ULTRACOPRODUCT CONTINUA AND THEIR REGULAR SUBCONTINUA

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ABSTRACT. We continue our study of ultracoproduct continua, focusing on the role played by the regular subcontinua–those subcontinua which are themselves ultracoproducts. Regular subcontinua help us in the analysis of intervals, composants, and noncut points of ultracoproduct continua. Also, by identifying two points when they are contained in the same regular subcontinua, we naturally generalize the partition of a standard subcontinum of $\mathbb{H}^*$ into its layers.

1. Introduction

The theme of this article is an examination of ultracoproduct continua from the perspectives of intervals, composants and varieties of noncut point, and is a continuation of the study of topological ultracoproducts begun in [5, 6]. (See [7] for a survey up to 2003, as well as [18] for a survey up to 1992 on the use of ultracoproducts of arcs.) A principal tool in our investigation is the employment of *regular subcontinua*, those subcontinua which are themselves ultracoproducts. We give a partial answer to when ultracoproducts of intervals are intervals; we also specify conditions under which the composant structure of an ultracopower of continuum X is like–or very much unlike–that of X. We consider the existence of various kinds of noncut point in nonmetrizable continua, in the aim of generalizing existence results known for the metrizable case. While the existence of nonblock points is assured for separable–but not all–continua, it is still true that each continuum has ultracopowers which are irreducible about their sets of nonblock points. Finally we investigate what happens when we define two points of an ultracoproduct to be $\mathcal{R}$-equivalent if they both lie in the same regular subcontinua. $\mathcal{R}$-classes in ultracoproducts of arcs via nonprincipal ultrafilters on a countable set are also known as *layers*, and are instrumental in the study of the Stone-Čech remainder $\mathbb{H}^* := \beta(\mathbb{H}) \setminus \mathbb{H}$ of the real half-line (see, e.g., [18]).

2. The Ultracoproduct Construction

Here we use the term **compactum** to refer to a compact Hausdorff topological space; a **continuum** is a nonempty compactum that is also connected. A **subcontinuum** of a topological space X is a subset that is a continuum in its subspace topology. If $x \in X$, then the **component of X at x** is the union $C(X, x)$ of all

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connected subsets of X that contain x. The components of a space are well known to partition it into connected closed subsets.

A point c of a connected space X is a **cut point** of X if its complement $X \setminus \{c\}$ is not connected; otherwise c is a **noncut point**.

A space is **nondegenerate** if it contains at least two points. If x is a point of a nondegenerate continuum X, the **composant of X at x** is the union $\kappa(X, x)$ of all proper subcontinua of X that contain x. The composants of a nondegenerate continuum are well known to be connected dense subsets (see [24]).

The topological ultracoproduct construction gives us an important source of nonmetrizable continua; it also furnishes an avenue for bringing model-theoretic methods to topology.

Start with an infinite discrete set I and let $\vec{X} = \langle X_i : i \in I\rangle$ be an I-sequence of compacta. Then each ultrafilter $\mathcal{D}$ on I gives rise to a new compactum $\vec{X}_{\mathcal{D}}$ (also denoted $\sum_{\mathcal{D}} X_i$), the **$\mathcal{D}$-ultracoproduct** of the family, as follows:

Step 1 Form the disjoint union $Y = \bigcup_{i\in I}(X_i \times \{i\})$, with $q : Y \to I$ the map taking a pair in Y to its second coordinate.

Step 2 Let $q^\beta : \beta(Y) \to \beta(I)$ be the Stone-Čech lift of q.

Step 3 Viewing the ultrafilter $\mathcal{D}$ as a point in $\beta(I)$, define $\vec{X}_{\mathcal{D}}$ to be the point pre-image $(q^\beta)^{-1}[\mathcal{D}]$.

When each X_i is the same compactum X, then $\vec{X}_{\mathcal{D}}$ is denoted $X_{\mathcal{D}}$ and is referred to as the **$\mathcal{D}$-ultracopower** of X. The space Y above is then $X \times I$, and the first-coordinate map $p : Y \to X$ induces a continuous surjection $p_{\mathcal{D}} := p^\beta | X_{\mathcal{D}} : X_{\mathcal{D}} \to X$, known as the **codiagonal map**.

