topological closure systems can fail to be thin. The following two results give sufficient conditions for a closure system to be thin.

3.4 Proposition. If $\mathcal{S}$ is a closure system on X, then $\mathcal{S} \cap Den(\mathcal{S}) = \{X\}$. Consequently, if one of $\mathcal{S}$, $Den(\mathcal{S})$ is coarse-residual (resp. fine-$|X|$-residual, where $|X|$ is regular), then the other is coarse-meager (resp. fine-$|X|$-meager). □

Let $(X, \mathcal{S})$ be a system of sets. A subsystem $\mathcal{P} \subseteq \mathcal{S} \setminus \{X\}$ is a *weak base* for $\mathcal{S}$ if every $\mathcal{S}$-distinguished proper subset of X is contained in a member of $\mathcal{P}$. (A weak base for a proper ideal of sets is the dual of a filterbase for the dual filter of the ideal; the dual of a weak base for a topological closure system is a π-base for the dual topology.) If κ is an infinite cardinal, we say $\mathcal{S}$ is *κ-based* if $\mathcal{S}$ has a weak base $\mathcal{P}$ of cardinality at most κ, such that every $\mathcal{P}$-co-distinguished set has cardinality at least κ.

3.5 Theorem. Let $(X, \mathcal{S})$ be a closure system, $\kappa \in \{\omega, |X|\}$, $|X|$ a regular cardinal (where necessary, for Baire category considerations).

(*i*) If $\mathcal{S}$ is κ-based, then $Den(\mathcal{S})$ is $\mathcal{T}_\kappa$-κ-residual; consequently $\mathcal{S}$ is $\mathcal{T}_\kappa$-κ-meager.

(*ii*) If $\mathcal{S}$ is $|X|$-algebraic, then $Den(\mathcal{S})$ is fine-$|X|$-residual if and only if $Den(\mathcal{S})$ is fine-dense, if and only if no proper $\mathcal{S}$-distinguished subset is co-small.

Proof. Ad (*i*). Assume $\kappa \in \{\omega, |X|\}$, and let $\mathcal{P}$ witness the fact that $\mathcal{S}$ is κ-based. We show $Den(\mathcal{S})$ is $\mathcal{T}_\kappa$-κ-residual; the rest follows by Proposition 3.4. Let $\mathcal{R} := \{Y \subseteq X : \text{for all } S \in \mathcal{P}, \text{ for all } \alpha < \kappa, |Y \setminus S| \geq |\alpha|\}$. Then clearly $\mathcal{R} \subseteq Den(\mathcal{S})$, so we need only show $\mathcal{R}$ is $\mathcal{T}_\kappa$-κ-residual. Fix $S \in \mathcal{P}$ and $\alpha < \kappa$, and define $\mathcal{R}_{S,\alpha} := \{Y \subseteq X : |Y \setminus S| \geq |\alpha|\}$. Because $|\mathcal{P}| \leq \kappa$ and $\mathcal{R}$ is the intersection of all the families $\mathcal{R}_{S,\alpha}$, we need to show now that each such family is dense open in $\mathcal{T}_\kappa$. So fix disjoint sets $F, G \subseteq X$ of cardinality $< \kappa$, and set $Y := X \setminus G$. Of course $Y \in [F, G]$. Since $|G| < \kappa$ and $|X \setminus S| \geq \kappa$, we have that $|Y \setminus S| \geq \kappa$ as well. Thus $Y \in \mathcal{R}_{S,\alpha}$, and we have density. Finally, suppose $Y \in \mathcal{R}_{S,\alpha}$; pick $F \subseteq Y \setminus S$ of cardinality $|\alpha|$. Then $[F, \emptyset]$ is a $\mathcal{T}_\kappa$-neighborhood of Y that is contained in $\mathcal{R}_{S,\alpha}$.

Ad (*ii*). Assume $Den(\mathcal{S})$ is fine-$|X|$-residual. Then $Den(\mathcal{S})$ is fine-dense; so for each small subset G, $[\emptyset, G]$ contains an $\mathcal{S}$-dense set. Thus $X \setminus G$ is $\mathcal{S}$-dense. This says no proper $\mathcal{S}$-distinguished subset is co-small.

