Topological extensions and subspaces of $\eta_\alpha$-sets

by

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Abstract. The $\eta_\alpha$-sets of Hausdorff have large compactifications (of cardinality $\geq \exp(\alpha)$; and of cardinality $\geq \exp(\exp(2^{\omega_1}))$ in the Stone-Cech case). If $Q_{\alpha}$ denotes the unique (when it exists) $\eta_\alpha$-set of cardinality $\alpha$, then $Q_{\alpha}$ can be decomposed (= partitioned) into homeomorphs of any prescribed nonempty subspace; moreover the subspaces of $Q_{\alpha}$ can be characterized as those which are regular $T_\gamma$, of cardinality and weight $\leq \alpha$, whose topologies are closed under $<\alpha$ intersections.

Let $<A, <$ be a linearly ordered set. If $B$ and $C$ are subsets of $A$, we use the notation $B < C$ to mean that $b < c$ for all $b \in B, c \in C$. If $\alpha$ is an infinite cardinal number, we say that $<A, <$ is an $\eta_\alpha$-set if whenever $B, C \subseteq A$ have cardinality $< \alpha$ and $B < C$ then there is an element $a \in A$ with $B < [a] < C$. Such ordered sets, invented by Hausdorff [8] (see also [5, 6, 7]), are the forerunners and prototypical examples of saturated relational structures in model theory (see [5, 6]). Our interest in the present note centers on topological issues related to $\eta_\alpha$-sets, considered as linearly ordered topological spaces (LOTS's) with the open interval topology.

Roughly stated, our results are these: (i) certain Hausdorff extensions of $\eta_\alpha$-sets must have cardinality $\geq 2^\omega$; and some (the compact $C^*$-extensions) must have cardinality $\geq \exp(\exp(2^{\omega_1}))$; (ii) the (unique when it exists; i.e., when $\omega = \omega_1$) $\eta_\alpha$-set $Q_{\alpha}$ of cardinality $\alpha$ can be decomposed (partitioned) into homeomorphs of any prescribed nonempty subspace; and (iii) the subspaces of $Q_{\alpha}$ are precisely the regular $T_\gamma$ spaces, of cardinality and weight $\leq \alpha$, whose topologies are closed under $<\alpha$ intersections.

1. Preliminaries. We follow the convention that ordinal numbers are the sets of their predecessors and that cardinals are initial ordinals. If $\alpha$ is an infinite cardinal, $\alpha^+$ denotes the cardinal successor of $\alpha$ ($\omega = \{0, 1, 2, \ldots\}$, $\omega_1 = \omega^+$, etc.) If $A$ is a set, $|A|$ denotes the cardinality of $A$. If $B$ is another set then $A^B$ is the set of all functions $f: B \to A$. For cardinals $\alpha, \beta$, we let $\alpha^\beta = |A^B|$ and $\alpha^{<\beta} = \sup\{\alpha^\gamma: \gamma < \beta\}, \exp(\alpha)$ sometimes denotes $2^\omega$, especially in interactions: $\exp(\omega) = \exp(\exp(\omega))$, etc. A useful application of Koenig's Lemma is the following.
1.1. Lemma (see [6]). Let \( \alpha \) be an infinite cardinal. Then \( \alpha = \alpha^{\text{cf} \alpha} \) iff \( \alpha \) is regular and \( \alpha = 2^{\aleph_0} \).

The basic properties of \( \eta_\alpha \)-sets can be summarized as follows.

1.2. Theorem. Let \( \alpha \) be an infinite cardinal.

(i) If \( \langle A, \prec \rangle \) is an \( \eta_\alpha \)-set then the ordering is dense without endpoints.

(ii) (Hausdorff) There exists an \( \eta_\alpha \)-set of cardinality \( 2^{\aleph_0} \).

(iii) (Hausdorff) Any two \( \eta_\alpha \)-sets of cardinality \( \alpha \) are order isomorphic (call this set \( Q_\alpha \), when it exists).

(iv) (Gillman) If \( \alpha < \aleph_2 \) then there are at least two nonisomorphic \( \eta_\alpha \)-sets of cardinality \( \aleph_2 \).