We use both vector notation and index notation in the sequel for ultracoproducts and their near-relatives, the ultraproducts. While vector notation has the advantage of compactness, the index notation is obviously better for working with coordinatewise operations.

As the terminology suggests, ultracoproducts and classical ultraproducts are dual notions from the viewpoint of category theory (see [7]), but the following is a more useful account of their connection. Given an I-sequence $\vec{A}$ of nonempty sets and $\mathcal{D}$ an ultrafilter on I, the **$\mathcal{D}$-ultraproduct** $\vec{A}^{\mathcal{D}}$ (also denoted $\prod_{\mathcal{D}} A_i$) consists of all equivalence classes that arise as $\vec{a}, \vec{b} \in \prod_{i\in I} A_i$ are identified whenever $\{i \in I : a_i = b_i\} \in \mathcal{D}$. Elements of $\vec{A}^{\mathcal{D}}$ are denoted $\vec{a}^{\mathcal{D}}$. If R_i is a finitary relation on A_i of fixed arity n, $i \in I$, then the $\mathcal{D}$-ultraproduct $\vec{R}^{\mathcal{D}}$ may be naturally viewed as an n-ary relation on $\vec{A}^{\mathcal{D}}$. In this way we extend ultraproducts of sets to ultraproducts of relational structures. (See, e.g., [16].)

When each X_i is a compactum, the points of $\vec{X}_{\mathcal{D}}$ are the maximal filters in the bounded lattice consisting of all ultraproducts $\vec{F}^{\mathcal{D}}$, where each F_i is closed in X_i. If $S_i \subseteq X_i$, $i \in I$, we denote by $(\vec{S}^{\mathcal{D}})^\sharp$ the set of points $\mu \in \vec{X}_{\mathcal{D}}$ such that some member of μ is contained in $\vec{S}^{\mathcal{D}}$. Subsets of $\vec{X}_{\mathcal{D}}$ of the form $(\vec{S}^{\mathcal{D}})^\sharp$ are called **regular**. The closed (resp., open) sets in $\vec{X}_{\mathcal{D}}$ are then basically generated by the closed (resp., open) regular subsets.

Remark 2.1. Indeed (see [7]), if $\mathcal{A}_i$ is a **lattice base** for X_i, $i \in I$; i.e., a closed-set base that is also a bounded lattice under finite unions and intersections, then the regular sets $(\vec{A}^{\mathcal{D}})^\sharp$, where each A_i is in $\mathcal{A}_i$, constitute a lattice base for $\vec{X}_{\mathcal{D}}$. The class of *Wallman lattices*, those bounded lattices isomorphic to a lattice base for some compactum, is axiomatized using simple first-order sentences. This fact provides an important gateway between the model-theoretic world and the topological one (see, e.g., Theorem 4.7 below).

Each $\vec{x}^{\mathcal{D}} \in \vec{X}^{\mathcal{D}}$ may be canonically identified with the single point in $(\prod_{\mathcal{D}}\{x_i\})^\sharp$, which we denote by $\vec{x}_{\mathcal{D}}$. We refer to such points as the **regular points** of $\vec{X}_{\mathcal{D}}$. The regular points show how each ultraproduct $\vec{S}^{\mathcal{D}}$ may be viewed as a (clearly dense) subset of $(\vec{S}^{\mathcal{D}})^\sharp$.

It is easy to show that when each closed $F_i \subseteq X_i$ is regarded as a compactum, the set $(\vec{F}^{\mathcal{D}})^\sharp$, as a subspace, is naturally homeomorphic to the ultracoproduct $\vec{F}_{\mathcal{D}}$. Because of its relative simplicity, we will use the latter notation when appropriate.

One may generally start with an I-sequence $\vec{X}$ of topological spaces and an ultrafilter $\mathcal{D}$ on I, and take ultraproducts $\vec{U}^{\mathcal{D}}$, where U_i is open in X_i, $i \in I$. These "open ultraboxes" provide an open-set base for the **topological ultraproduct** $\vec{X}^{\mathcal{D}}$ (see [7, Section 3]). Often the topologies on the constituent spaces X_i are induced by other structures; e.g., by total orderings $\leq_i$. In this case the ultraproduct topology on $\vec{X}^{\mathcal{D}}$ is induced by the ultraproduct total ordering $\prod_{\mathcal{D}} \leq_i$. (See [7]. This also works when the other structures are uniformities, but not when they are metrics.)