For the third leg of the proof, we need $\mathcal{S}$ to be $|X|$-algebraic. Assume no proper $\mathcal{S}$-distinguished subset is co-small, so every co-small subset of X is in $Den(\mathcal{S})$. For each $x \in X$, set $\mathcal{R}_x := \{Y \subseteq X : x \in \langle Y \rangle\}$. Then $Den(\mathcal{S}) = \bigcap_{x \in X} \mathcal{R}_x$, so we need to show each $\mathcal{R}_x$ is dense open in the fine topology. For density, we have $X \setminus G \in [F, G] \cap \mathcal{R}_x$ whenever F and G are disjoint small subsets, because co-small subsets are $\mathcal{S}$-dense. For openness, if $Y \in \mathcal{R}_x$, use $|X|$-algebraicity to find a small $F \subseteq Y$ with $x \in \langle F \rangle$. Then $[F, \emptyset]$ is a fine-open neighborhood of Y contained in $\mathcal{R}_x$. □

Any positive result, such as Theorem 3.5, asserting that a certain system $\mathcal{R} \subseteq \wp(X)$ is thick, implies some cardinality bounds on $\mathcal{R}$ (with less fuss than in Theorem 3.1(iii)). For instance, if $\mathcal{R}$ is coarse-residual, then $|\mathcal{R}| \geq 2^{\omega}$, by standard Baire category folklore. (One builds an ω-height binary tree.) If $|X|$ is regular and not weakly compact, and $\mathcal{R}$ is fine-$|X|$-residual, then $|\mathcal{R}| = 2^{|X|}$ (indeed $\mathcal{R}$ and $(\wp(X), \mathcal{T}_{|X|})$ are homeomorphic), by the Hung-Negrepontis category theorem [3].

The following is a known fact about ultrafilters (see, e.g., Exercises 4G.1,2 in [4]). It is nonetheless somewhat amusing that it follows easily from Corollary 2.2 and Theorem 3.5(i) (keeping in mind that the Baire category argument in the uncountable case applies only for regular cardinality). Recall that an ultrafilter $\mathcal{U}$ on an infinite set X is *uniform* if every element of $\mathcal{U}$ has cardinality $|X|$.

3.6 Theorem [4]. Let $\mathcal{U}$ be a nonprincipal ultrafilter on an infinite set X, with filterbase $\mathcal{F}$. Then $\mathcal{F}$ is uncountable. If, in addition, $\mathcal{U}$ is uniform, Then $|\mathcal{F}| > |X|$. □

Next we examine $Dis(\mathcal{S})$, the system of $\mathcal{S}$-discrete subsets of X. As mentioned earlier, $Dis(\mathcal{S})$ is closed under subsets. Moreover, if $Y \in Dis(\mathcal{S})$ and $U, V \subseteq Y$ are distinct, say $u \in U \setminus V$, then $u \notin \langle Y \setminus \{u\} \rangle$; hence $u \notin \langle V \setminus \{u\} \rangle = \langle V \rangle$. Thus $u \in \langle U \rangle \setminus \langle V \rangle$; consequently the closure operator is injective on $\wp(Y)$ for any $Y \in Dis(\mathcal{S})$. This implies $|\mathcal{S}| \geq 2^{|Y|}$ for any $Y \in Dis(\mathcal{S})$.

Define $Bas(\mathcal{S}) := Den(\mathcal{S}) \cap Dis(\mathcal{S})$. (A member of $Bas(\mathcal{S})$ is called an $\mathcal{S}$-*basis*.) When $\mathcal{S}$ is the system of vector subspaces of a vector space structure on X, $Bas(\mathcal{S})$ is the collection of vector space bases. Since bases occur only under very special circumstances, one should expect $Bas(\mathcal{S})$ to be thin; in particular one should not expect both $Den(\mathcal{S})$ and $Dis(\mathcal{S})$ to be thick. This expectation is justified, as we see presently. Let $MaxDis(\mathcal{S})$ and $MinDen(\mathcal{S})$ be respectively the maximal $\mathcal{S}$-discrete and the minimal $\mathcal{S}$-dense subsets of X. We collect some elementary facts in the following.