(v) (Gillman, B. Jóbasz) \( Q_\alpha \) exists iff \( \alpha = \alpha^{\text{cf} \alpha} \).

1.3. Remark. \( Q_\alpha \) is, of course, the rational line \( Q \). From (v) above plus Lemma 1.1 we can also infer that \( Q_\alpha \) exists iff \( \alpha = \alpha^{\text{cf} \alpha} \); \( Q_\alpha \) does not exist for \( \alpha \) a singular cardinal, and \( Q_\alpha \) always exists for \( \alpha \) strongly inaccessible. Thus if the Generalized Continuum Hypothesis holds then \( Q_\alpha \) exists if \( \alpha \) is a regular cardinal.

Let \( X \) be a topological space. Then:

(i) \( X \) is \( \omega \)-compact if every open cover has a subcover of cardinality \( < \omega \) (compact = \( \omega \)-compact, Lindelöf = \( \omega_1 \)-compact);

(ii) \( X \) is \( \omega \)-additive [10] (= a \( P \)-space [6]) if intersections of \( < \alpha \) open sets are open (\( P \)-space = \( P_{\omega_1} \)-space); and

(iii) \( X \) is \( \omega \)-Baire if intersections of \( < \alpha \) dense open sets are dense (Baire = \( \omega_1 \)-Baire).

2. Extensions. A topological space \( X \) is an \( \eta_\alpha \)-LOTS if there is a linear ordering on the underlying set of \( X \) which makes that set an \( \eta_\alpha \)-set and whose open intervals basically generate the topology of \( X \). The following is an easy application of the definition of \( \eta_\alpha \)-set.

2.1. Lemma. Let \( X \) be an \( \eta_\alpha \)-LOTS, and let \( \mathcal{B} \) be the open interval basis arising from a suitable \( \eta_\alpha \)-order on \( X \).

(i) (\( \eta \)-intersection condition) If \( \mathcal{U} \subseteq \mathcal{B} \) has cardinality \( < \alpha \) and \( \bigcap \mathcal{U} \neq \emptyset \) then \( \bigcup \mathcal{U} \neq \emptyset \) for some finite \( \mathcal{U} \subseteq \mathcal{U} \).

(ii) (\( \eta \)-cover condition) If \( \mathcal{U} \subseteq \mathcal{B} \) has cardinality \( < \alpha \) and \( \bigcup \mathcal{U} = X \) then \( \bigcup \mathcal{U} = X \) for some finite \( \mathcal{U} \subseteq \mathcal{U} \).

(iii) (\( \eta \)-additivity condition) If \( \mathcal{U} \subseteq \mathcal{B} \) has cardinality \( < \alpha \) then \( \bigcup \mathcal{U} \) is an open set.

2.2. Theorem. Let \( X \) be an \( \eta_\alpha \)-LOTS.

(i) \( X \) is \( \omega \)-additive, hence strongly zero-dimensional when \( \alpha > \omega_0 \).

(ii) \( X \) is \( \omega \)-Baire.

Proof. (i) That \( X \) is \( \omega \)-additive follows immediately from Lemma 2.1 (iii).

(ii) If \( \alpha > \omega_0 \), \( X \) is a regular \( T_\omega \)-\( P \)-space; and such spaces are well known to be strongly zero-dimensional.

(iii) This is essentially the proof of Theorem 2.2 in [3]: use Lemma 2.1 (i, iii).

2.3. Theorem. Let \( X \) be an \( \eta_\alpha \)-LOTS.

(i) There is a family of \( 2^{\aleph_0} \) pairwise disjoint open subsets of \( X \).

(ii) \( X \) has a closed discrete set of cardinality \( 2^{\aleph_0} \).

(iii) \( X \) is not \( 2^{\aleph_0} \)-compact.