When the spaces under consideration are compacta, the topological ultracoproduct is a compactification of the corresponding topological ultraproduct.

In model theory, when each A_i is the same relational structure A, we write $A^{\mathcal{D}}$ to denote the **$\mathcal{D}$-ultrapower** $\prod_{\mathcal{D}} A_i$. Here we have a canonical **diagonal embedding** $d_{\mathcal{D}} : A \to A^{\mathcal{D}}$, given by $a \mapsto a^{\mathcal{D}}$ As a direct consequence of the Łoś Ultraproduct Theorem [16, Theorem 4.1.9], diagonal embeddings are *elementary*, in the model-theoretic sense. However, they are almost never continuous as functions from a topological space into one of its topological ultrapowers.

Remark 2.2. In the compact Hausdorff setting, the codiagonal map $p_{\mathcal{D}}$ is specified by taking a given $\mu \in X_{\mathcal{D}}$ to the unique $x \in X$ such that $\mu \in (U^{\mathcal{D}})^\sharp$ for every open neighborhood U of x. $p_{\mathcal{D}}$ is thus seen to be a left-inverse for the diagonal $d_{\mathcal{D}}$; moreover, when it is restricted to the ultrapower $X^{\mathcal{D}}$, we obtain what is known in nonstandard analysis as the *standard part* map (see [7, Theorem 3.8]).

A basic fact (see [5, Lemma 4.6]) about ultracoproducts of compacta is that the Boolean lattice of clopen subsets of the ultracoproduct is isomorphic to the corresponding ultraproduct of the clopen-set lattices of the factor spaces. As an immediate consequence of this, we see that $\vec{X}_{\mathcal{D}}$ is a continuum if and only if

$$\{i \in I : X_i \text{ is a continuum}\} \in \mathcal{D}.$$

(What is more, when the factor spaces are continua, the family $\{\vec{X}_{\mathcal{D}} : \mathcal{D} \in \beta(I)\}$ of ultracoproducts comprises the components of $\beta(\bigcup_{i\in I}(X_i \times \{i\}).)$

Remark 2.3. Ultracoproducts of *arcs*, i.e.,homeomorphs of the closed unit interval $\mathbb{I} := [0, 1]$ in the real half-line $\mathbb{H} := [0, \infty)$, were first investigated by J. Mioduszewski [22], who was motivated to study the Stone-Čech remainder $\mathbb{H}^*$. With $\omega := \{0, 1, 2, \dots\}$, $\mathcal{D}$ a nonprincipal ultrafilter on ω, and $\vec{X}$ an ω-sequence of arcs, the ultracoproduct $\vec{X}_{\mathcal{D}}$ is what we refer to here as an *ultra-arc*. If each X_n is of the form $[a_n, b_n]$, where $a_0 < b_0 < a_1 < b_1 < \dots$ is an unbounded sequence in $\mathbb{H}$, then the ultra-arc $\mathbb{I}_{\mathcal{D}}$ is homeomorphic to $\sum_{\mathcal{D}}[a_n, b_n]$, which in turn is naturally homeomorphic to the set

$$\bigcap_{J\in\mathcal{D}} \mathrm{cl}_{\beta(\mathbb{H})}(\bigcup_{n\in J}[a_n,b_n]) \subseteq \mathbb{H}^*.$$

Such ultra-arcs are commonly referred to as the *standard subcontinua* of $\mathbb{H}^*$ (see [18]), and have proven to be key to understanding its fine structure.

3. Intervals

Road systems were introduced in [10] to provide a uniform framework in which to describe various classical betweenness notions. If X is a continuum, then its family of subcontinua qualifies as a road system because: (i) each singleton set is a subcontinuum; and (ii) each doubleton set is contained in a subcontinuum. The point z is said to **lie between** points x and y if each subcontinuum containing $\{x,y\}$ contains z as well. The **interval** $[x,y]$ in this interpretation of betweenness consists of all points lying between x and y (so $y \in \kappa(X,x)$ if and only if $[x,y] \neq X$).

The following useful fact about components of ultracoproduct compacta was first proved by R. Gurevič [17].

Lemma 3.1. ([17, Lemma 10]) *Let $x_i \in X_i$, $i \in I$. Then $C(\vec{X}_{\mathcal{D}},\vec{x}_{\mathcal{D}}) = \sum_{\mathcal{D}} C(X_i,x_i)$. Thus components of ultracoproducts at regular points are regular sets.*

We will consider analogues of Lemma 3.1 for composants in the next section.