3.7 Proposition. Let $(X, \mathcal{S})$ be a closure system.

(i) $Bas(\mathcal{S}) = MinDen(\mathcal{S}) \subseteq MaxDis(\mathcal{S})$.

(ii) If $(X, \mathcal{S})$ is algebraic, then $MaxDis(\mathcal{S})$ is nonempty; however $Bas(\mathcal{S})$ can be empty.

(iii) If $(X, \mathcal{S})$ satisfies the EP, then $Bas(\mathcal{S}) = MaxDis(\mathcal{S})$.

(*iv*) $MaxDis(\mathcal{S})$ is not coarse-residual; if $|X|$ is regular, it is not fine-$|X|$-residual either.

Proof. Ad (*i*). If $Y \in Bas(\mathcal{S})$, then $Y \in Den(\mathcal{S})$. Suppose $U \subseteq Y$, $y \in Y \setminus U$. Since $Y \in Dis(\mathcal{S})$, $y \notin \langle Y \setminus \{y\}\rangle$; hence $y \notin \langle U \setminus \{y\}\rangle = \langle U\rangle$. Thus $Y \in MinDen(\mathcal{S})$. For the reverse inclusion, suppose $Y \in MinDen(\mathcal{S})$, $y \in Y$. Then $\langle Y \setminus \{y\}\rangle \neq X = \langle Y\rangle$. If $y \in \langle Y \setminus \{y\}\rangle$, then $Y \subseteq \langle Y \setminus \{y\}\rangle$, so $X = \langle Y\rangle \subseteq \langle\langle Y \setminus \{y\}\rangle\rangle = \langle Y \setminus \{y\}\rangle$, an impossibility. Thus $y \notin \langle Y \setminus \{y\}\rangle$; hence $Y \in Dis(\mathcal{S})$. This proves $Bas(\mathcal{S}) = MinDen(\mathcal{S})$. To see that $Bas(\mathcal{S}) \subseteq MaxDis(\mathcal{S})$, suppose $Y \in Den(\mathcal{S})$, $Y \subseteq Z \subseteq X$. If $z \in Z \setminus Y$, then $X = \langle Y\rangle \subseteq \langle Y \setminus \{z\}\rangle \subseteq \langle Z \setminus \{z\}\rangle$. Thus $Z \notin Dis(\mathcal{S})$.

Ad (*ii*). Every singleton set is $\mathcal{S}$-discrete; so $Dis(\mathcal{S})$ is never empty. Algebraicity, then, allows a straightforward application of Zorn's lemma to obtain maximal elements in $Dis(\mathcal{S})$ partially ordered by set inclusion.

To prove the second assertion, let X be the ordered set of rational numbers, and $\mathcal{S}$ be the closed left-unbounded intervals (as in the paragraph before Corollary 3.2). Then $(X, \mathcal{S})$ is algebraic, the only $\mathcal{S}$-discrete subsets being those of cardinality ≤ 1. However, $Den(\mathcal{S})$ consists of the right-unbounded sets. (We are grateful to the referee for pointing out these features of this example.)

Ad (*iii*). Suppose $(X, \mathcal{S})$ satisfies the EP, with $Y \in MaxDis(\mathcal{S})$. We show $Y \in Den(\mathcal{S})$. Assume otherwise, and fix $x \in X \setminus \langle Y\rangle$. Then $Y \cup \{x\}$ is not $\mathcal{S}$-discrete, and must contain an element y witnessing this. Since $x \notin \langle Y\rangle$ and $(Y \cup \{x\}) \setminus \{x\} \subseteq Y$, we know $y \neq x$. Thus $y \in Y$. Since Y is $\mathcal{S}$-discrete, we know $y \notin \langle Y \setminus \{y\}\rangle$. Now $(Y \cup \{x\}) \setminus \{y\} = (Y \setminus \{y\}) \cup \{x\}$, since $y \neq x$. Hence $y \in \langle (Y \setminus \{y\}) \cup \{x\}\rangle$, so by the EP, $x \in \langle (Y \setminus \{y\} \cup \{y\}\rangle = \langle Y\rangle$. From this contradiction, we infer that Y is $\mathcal{S}$-dense after all.