Proof. (i) Pick an appropriate \( \eta_\alpha \)-ordering for \( X \) and let \( \langle x_\xi : \xi < \lambda \rangle \) be an increasing well ordered cofinal sequence in \( X \). Then \( \lambda > \omega \), so let \( I_\xi = \{ x \in X : x < x_{\xi+1} \} \) for \( \xi < \alpha \). For each \( \xi < \alpha \) use the properties of the open interval basis given in Lemma 2.1 to construct in \( I_\xi \) a binary tree, of height \( \alpha \), consisting of \( 2^{\text{th}} \) pairwise disjoint open sub-intervals at each level \( \gamma < \xi \) (ordering is reverse inclusion). This gives \( \bigcup \{ I_\xi : \xi < \alpha \} = X \).

(ii) If \( \alpha = \omega_0 \), use the increasing sequence from (i) above and stop at \( \omega \); i.e., use the closed discrete set \( S = \{ x_\xi : \xi < \omega \} \), of cardinality \( \omega = 2^{\aleph_0} \). If \( \alpha > \omega_0 \) use zero-dimensionality: by (i) there is a set \( \mathcal{U} \) of \( 2^{\aleph_0} \) pairwise disjoint clopen subsets of \( X \). Let \( S \) consist of one point from each member of \( \mathcal{U} \).

(iii) This is immediate from (ii).

The main result in this section can now be stated.

2.4. Theorem. Let \( X \) be an \( \eta_\alpha \)-LOTS and let \( Y \) be an \( \omega \)-compact, regular \( T_\omega \)-topological extension of \( X \).

(i) \( | Y | > 2^{\aleph_0} \).

(ii) If \( X \) is dense in \( Y \) then \( Y \) is \( \omega^{\text{th}} \)-Baire.

(iii) If \( Y \) is compact Hausdorff and \( X \) is \( C^* \)-embedded in \( Y \) then \( | Y | \geq 2^{\exp(2^{\aleph_0})} \).

Proof. (i) Since \( \omega \)-compactness is a closed-hereditary property, we can assume \( X \) is dense in \( Y \). For each \( S \subseteq Y \) let \( S^\alpha \), \( S^\beta \) denote respectively the closure and the interior of \( S \) in \( Y \). Using Lemma 2.1, let \( \mathcal{B} \) be an open basis for \( X \) with the \( \alpha \)-intersection condition. We show first that for \( U \subseteq Y \) open, \( U \cap X \subseteq (U \cap X)^\alpha \) to see this, let \( x \in U^\alpha \). Since \( X \) is dense and every open \( V \) containing \( x \) intersects \( U \), we have \( x \in (U \cap X)^\alpha \). Thus \( (U \cap X)^\alpha = U^\alpha \) and we get \( U \cap X \subseteq U \cap X \cap (U \cap X)^\alpha \).

We build an \( \alpha \)-level tree \( T \) in \( Y \) by induction satisfying: (a) \( T \) is a binary tree of sets, ordered by reverse inclusion; (b) each member of \( T \) is of the form \( B^\alpha \) where \( B \neq \emptyset \) \( \leftrightarrow \emptyset \); (c) the members of each level of \( T \) are pairwise disjoint; and (d) whenever \( B_1 \subseteq B_2 \) in \( T \), it is also true that \( B_1 \subseteq B_2 \).

For each ordinal \( \xi < \alpha \) define the \( \beta \)-th level \( T^\xi \) inductively: \( T^0 = [B^\beta] \)