A subset of the ultracoproduct $\vec{X}_{\mathcal{D}}$ of compacta is **semiregular** if it contains at least one regular point, and is **irregular** otherwise. Because $\vec{X}^{\mathcal{D}}$ is dense in $\vec{X}_{\mathcal{D}}$, any subset with nonempty interior is semiregular. On the other hand, basic results in the study of topological ultraproducts (see, e.g., [7]) imply that if $\mathcal{D}$ is a countably incomplete ultrafilter (i.e., one not closed under countable intersections), then every infinite compact subset–as well as every nondegenerate connected subset–of $\vec{X}_{\mathcal{D}}$ must contain irregular points.

For any subset $S \subseteq \vec{X}_{\mathcal{D}}$, define $\mathcal{R}(S)$ to be the family of all regular subcontinua of $\vec{X}_{\mathcal{D}}$ that contain S, and let $R(S)$ be the intersection $\bigcap\mathcal{R}(S)$. $R(S)$ is the **regular hull** of S, evidently a subcompactum of the ultracoproduct. $R(\mu)$ is shorthand for $R(\{\mu\})$, as per convention, when $\mu \in \vec{X}_{\mathcal{D}}$. We refer to regular hulls of singleton sets as **point hulls**. The following simple result is used frequently in the sequel.

Theorem 3.2. *If K is a semireglar subcontinuum of $\vec{X}_{\mathcal{D}}$, then $R(K) = K$. In particular, point hulls of regular points are singletons.*

Proof. Let $K \subseteq \vec{X}_{\mathcal{D}}$ be a subcontinuum that is semiregular; say K contains the regular point $\vec{x}_{\mathcal{D}}$. Let $\mathcal{F}$ be the collection of all closed regular sets containing K. Then $K = \bigcap\mathcal{F}$. If $\vec{F}_{\mathcal{D}} \in \mathcal{F}$ and $C_i = C(F_i,x_i)$, then $\vec{C}_{\mathcal{D}} = C(\vec{F}_{\mathcal{D}},\vec{x}_{\mathcal{D}})$, by Lemma 3.1. Hence $K \subseteq \vec{C}_{\mathcal{D}} \in \mathcal{R}(K) \subseteq \mathcal{F}$, and we infer that $K = R(K)$. □

We will see below (Remark 4.8 (ii, iii)) that the semiregularity assumption in Theorem 3.2 cannot be dropped.

If X is a continuum and $x,y \in X$, the interval $[x,y]$ is manifestly the intersection of all subcontinua of X that contain both x and y. The points x and y are **bracket points** for the interval (bearing in mind that an interval may have many sets of bracket points). In the case of an ultracoproduct continuum, an interval is **bracket-regular** if it has a set of regular bracket points.

Theorem 3.3. *Let $x_i, y_i \in X_i$, $i \in I$. Then $[\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}] \supseteq \sum_{\mathcal{D}}[x_i, y_i]$. Moreover, if there exists a natural number $n \geq 1$ such that each interval $[x_i, y_i]$ is the intersection of at most n subcontinua of X_i, then $[\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}] = \sum_{\mathcal{D}}[x_i, y_i]$. In particular, bracket-regular intervals are regular sets, and $[\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}]$ is connected if and only if*

$$\{i \in I : [x_i, y_i] \text{ is connected}\} \in \mathcal{D}.$$

Proof. Let $K \subseteq \vec{X}_{\mathcal{D}}$ be any subcontinuum containing $\{\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}\}$. By Theorem 3.2, $K = \bigcap \mathcal{R}(K)$. If $\vec{L}_{\mathcal{D}} \in \mathcal{R}(K)$, then we have $\{i \in I : \{x_i, y_i\} \subseteq L_i\} \in \mathcal{D}$. Thus $\{i \in I : [x_i, y_i] \subseteq L_i\} \in \mathcal{D}$ too. Hence $\sum_{\mathcal{D}}[x_i, y_i] \subseteq \vec{L}_{\mathcal{D}}$. This gives us $\sum_{\mathcal{D}}[x_i, y_i] \subseteq K$, and we infer that $\sum_{\mathcal{D}}[x_i, y_i] \subseteq [\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}]$.