Ad (*iv*). $MaxDis(\mathcal{S})$ is clearly an antichain in $\wp(X)$. The result is immediate from Proposition 2.5. □

$MaxDis(\mathcal{S})$ is never thick, by Proposition 3.7(*iv*); it can easily be made to be thin, however, as the following shows.

3.8 Proposition. Let $(X, \mathcal{S})$ be a closure system, with $\omega \leq \kappa \leq |X|$. If $\mathcal{S}$ is κ-algebraic or if $\mathcal{S}$ has a weak base $\mathcal{P}$ such that every $\mathcal{P}$-co-distinguished set has cardinality $< \kappa$, then $Bas(\mathcal{S})$ is $\mathcal{T}_\kappa$-nowhere dense.

Proof. Set $\mathcal{M} := Bas(\mathcal{S})$, and let $[F, G]$ be an arbitrary nonempty κ-interval. It suffices to find a nonempty κ-interval $[F', G'] \subseteq [F, G]$ that misses $\mathcal{M}$. We may therefore assume $[F,G] \cap \mathcal{M} \neq \emptyset$.

Assume first that $\mathcal{S}$ is κ-algebraic.

Case 1a. $X \setminus G$ is $\mathcal{S}$-discrete. Then $[F,G] \cap \mathcal{M} = \{X \setminus G\}$. $|F \cup G| < \kappa \leq |X|$; so pick $x \in X \setminus (F \cup G)$. Then $\emptyset \neq [F,G'] \subseteq [F,G]$, and $[F,G'] \cap \mathcal{M} = \emptyset$, where $G' := G \cup \{x\}$.

Case 1b. $X \setminus G$ is not $\mathcal{S}$-discrete. Then there is some $x \in X \setminus G$ such that $x \in \langle X \setminus (G \cup \{x\})\rangle$. Since $\mathcal{S}$ is κ-algebraic, there is some $H \subseteq X \setminus (G \cup \{x\})$, $|H| < \kappa$, such that $x \in \langle H \rangle$. Then $F' := F \cup H \cup \{x\}$ is not $\mathcal{S}$-discrete, so no superset can be $\mathcal{S}$-discrete either. Thus $\emptyset \neq [F',G] \subseteq [F,G]$, and $[F',G] \cap \mathcal{M} = \emptyset$.

Next assume $\mathcal{S}$ has a weak base as in the hypothesis.

Case 2a. F is $\mathcal{S}$-dense. Then $[F,G] \cap \mathcal{M} = \{F\}$. Pick $x \in X \setminus (F \cup G)$. Then $[F \cup \{x\}, G]$ is our desired $[F',G']$.

Case 2b. F is not $\mathcal{S}$-dense. Then by our assumption, there is a nonempty $\mathcal{S}$-co-distinguished set H of cardinality $< \kappa$ such that $F \cap H = \emptyset$. Thus $[F, G \cup H]$ is our desired $[F',G']$. □

An immediate consequence of Theorem 3.5(i) and Proposition 3.8 is the following.

3.9 Theorem. Let $(X,\mathcal{S})$ be a κ-based κ-algebraic closure system, where $\kappa \in \{\omega, |X|\}$, $|X|$ a regular cardinal (where necessary for Baire category considerations). Then $Dis(\mathcal{S})$ is $\mathcal{T}_\kappa$-κ-meager. □

3.10 Examples.

(i) Let X be a countably infinite set, with $\mathcal{I} := \{Y \subseteq X : Y \text{ is finite}\}$. Set $\mathcal{S} := \mathcal{I} \cup \{X\}$. Then $Den(\mathcal{S}) = \{Y \subseteq X : Y \text{ is infinite}\}$, a residual system, and $Dis(\mathcal{S}) = \mathcal{I}$, a meager system.