where \( \emptyset \neq B \in \mathcal{B} \). Assuming \( T^\gamma \subseteq [B^\gamma] \gamma < \xi \) has been defined, define \( T^\xi \gamma + 1 \), as follows: Let \( B^\beta \subseteq T^\gamma \). Then there is an open \( U \subseteq Y \) with \( B = U \cap X \subseteq (U \cap X)^\alpha \cap B^\beta \). Use regularity to find open sets \( U_1, U_2 \neq \emptyset \) (all nonempty open sets are self-dense) with \( U_1 \subseteq U \subseteq U_2 \subseteq B^\beta \) and \( U_1 \cap U_2 = \emptyset \). Since \( B \) is dense in \( B^\beta \), there are \( B_1, B_2 \neq \emptyset \), nonempty, such that
\( B_i \subseteq U_i \cap B, \ i = 1, 2 \). So define 
\( T'_{\xi+1} = \bigcup \{ \langle B_i^\gamma, B_i^\zeta \rangle : B^{\gamma, \zeta} \in T_\xi \} \). In the case 
where \( \xi \) is a limit ordinal and \( T \cap \xi \) is already constructed, let 
\( B_{\xi} = \bigcap \{ B^{\gamma, \zeta} : \gamma < \xi \} \) be a branch in \( T \cap \xi \). By the 
inductive hypothesis, \( \bigcap \{ B^{\gamma, \zeta} : B^{\gamma, \zeta} \in T_\xi \} \neq \emptyset \) and 
contains a nonempty \( B_\xi \in \mathcal{B} \). Let 
\( T_\xi = \{ \langle B^{\gamma, \zeta} \rangle : \mathcal{B} \text{ is a branch of } T \cap \xi \} \), and 
\( T = \bigcup \{ T_\xi \}. \) By \( \omega \)-compactness, each branch of \( T \) has nonempty 
intersection.

Since \( T \) has \( \omega \) branches, we conclude that \( |T| \geq \omega \).

(ii) Let \( X, Y, \mathcal{B} \) be as above. Let \( \langle U_\xi : \xi < \alpha \rangle \) be a family of \( \alpha \)-dense open subsets of \( Y \), with \( S = \bigcup U_\xi \). We show \( S \) is dense in \( Y \). To this end let 
\( V \subseteq Y \) be nonempty open. To show \( V \cap S \neq \emptyset \), use induction on \( \alpha \). We 
construct a decreasing chain \( \langle B^{\gamma, \zeta} : \gamma < \alpha, \xi < \alpha, B^{\gamma, \zeta} \in \mathcal{B} \rangle \) for \( \gamma < \xi < \alpha \), 
\( \emptyset \neq B^{\gamma, \zeta} \subseteq B^{\gamma, \zeta} \subseteq V \cap \bigcap U_\xi \). This is possible since \( X \) is dense in \( Y \), \( \xi \) is regular, and \( \mathcal{B} \) satisfies the \( \alpha \)-intersection condition. Using \( \omega \)-compactness we get \( \emptyset \neq \bigcap B^{\gamma, \zeta} \subseteq V \cap S \).

(iii) Assume \( Y \) is compact Hausdorff and \( X \) is \( C^* \)-embedded in \( Y \). Using 
Theorem 2.3 (ii), let \( S \) be any closed discrete subset of \( X \) of cardinality \( 2^{\omega_1} \).

Since \( X \) is normal, \( S \) is \( C^* \)-embedded in \( X \), hence in \( Y \). Therefore \( S \) is 
homeomorphic to the Stone-Čech compactification of \( S \), so \( |Y| \geq |S|^{\omega_1} = \exp^\omega(2^{\omega_1}) \).

25. Remark. Both estimates in Theorem 2.4 (i, iii) can be realized as follows: (i) the order 
compactification (= Dedekind-completion-plus-endpoints) of \( Q_\omega \) has cardinality \( 2^\omega \); (ii) the Stone-Čech compactification of \( Q_\omega \) has cardinality \( \exp^\omega(2^\omega) \).

3. Subspaces. In this section we will focus on topological subspaces of the 
spaces \( Q_\alpha \).

A space \( X \) partitions a space \( Y \) (see [4]) if there is a family of 
embeddings of \( X \) into \( Y \) whose images form a cover of \( Y \) by pairwise disjoint 
sets. Our first aim is to show that any nonempty subspace of \( Q_\omega \) partitions 
\( Q_\omega \) (a property shared by the space of irrational numbers and the Cantor 
discontinuum, but not the real line [4]). The proof for \( \alpha = \omega \) is quite easy 
and rests on the following well known result [9].

3.1. Lemma (Sierpiński). Let \( X \) be countable, first countable, regular \( T_1 \), 
and self-dense. Then \( X \) is homeomorphic to \( Q \) (\( X \cong Q \)).

3.2. Theorem. Let \( X \) be a nonempty subspace of \( Q_\omega \). Then \( X \) 
partitions \( Q_\omega \).

Proof. Simply note that by Lemma 3.1, \( X \times Q \cong Q \).