Suppose that each $[x_i, y_i]$ is the intersection of at most n subcontinua of X_i; without loss of generality we may assume $n = 2$, and write $[x_i, y_i]$ as the intersection $K_i \cap M_i$ of subcontinua of X_i, $i \in I$. Then $\vec{K}_{\mathcal{D}}$ and $\vec{M}_{\mathcal{D}}$ are subcontinua of $\vec{X}_{\mathcal{D}}$ containing $\{\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}\}$; so we have

$$\sum_{\mathcal{D}}[x_i, y_i] \subseteq [\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}] \subseteq \vec{K}_{\mathcal{D}} \cap \vec{M}_{\mathcal{D}} = \sum_{\mathcal{D}}(K_i \cap M_i) = \sum_{\mathcal{D}}[x_i, y_i],$$

and the desired equality holds. □

Question 3.4. Are bracket-regular intervals always regular sets?

The next result is an immediate corollary of Theorem 3.3. Recall that a continuum is **unicoherent** if it cannot be the union of two subcontinua whose intersection is disconnected; it is **hereditarily unicoherent** if each of its subcontinua is unicoherent. It is easily shown that a continuum is hereditarily unicoherent if and only if each of its intervals is connected.

Corollary 3.5. *Let $x_i, y_i \in X_i$, $i \in I$, where each X_i is a hereditarily unicoherent continuum. Then $[x_{\mathcal{D}}, y_{\mathcal{D}}] = \sum_{\mathcal{D}}[x_i, y_i]$. In particular, bracket-regular intervals are regular subcontinua.*

Remark 3.6. In [22], the *layer* of a point $\mu \in \mathbb{I}_{\mathcal{D}}$ is defined to be the intersection of all bracket-regular intervals containing μ. This is clearly the point hull $R(\mu)$, by Corollary 3.5.

Ultracoproducts both preserve and reflect unicoherence of continua [8, Theorem 5.1]; also if $\{i \in I : X_i \text{ is not hereditarily unicoherent}\} \in \mathcal{D}$, then it is easy to form two regular subcontinua of $\vec{X}_{\mathcal{D}}$ with disconnected intersection. So hereditary unicoherence is reflected by the ultracoproduct construction, but we do not currently know whether it is also preserved. Corollary 3.5 provides only a weak affirmative answer; here is a second one.

Theorem 3.7. *In an ultracoproduct of hereditarily unicoherent continua, the intersection of any two semiregular subcontinua is connected. Hence any semiregular interval is connected as well.*

Proof. Assume each X_i is hereditarily unicoherent, $i \in I$, with K and M two overlapping semiregular subcontinua of $\vec{X}_{\mathcal{D}}$. By Theorem 3.2, we may write $K = \bigcap \mathcal{R}(K)$ and $M = \bigcap \mathcal{R}(M)$. If $\vec{A}_{\mathcal{D}}$ and $\vec{B}_{\mathcal{D}}$ are both in $\mathcal{R}(K) \cup \mathcal{R}(M)$, then–because each X_i is hereditarily unicoherent–$\vec{A}_{\mathcal{D}} \cap \vec{B}_{\mathcal{D}} = \sum_{\mathcal{D}}(A_i \cap B_i)$ is a subcontinuum containing $K \cap M \neq \emptyset$. Let $\mathcal{P}$ be the family of pairwise intersections of sets from

$\mathcal{R}(K) \cup \mathcal{R}(M)$. Then $\mathcal{P}$ is a downwardly directed family of subcontinua of $\vec{X}_{\mathcal{D}}$; hence $\bigcap \mathcal{P} = K \cap M$ is a subcontinuum of $\vec{X}_{\mathcal{D}}$.

Now suppose $[\mu, \nu]$ is a semiregular interval in $\vec{X}_{\mathcal{D}}$, and let $\mathcal{K}$ be the family of subcontinua of $\vec{X}_{\mathcal{D}}$ that contain $\{\mu, \nu\}$. Then each subcontinuum in $\mathcal{K}$ is semiregular; hence, by the argument above, the family $\mathcal{K}$ is downwardly directed. Thus $[\mu, \nu] = \bigcap \mathcal{K}$ is connected. □

Question 3.8. Does the ultracoproduct construction preserve hereditary unicoherence? (In the very special situation with ultra-arcs, the answer is yes: $\mathbb{H}^*$ is well known to be hereditarily unicoherent, by an old result of L. Gillman and M. Henriksen [18, Theorem 5.6]. Ultra-arcs embed in $\mathbb{H}^*$, and are therefore hereditarily unicoherent too.)