(ii) Let X again be countably infinite, with $\mathcal{I}$ now a free maximal ideal of sets; $\mathcal{S} := \mathcal{I} \cup \{X\}$, and $\mathcal{U} := \Delta_X(\mathcal{I})$, the corresponding ultrafilter. Then $Den(\mathcal{S}) = \mathcal{U}$ and $Dis(\mathcal{S}) = \mathcal{I}$. By Corollary 2.2, both of these systems are nonmeager and nonresidual. Note that in this example, as well as the one above, $Bas(\mathcal{S}) = \emptyset$ and both hypotheses of Proposition 3.8 fail.

(iii) Suppose $(X,\mathcal{S})$ is a topological closure system satisfying the T_1 axiom; i.e., the singleton subsets of X are $\mathcal{S}$- distinguished. Then $Bas(\mathcal{S}) \neq \emptyset$ if and only if there is an $\mathcal{S}$-dense set of $\mathcal{S}$-isolated points; in which case that set is the unique element of $Bas(\mathcal{S})$. □

4. EP-Systems of Sets.

By an *EP-system*, we mean a closure system satisfying the exchange property. As an immediate consequence of Lemma 1.2 and

Theorem 3.1(i, ii), we have the following.

4.1 Theorem. Let $(X, \mathcal{S})$ be an algebraic EP-system of infinite regular cardinality, such that no small subset of X is $\mathcal{S}$-dense. Then $\mathcal{S}$, with the topology inherited from $\mathcal{T}_{|X|}$, is homeomorphic to $(\wp(X), \mathcal{T}_{|X|})$, and is hence a $|X|$-Baire space. □

Algebraic EP-systems are also called "independence spaces," or "infinite matroids" in the literature (see, e.g., [5, 9]). Since the notion of "independence" makes sense in contexts far removed from what these systems encompass, we avoid using the first alias. Since finiteness is almost universally *built in* to the definition of "matroid," we avoid using the second as well. A celebrated feature of algebraic EP-systems $(X, \mathcal{S})$ is that bases (i.e., subsets that are dense and discrete/minimal dense/maximal discrete) exist (as shown in Proposition 3.7(i, ii)), and that any two bases have the same cardinality [9], called the *dimension* ($dim(\mathcal{S})$) of the system. Furthermore, every $\mathcal{S}$-dense (resp. $\mathcal{S}$-discrete) subset of X contains (resp. is contained in) an $\mathcal{S}$-basis.

Theorem 4.1 allows us to relativize our fine topology scale to subsystems of algebraic EP-systems of large (i.e., not small) dimension. (Of course, there is no problem relativizing the coarse topology scale to subsystems of arbitrary algebraic systems.) If $(X, \mathcal{S})$ is any system of sets and $S \in \mathcal{S}$, let $\mathcal{S}_S := \{T \cap S : T \in \mathcal{S}\}$. It is clear that if $(X, \mathcal{S})$ is a closure system, then so is $(S, \mathcal{S}_S)$; further refinements of closure, e.g., being algebraic, satisfying the EP, are also inherited. The following is a relativized algebraic version of Theorem 2.3.

4.2 Theorem. Let $(X, \mathcal{S})$ be an algebraic EP-system of infinite cardinality κ.

(i) If $dim(\mathcal{S}) \geq \omega$, then $\mathcal{S}_i := \{S \in \mathcal{S} : dim(\mathcal{S}_S) \geq \omega\}$ is coarse-residual in $\mathcal{S}$; if $dim(\mathcal{S}) = \kappa$, where κ is regular, then $\mathcal{S}_i$ is fine-residual in $\mathcal{S}$ as well.