To prove an analogue to Theorem 3.2 for \( \alpha > \omega \), we will need some 
machinery a bit more involved, namely the ultrafilter construction [1, 3, 
5] of which we give only a sketch here.

Let \( \langle A_i : i \in I \rangle \) be an indexed family of sets, with \( D \) an ultrafilter of 
subsets of \( I \). \( \Pi_{i \in I} A_i \) (respectively \( A^{(i)} \), when \( A_i = A \) for all \( i \in I \)) denotes the \( D \)- 
ultraproduct (respectively \( D \)-ultrafilter), namely the set of equivalence 
classes \( \equiv_{D}^{A} \) where \( \equiv_{D}^{A} : \mathcal{A} \to \mathcal{D} \) is a \( D \)- 
ultrafilter \( \equiv_{D}^{A} : \mathcal{A} \to \mathcal{D} \) and \( \equiv_{D}^{A} \) is a relation on \( \mathcal{A} \) 
and \( \equiv_{D}^{A} \) is a \( D \)-ultrafilter of \( \mathcal{A} \) in \( \mathcal{D} \). When \( A \) carries 
additional finitary relations (e.g., order structure, algebraic structure) this 
mapping is an "elementary embedding", in the parlance of model theory.

Since some of the following arguments use techniques from model 
theory, in particular the theory of ultraproducts and saturated models, we 
refer the reader to [5] for the basic theory and terminology. Regrettably 
we cannot make the paper self-contained for topologists who do not have some 
grounding in model theory.

When \( X \) is a topological space ("\( X^\omega \) also stands for the underlying 
point set) and \( \mathcal{B} \) is a basis for the open sets of \( X \), we use \( \langle X, \mathcal{B} \rangle \) to denote the \( \omega \)- 
ultraproduct of its separate clopen sets \( U \in \mathcal{B} \). This is a \( \omega \)- 
ultraproduct of \( \omega \)-ultrafilters \( \equiv_{D}^{U} \). When \( A \) is an \( \omega \)-ultrafilter 
and \( A \) is a \( \omega \)-ultrafilter on \( A \), then \( A \) is an \( \omega \)-ultrafilter.

A very simple but important result from [1] is that if \( \alpha \) is a linear order on 
\( X, \mathcal{B} \) is a topological basis for the order topology, and \( D \) is any ultrafilter then \( \equiv_{D}^{A} \) is a \( \omega \)-ultrafilter for \( X \) arising from 
\( \mathcal{B}^{(i)} \).

By way of a brief digression into general model theory, suppose \( A = \langle A_i : i \in I \rangle \) is a relational structure (over a countable language). If \( D \) is an 
ultrafilter then \( \equiv_{D}^{A} \) is an elementary embedding. In particular, if \( \langle X, \mathcal{B} \rangle \) is a \( \omega \)- 
ultraproduct basis structure then for each \( B \in \mathcal{B} \), \( A_B \equiv_{D}^{A} \mathcal{B} \) is 
also a \( \omega \)-ultrafilter on \( A \). Thus \( A_B \equiv_{D}^{A} X \) is a \( \omega \)-ultraproduct embedding, 
provided it is continuous. We will come back to this later.

We assume the reader to be familiar with what it means for a relational 
structure \( A \) to be \( \alpha \)-saturated, for \( \alpha \) a cardinal number. In particular, the \( \eta_\alpha \) 
sets are precisely the \( \alpha \)-saturated dense linearly ordered sets without 
endpoints.

Of major importance to us are the following well known results.

3.3. Lemma. Let \( A \) be a relational structure and let \( D \) be a \( \beta \)-good 
countably incomplete ultrafilter on a set of cardinality \( \beta \). Then \( A^{(i)} \) is a \( \beta \)- 
saturated, and of cardinality \( |A|^\beta \).
3.4. Lemma. Any two $\alpha$-saturated elementarily equivalent relational structures of cardinality $\alpha$ are isomorphic.