The argument in the last proof gives us information about regular hulls.

Corollary 3.9. *In an ultracoproduct of hereditarily unicoherent continua, all regular hulls of subsets are subcontinua.*

In the sequel we will be interested in whether regular hulls of subcontinua are connected; and for this, we do not need the full power of hereditary unicoherence. Given finite cardinal $n \geq 1$, define continuum X to be **hereditarily n-coherent** if the intersection of any two subcontinua of X has $\leq n$ components. (So hereditary unicoherence is synonymous with hereditary 1-coherence; simple closed curves are hereditarily 2-coherent.)

Theorem 3.10. *Let $n \geq 1$ be finite. In an ultracoproduct of hereditarily n-coherent continua, all regular hulls of subcontinua are subcontinua.*

Proof. Let K be a subcontinuum of $\vec{X}_{\mathcal{D}}$, where each constituent continuum is hereditarily n-coherent. It suffices to show that the collection $\mathcal{R}(K)$ of regular subcontinua containing K is downwardly directed. But if $M = \vec{P}_{\mathcal{D}}$ and $N = \vec{Q}_{\mathcal{D}}$ are in $\mathcal{R}(K)$, then $M \cap N = \sum_{\mathcal{D}}(P_i \cap Q_i)$. Because of hereditary n-coherence, we have some $1 \leq m \leq n$ such that for $\mathcal{D}$-almost every $i \in I$, $P_i \cap Q_i = P_{i,1} \cup \cdots \cup P_{i,m}$, a union of m pairwise disjoint subcontinua of X_i. Thus $M \cap N = L_1 \cup \cdots \cup L_m$, where L_j is the subcontinuum $\sum_{\mathcal{D}} P_{i,j}$. For some unique $k \in \{1, \ldots, m\}$, we have $K \subseteq L_k$; hence $L_k \in \mathcal{R}(K)$. Thus $\mathcal{R}(K)$ is downwardly directed, and $R(K) = \bigcap \mathcal{R}(K)$ is a subcontinuum of $\vec{X}_{\mathcal{D}}$. □

A continuum X is **irreducible about** $S \subseteq X$ if no proper subcontinuum of X contains S. (So X is irreducible about $\{a, b\}$ just in case $X = [a, b]$.) X is **irreducible** if it is irreducible about some two-point subset. The next result is another easy consequence of Theorem 3.2.

Proposition 3.11. *If each continuum X_i is irreducible about $S_i \subseteq X_i$, $i \in I$, then $\vec{X}_{\mathcal{D}}$ is irreducible about $\vec{S}^{\mathcal{D}}$. In particular, if X_i is irreducible about $\{x_i, y_i\}$, then $\vec{X}_{\mathcal{D}}$ is irreducible about $\{\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}\}$; so ultracoproducts of irreducible continua are irreducible.*

Proof. If K is a proper subcontinuum of $\vec{X}_{\mathcal{D}}$ containing $\vec{S}^{\mathcal{D}}$, then K is semiregular; hence Theorem 3.2 affords us a proper regular subcontinuum $\vec{K}_{\mathcal{D}} \supseteq K$. But then

$$\{i \in I : K_i \text{ is a proper subcontinuum of } X_i \text{ containing } S_i\} \in \mathcal{D},$$

a contradiction. □

Question 3.12. To what extent is it true that the ultracoproduct construction reflects irreducibility in continua?

We will see in the next section (i.e., Remark 4.8 (i)) that $X_{\mathcal{D}}$ can be irreducible, while X is not. However, we can produce an interesting scenario in which there is an affirmative answer.

Recall that a continuous surjection between topological spaces is **monotone** if pre-images of subcontinua of the range are subcontinua of the domain. A space is **locally connected** if each point has a neighborhood base consisting of connected open sets.