(ii) If $dim(\mathcal{S}) = \kappa$, then $\mathcal{S}_l := \{S \in \mathcal{S} : dim(\mathcal{S}_S) = \kappa\}$ is fine-κ-residual, provided κ is regular, and either a limit cardinal or equal to $2^{<\kappa}$.

(iii) If $dim(\mathcal{S}) = \kappa$ and κ is uncountable, then $\mathcal{S}_l$ is neither coarse-meager nor coarse-residual in $\mathcal{S}$.

Proof. Ad (i). For each $\alpha < \omega$, let $\mathcal{R}_\alpha := \{S \in \mathcal{S} : S$ contains an $\mathcal{S}$-discrete subset of cardinality $|\alpha|\}$. Then $\mathcal{S}_i = \bigcap_{\alpha<\omega} \mathcal{R}_\alpha$, so it suffices to show each $\mathcal{R}_\alpha$ is coarse-open and coarse-dense in $\mathcal{S}$. First let $S \in \mathcal{R}_\alpha$, say $F \subseteq S$ is an $|\alpha|$-element $\mathcal{S}$-discrete set. Then $S \in [F, \emptyset] \cap \mathcal{S} \subseteq \mathcal{R}_\alpha$, so $\mathcal{R}_\alpha$ is coarse-open in $\mathcal{S}$.

Next let $[F, G] \cap \mathcal{S}$ be a nonempty relatively coarse-open set. Then $\langle F \rangle \cap G = \emptyset$. Since $dim(\mathcal{S}) \geq \omega$, we can argue, much as in the proof of Theorem 3.1(ii), that there is some $x_0 \in X \setminus \langle F \rangle$ with $\langle F \cup \{x_0\} \rangle \cap G = \emptyset$. (Indeed, if not, then for all $x \in X \setminus \langle F \rangle$, we have some $g(x) \in G \cap \langle F \cup \{x\} \rangle$. Since $g(x) \notin \langle F \rangle$, the EP ensures that $x \in \langle F \cup \{g(x)\} \rangle$; hence $X \setminus \langle F \rangle \subseteq \langle F \cup G \rangle$. But then $F \cup G$ is a finite $\mathcal{S}$-dense set, which must then contain an infinite $\mathcal{S}$-basis.) Proceeding by induction, we obtain $I := \{x_\beta : \beta < \omega\}$ such that for each $\beta < \omega$, $x_\beta \in X \setminus \langle F \cup \{x_\gamma : \gamma < \beta\} \rangle$ and $\langle F \cup \{x_\gamma : \gamma \leq \beta\} \rangle \cap G = \emptyset$. Algebraicity ensures that $\langle F \cup I \rangle \cap G = \emptyset$; we need to show that I is an $\mathcal{S}$-discrete set. Indeed, suppose $\beta < \omega$ is fixed and $x_\beta \in \langle I \setminus \{x_\beta\} \rangle$. Then there is some finite $I_0 \subseteq I \setminus \{x_\beta\}$ such that $x_\beta \in \langle I_0 \rangle$. Assume I_0 has the least possible cardinality for this to happen, and let $\gamma < \omega$ be largest such that $x_\gamma \in I_0$. By the inductive definition of I, it follows that $\beta < \gamma$. Because $|I_0|$ is minimal, we know that $x_\beta \notin \langle I_0 \setminus \{x_\gamma\} \rangle$. By the EP, however, we then infer that $x_\gamma \in \langle (I_0 \setminus \{x_\gamma\}) \cup \{x_\beta\} \rangle$. But this contradicts the inductive definition of I, and establishes the claim.

Set $S := \langle F \cup I \rangle$. Then $S \in [F, G] \cap \mathcal{S}_l \subseteq [F, G] \cap \mathcal{R}_\alpha$, so $\mathcal{S}_l$ (and therefore $\mathcal{R}_\alpha$) is coarse-dense in $\mathcal{S}$.

Now assume $dim(\mathcal{S}) = \kappa$, where κ is regular. We have only to show each $\mathcal{R}_\alpha$ is fine-dense in $\mathcal{S}$. The proof is essentially identical to that just given above, and in fact shows that $\mathcal{S}_l$ is fine-dense in $\mathcal{S}$.