Fix $\alpha = \omega^{\omega}$. If $\alpha$ is a successor we fix $\alpha = \beta^+ = 2^\beta$ and let $D$ be a $\beta^+$-good countably incomplete ultrafilter on $\beta$. If $\alpha$ is a limit cardinal, we let $\langle \beta_\xi : \xi < \alpha \rangle$ be a fixed increasing sequence of cardinals which is cofinal in $\alpha$ (note: $\alpha$ is regular; and for $\xi < \alpha$, $\beta_\xi < \alpha$, and $2^\beta < \alpha$); and for each $\xi < \alpha$ we let $D_\xi$ be a $\beta_\xi^+$-good countably incomplete ultrafilter on $\beta_\xi$. If $A$ is a relational structure of cardinality $\leq \alpha$ we form an elementary extension $A^{(0)}$ of $A$, which is $\alpha$-saturated and of cardinality $\alpha$, as follows: If $\alpha = \beta^+$, set $A^{(0)} = A^{(0)}$. If $\alpha = \sup \{ \beta_\xi : \xi < \alpha \}$ let $A^{(0)} = \bigcup \{ A^{(0)}(\beta_\xi) : \beta_\xi < \alpha \}$ where $A^{(0)}(\beta_\xi) = A^{(0)}$, $A^{(0)}(\omega_1) = A^{(0)}(\omega_1^{(1)})$, and $A^{(0)} = \bigcup A^{(0)}$ where $\gamma$ is a limit ordinal.

3.5. Theorem. Let $\alpha = \omega^{\omega}$. Then $Q_\alpha$ is an $\aleph_1$-closed field (i.e., a field which is ordered by an $\aleph_1$-set). Hence $Q_\alpha$ is a homogeneous LOTS.

Proof. Letting $\mathbb{A}$ be the ordered field of rational numbers, we obtain $A^{(0)}$ via the machinery outlined above. Then the order structure on $A^{(0)}$ is an $\aleph_1$-set of cardinality $\alpha$, hence $Q_\alpha$. To get (point) homogeneity, we use the additive abelian group structure on $Q_\alpha$ to translate points.

3.6. Theorem. Let $\alpha = \omega^{\omega}$ and let $X$ be a regular $T_1$ space which is self-dense, and of cardinality and weight $\leq \alpha$. Then $X^{(\alpha)} \cong Q_\alpha$.

Proof. Choose a basis $\mathcal{B}$ for $X$ which has cardinality $\leq \alpha$, and let $\langle X_\xi : \eta \rangle$ be a countable elementary substructure of $\langle X ; \mathcal{B} \rangle$. Then $\langle X_\xi : \eta \rangle$ generates a regular $T_1$ space which is self-dense, and of countable cardinality and weight. By Lemma 3.1 there is a basis $\mathcal{C}$ for the open sets of $Q_\alpha$ such that $\langle X_\xi : \eta \rangle \cong Q_\alpha$. Therefore $\langle X ; \mathcal{B} \rangle$ and $\langle Q_\alpha ; \mathcal{C} \rangle$ are elementarily equivalent. So we use Lemma 3.4, plus the machine for constructing the $A^{(0)}$, and conclude that $\langle X ; A^{(0)} \rangle \cong Q_\alpha$. Thus $X^{(\alpha)}$ is homeomorphic with $Q_\alpha$.

3.7. Corollary. $Q_\alpha \cong Q_\beta$ (with the usual product topology).

Proof. Simply use Theorem 3.6 to conclude $Q_\alpha \cong Y^{(\omega)}$. It is then easy to verify (since ultraproducts commute with finite cartesian products) that $Q^{(\omega)} \cong Q^{(\omega)} \cong Q^{(\omega)}$. We can now prove our analogue to Theorem 3.2 for uncountable $\alpha$.

3.8. Theorem. Let $\alpha \leq \omega$, be nonempty. Then $X$ partitions $Q_\alpha$.

Proof. We actually prove that $X$ partitions $Q_\beta$ and then invoke Corollary 3.7. We use the technique of Theorem 2.5 in [3], in analogy with the question of subsets of the real line partitioning Euclidean 3-space.

By Theorem 3.5 we can use the $\mathcal{B}$-field structure of $Q_\alpha$ to treat $Q_\alpha$ as affine 3-space. Thus we can talk of affine lines and planes in $Q_\alpha$ as if we were in Euclidean space. In particular, lines are (affine) homeomorphs of $Q_\alpha$, a line $L$ not contained in a plane $P$ must intersect $P$ in at most one point, each point $p \in P$ is contained in $n$ distinct lines in $P$, and each point $p \in Q_\beta$ is contained in $n$ distinct planes.

Let $\emptyset \neq X \subseteq Q_\alpha$, and let $\langle P_\xi : \xi < \alpha \rangle$ be a well ordering of the points of $Q_\alpha$. Inducting on $\alpha$, we assume that $P_\xi$ is the first point not covered by a copy of $X$, that $P_\xi \subseteq X_\xi \subseteq X$ for $\gamma < \xi$ and that distinct $X_i$'s are disjoint and embedded in affine lines $L_i \subseteq Q_\alpha$. Since $2^\beta < \alpha$ there is a plane $P_\xi$ containing $P_\xi$ but failing to contain any $L_i$ for $\gamma < \xi$.

Thus $|P_\xi \cap X_\xi| \leq 1$ for $\gamma < \xi$, so $|P_\xi \cap X_\xi| < \alpha$. Since there are $\alpha$ lines in $P_\xi$ containing $P_\xi$, there is one, say $L_\xi$, which misses $\bigcup X_\xi$ altogether. Since $Q_\alpha$ is homogeneous, there is a copy $X_\xi$ of $X_\xi$ such that $P_\xi \subseteq X_\xi \subseteq X_\xi$, and the induction is complete.

We next turn to characterizing those topological spaces $X$ which embed as subspaces of $Q_\alpha$, for $\alpha = \omega^{\omega}$. Clearly if $X$ does embed in $Q_\alpha$, then (i) $X$ is regular; (ii) both the cardinality and the weight of $X$ are $\leq \alpha$; and (iii) $X$ is $\omega$-additive. We will show that these three conditions suffice for $X$ to embed in $Q_\alpha$. When $\alpha = \omega$, a simple application of Lemma 3.1 does the trick. For uncountable $\alpha$, however, it seems necessary to resort again to model-theoretic methods.

3.9. Theorem. Assume $\alpha = \omega^{\omega}$ and suppose $X$ is a space which is regular $T_1$, both of whose cardinality and weight are $\leq \alpha$, and which is $\omega$-additive. Then $X$ embeds in $Q_\alpha$.

Proof. First let $Y = X \times Q_\alpha$. Then $Y$ has all of the above properties and is self-dense as well. By Theorem 3.6, then, $Y^{(\omega)} \cong Q_\alpha$; so it remains to show that $Y$ embeds in $Y^{(\omega)}$. This will suffice since $X$ clearly embeds in $Y$.

Suppose $\alpha = \beta^+ = 2^\beta$. Then $Y^{(\omega)} \cong Y^{(\omega)}$. To show that $A_{\alpha} : Y \rightarrow Y^{(\omega)}$ is a topological embedding we need only show continuity. Let $\mathcal{A}$ be a basis for the topology on $X$, and let $[[B_{\mathcal{A}}]] \in \mathcal{A}_{\alpha}$. Then $A_{\alpha}^{-1}[[B_{\mathcal{A}}]] = \bigcup \{ B(x) : x \in \alpha \}$, an open set in $Y$ since $Y$ is $\omega$-additive.

Suppose $\alpha = \sup \{ \beta_\xi : \xi < \alpha \}$, and $Y^{(\omega)}$ is constructed as a chain union of the $Y^{(\omega)}$s using the ultrafilters $D_\xi$, $\xi < \alpha$. For each $\xi < \alpha$, let $d_\xi = d_{\alpha_\xi} : Y^{(\omega)} \rightarrow Y^{(\omega+1)}$, and let $\epsilon_\xi : Y^{(\omega)} \rightarrow Y^{(\omega)}$ be the natural elementary embedding. Since $\alpha$ is a limit ordinal, $\epsilon_\xi : Y \rightarrow (\epsilon_\omega)^{Y^{(\omega)}}$ will be continuous provided the same is true for each $\xi$. The only difficulty in a proof by induction on $\alpha$ is at the successor stages, but that case has essentially been taken care of: Let $\mathcal{A}$ be a basis for the topology on $Y^{(\omega)}$, and let $[[B_{\mathcal{A}}]] \in \mathcal{A}_{\alpha}$. Then