Ad (ii). Assuming $dim(\mathcal{S}) = \kappa$, where κ is a regular limit cardinal, repeat the argument in (i) with κ replacing ω. (The weak inaccessibility of κ ensures that $\mathcal{S}_l = \bigcap_{\alpha<\kappa} \mathcal{R}_\alpha$.) In the event $\kappa = 2^{<\kappa}$, note that $|\mathcal{S} \setminus \mathcal{S}_l| \leq \kappa$.

Ad (iii). Since $\mathcal{S}$ is coarse-closed in $\wp(X)$, and is hence a Baire space, the argument in the proof of Proposition 1.1(i) relativizes to $\mathcal{S}$. Thus $\mathcal{S}_l$ cannot be both coarse-meager and fine-dense in $\mathcal{S}$. Now the argument in (ii) above (adapted from that in (i)) showing each $\mathcal{R}_\alpha$ to be fine-dense in $\mathcal{S}$ actually shows $\mathcal{S}_l$ is fine-dense in $\mathcal{S}$. Thus $\mathcal{S}_l$ is not coarse-meager in $\mathcal{S}$.

To see that $\mathcal{S}_l$ is not coarse-residual in $\mathcal{S}$, we mimic the proof of the corresponding statement in Theorem 2.3(iii). So let $\mathcal{F} := \{S \in \mathcal{S} : dim(\mathcal{S}_S) < \kappa\}$. It suffices to show that $\mathcal{F}$ is not coarse-meager; for this it suffices to show that $\mathcal{F}$ is both coarse-dense in $\mathcal{S}$ as well as a countably compact subspace of $\mathcal{S}$ relative to the coarse topology. Re the question of coarse-denseness, let $[F, G]$ be an ω-interval intersecting $\mathcal{S}$. Then $\langle F \rangle \in \mathcal{F} \cap [F, G]$. Re the question of countable compactness, it suffices to show that if a countable intersection of ω-intervals intersects $\mathcal{S}$ (a compact space because of the algebraic condition, by Theorem 3.1(i)), then, it also intersects $\mathcal{F}$. But this is easy. Let $\{[F_n, G_n] : n < \omega\}$ be such a countable collection of ω-intervals. Then the intersection must contain $\langle \bigcup_{n<\omega} F_n \rangle$, a member of $\mathcal{F}$. $\square$

Another application of Lemma 1.2 and Theorem 3.1(ii) allows us to measure topologically families of pairs of subsets.

4.3 Theorem. Let $(X, \mathcal{S})$ be an infinite algebraic EP-system, $\kappa \in \{\omega, |X|\}$. If $dim(\mathcal{S}) = \kappa$, then $\mathcal{S}$, in the $\mathcal{T}_\kappa$-topology, is homeomorphic to its own topological square; and, if we assume that $\langle\{a,b\}\rangle \neq \langle\{a\}\rangle \cup \langle\{b\}\rangle$ whenever $\{a,b\}$ is an $\mathcal{S}$-discrete doubleton subset of X, then $\mathcal{R} := \{(S,T) \in \mathcal{S}^2 : \langle S \cup T\rangle = X\}$ is $\mathcal{T}_\kappa$-κ-residual in $\mathcal{S}^2$.

Proof. That $\mathcal{S}$ and $\mathcal{S}^2$ are homeomorphic is an easy consequence of Lemma 1.2 and Theorem 3.1(ii) ($|X|$ being assumed to be regular, where appropriate). To see that $\mathcal{R}$ is $\mathcal{T}_\kappa$-κ-residual, let $B \in Bas(\mathcal{S})$ have cardinality κ. For each $b \in B$, let $\mathcal{R}_b := \{(S,T) \in \mathcal{S}^2 : b \in \langle S \cup T\rangle\}$. Clearly $\mathcal{R} = \bigcap_{b\in B} \mathcal{R}_b$, so it suffices to show each $\mathcal{R}_b$ is open and dense in $\mathcal{S}^2$.

If $(S,T) \in \mathcal{R}_b$, let $F \subseteq S, G \subseteq T$ be finite such that $b \in \langle F \cup G\rangle$. Then $(S,T) \in ([F,\emptyset] \times [G,\emptyset]) \cap \mathcal{S}^2 \subseteq \mathcal{R}_b$, so $\mathcal{R}_b$ is coarse-open in $\mathcal{S}^2$.

Next suppose $[F_1, G_1] \times [F_2, G_2]$ is a product of κ-intervals that intersects $\mathcal{S}^2$. Then $\langle F_i\rangle \cap G_i = \emptyset, i = 1,2$. Let $a \in X \setminus \langle F_1 \cup F_2 \cup G_1 \cup G_2 \cup \{b\}\rangle$ (since $dim(\mathcal{S}) = \kappa$). Then $\{a,b\}$ is $\mathcal{S}$-discrete, so fix $c \in \langle\{a,b\}\rangle \setminus (\langle\{a\}\rangle \cup \langle\{b\}\rangle)$, and set $S := \langle F_1 \cup \{a\}\rangle, T := \langle F_2 \cup \{c\}\rangle$. By the EP, $b \in \langle\{a,c\}\rangle$, so $(S,T) \in \mathcal{R}_b$. Also, $S \cap G_1 = \emptyset$; since otherwise $a \in \langle F_1 \cup G_1\rangle$, again by the EP, a contradiction. Thus $S \in [F_1, G_1]$. It remains to show $T \cap G_2 = \emptyset$. Suppose otherwise. Then, by the EP, we have $c \in \langle F_2 \cup G_2\rangle$. The EP further tells us that $a \in \langle\{b,c\}\rangle \subseteq \langle\langle F_2 \cup G_2\rangle \cup \{b\}\rangle = \langle F_2 \cup G_2 \cup \{b\}\rangle$; another contradiction, and the proof is complete. $\square$

4.4 Example. The hypothesis on doubleton subsets cannot be dropped in Theorem 4.3: Let X be any infinite set, with $\mathcal{S} := \wp(X)$. Then $\mathcal{S}$ is trivially an algebraic EP-system of dimension $|X|$. But $\mathcal{R}$ misses any coarse-open set of the form $[\emptyset, \{b\}] \times [\emptyset, \{b\}]$, so cannot be coarse-dense in $\mathcal{S}^2$. $\square$

4.5 Concluding Remarks. The Baire category theorem, while strictly speaking a result in general topology, has been applied in areas far beyond the boundaries of its discipline. In terms of its use as a tool for producing existence theorems, it may well be compared to Zorn's lemma, itself a result in the theory of partial orders. Like Zorn's lemma, the Baire theorem converts "local existence" ($\forall x \exists y$...) into "global existence" ($\exists y \forall x$...); however, unlike Zorn's lemma, the Baire theorem has a countability bound on its universal quantifier. This cardinality restriction is somewhat redressed in the Hung-Negrepontis category theorem [3], which can be viewed as a "theorem schema," one theorem for each regular cardinal κ. In [1] and [2] we put this view to work in the case $\kappa = 2^\omega$ (assuming the regularity of the continuum in the process); in the present paper we lean more toward the view that these theorems of Baire and

Hung-Negrepontis provide us with a more refined notion of universal and existential quantification.

We close with two questions in this vein, which we were unable to answer during the course of our investigation.

Let $(X, \mathcal{S})$ be an infinite closure system, $\omega \leq \kappa \leq |X|$.

(i) Is $Bas(\mathcal{S})$ necessarily thin in the coarse and fine topologies?

(ii) It is easy to show, by Theorem 3.1(iv), that $Dis(\mathcal{S})$ is $\mathcal{T}_\kappa$-dense if and only if $\mathcal{S}$ itself is $\mathcal{T}_\kappa$-dense. Can one replace "dense" with "κ-residual" when $\kappa \in \{\omega, |X|\}$ is regular?

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