$$
eq \epsilon_\xi^{-1} \left( \bigcup \bigcap B(y) \right)_{\mathcal{A}_{\alpha}}$$

$$= \bigcup \bigcap \epsilon_\xi^{-1} \left( B(y) \right)_{\mathcal{A}_{\alpha}}.$$
an open set in $Y$ since $e_i$ is continuous by the inductive hypothesis, $|J| < \beta < \alpha$ for all $J \in D_\alpha$, $\xi < \alpha$, and $Y$ is $\alpha$-additive (note: the maps $d_\alpha$ are generally not continuous).

3.10. Remark. Although the subspaces of $Q_\alpha$ can be characterized in a purely topological manner, it seems the same cannot be said for $Q_\alpha$: itself: some sort of saturatedness condition must be imposed; and that involves the semantics of artificial language. The following example dashes any hope of achieving the obvious analogue to Lemma 3.1 for uncountable $\alpha$.

Call a space $X$ $Q_\alpha$-like if $X$ is regular $T_\alpha$, of cardinality and weight $\alpha$, which is $\alpha$-additive and self-dense. Clearly, there are no $Q_\alpha$-like spaces of singular cardinality, and $Q$ is the only $Q_\alpha$-like space.

3.11. Example. For any regular uncountable $\alpha$ there exists a $Q_\alpha$-like space which is not a Baire space.

Construction. We use a well known example due to Sikorski [10]. Let $(2^\alpha)_\alpha$ (see also [6]) denote the space formed by allowing as basis all $\alpha$ intersections of open sets in the usual product topology on $2^\alpha$; and let

$$\mathcal{B}_\alpha = \{ f \in 2^\alpha : f(\xi) = 0 \text{ for all but finitely many } \xi < \alpha \} \subseteq (2^\alpha)_\alpha.$$  

Then $\mathcal{B}_\alpha$ is $Q_\alpha$-like but is the union of countably many nowhere dense subsets.

3.12. Question. Are there $Q_\alpha$-like spaces which are not homeomorphic to $Q$, but which have open bases with the $\alpha$-intersection condition (see Lemma 2.1 (i))? To end on a more positive note, it is easy to prove that every subspace $X$ of $Q$ can be embedded as a closed subspace of $Q$. $X$ is closed in $X \times Q \simeq Q$. A similar statement can be made for $Q$, when $\alpha$ is uncountable. (We are thankful to R. L. Levy for bringing this question to our attention.) We will first need a lemma, the proof of which can be easily adapted from the proof of New Theorem 7.7 in [2].

3.13. Lemma ([2]). Let $X$ be an $\alpha$-additive regular $T_\alpha$ space and let $D$ be an ultrafilter on a set of cardinality $< \alpha$. Then $D_\alpha$ embeds $X$ as a closed subset of $X^{(\alpha)}$.

3.14. Theorem. (i) Every subspace of $Q_\alpha$ embeds as a closed subspace of $Q$.

(ii) Let $X$ be a nonempty subspace of $Q_\alpha$. Then $Q_\alpha$ can be partitioned into homeomorphs of $X$, each of which is closed and nowhere dense in $Q_\alpha$.

Proof. (i) Let $X \subseteq Q_\alpha$ and refer to the proof of Theorem 3.9. We show that $Y = X \times Q_\alpha$ embeds as a closed subspace of $Y^{(\alpha)}$. In the case $\alpha = \beta^* = 2^\beta$, we apply Lemma 3.13 directly. When $\alpha = \sup \{ \beta : \xi < \alpha \}$, use induction on $\alpha$: at the successor stages, the only stages where difficulties may arise, use Lemma 3.13 again.

(ii) This follows easily from (i) above plus the proof of Theorem 3.8.