Lemma 3.13. ([9, Propositions 2.2, 2.3]) *A compactum X is locally connected if and only if every codiagonal map $p_{\mathcal{D}} : X_{\mathcal{D}} \to X$ is monotone.*

Theorem 3.14. *Let X be a locally connected continuum. Then X is irreducible if and only if every (some) ultracopower of X is irreducible.*

Proof. First assume X is an irreducible continuum. Then $X_{\mathcal{D}}$ is irreducible, by Proposition 3.11, regardless of whether X is locally connected.

As for the converse, assume X is a locally connected continuum and that $X_{\mathcal{D}}$ is irreducible. By Lemma 3.13, $p_{\mathcal{D}}$ is a monotone map. If $X_{\mathcal{D}}$ is irreducible about $\{\mu, \nu\}$, then X is plainly irreducible about $\{p_{\mathcal{D}}(\mu), p_{\mathcal{D}}(\nu)\}$. $\square$

4. Composants

As promised after Lemma 3.1, we have the following.

Proposition 4.1. *For each $i \in I$, let x_i be a point in continuum X_i. Then $\prod_{\mathcal{D}} \kappa(X_i, x_i) \subseteq \kappa(\vec{X}_{\mathcal{D}}, \vec{x}_{\mathcal{D}}) \subseteq (\prod_{\mathcal{D}} \kappa(X_i, x_i))^{\sharp}$. (Hence $(\prod_{\mathcal{D}} \kappa(X_i, x_i))^{\sharp}$ is a connected dense subset of $\vec{X}_{\mathcal{D}}$.)*

Proof. Set $C_i = \kappa(X_i, x_i)$, and let $\vec{y}_{\mathcal{D}} \in \vec{C}^{\mathcal{D}}$. For $\mathcal{D}$-almost each $i \in I$, we have a proper subcontinuum $K_i \subseteq X_i$ with $x_i, y_i \in K_i$. Then $\vec{K}_{\mathcal{D}}$ is a proper subcontinuum of $\vec{X}_{\mathcal{D}}$ containing both $\vec{x}_{\mathcal{D}}$ and $\vec{y}_{\mathcal{D}}$; hence $\vec{y}_{\mathcal{D}} \in \kappa(\vec{X}_{\mathcal{D}}, \vec{x}_{\mathcal{D}})$, establishing the first inclusion.

Now suppose $\mu \in \kappa(\vec{X}_{\mathcal{D}}, \vec{x}_{\mathcal{D}})$. Then there is a proper subcontinuum K of $\vec{X}_{\mathcal{D}}$ containing both μ and $\vec{x}_{\mathcal{D}}$. K is semiregular; hence there is a proper regular subcontinuum $\vec{M}_{\mathcal{D}} \supseteq K$, by Theorem 3.2. We then have

$$\{i \in I : M_i \text{ is a proper subcontinuum of } X_i \text{ containing } x_i\} \in \mathcal{D},$$

thus $\{i \in I : M_i \subseteq C_i\} \in \mathcal{D}$; and hence $\mu \in K \subseteq \vec{M}_{\mathcal{D}} \subseteq (\vec{C}^{\mathcal{D}})^{\sharp}$. This gives us the second inclusion. The parenthetical claim is immediate because composants are always connected dense subsets. $\square$

The composant structure of a continuum is closely tied to whether the continuum is **decomposable**; i.e., expressible as the union of two of its proper subcontinua. A continuum that is not decomposable is deemed **indecomposable**. The following is an old result whose proof makes essential use of Lemma 3.1.

Proposition 4.2. ([17, Proposition 11]) *The ultracoproduct $\vec{X}_{\mathcal{D}}$ is a decomposable continuum if and only if $\{i \in I : X_i$ is a decomposable continuum$\} \in \mathcal{D}$.*

The basic facts about decomposability and composant structure are well known, and may be summarized as follows.

Lemma 4.3. *Let X be a nondegenerate continuum.*

(i) (See [24]) *If X is decomposable but not irreducible, then the only composant of X is X itself; i.e., $\kappa(X,x)=X$ for all $x \in X$.*
(ii) (See [24]) *If X is decomposable and irreducible about $\{x,y\}$, then X has exactly three composants: $\kappa(X,x)$, $\kappa(X,y)$, and X.*
(iii) (See [21]) *If X is indecomposable, then any two composants of X are disjoint. If X is also metrizable, then the number of its composants is $\mathfrak{c} := 2^{\aleph_0}$.*

Our case for the composants of ultracoproducts of decomposable continua is the following.

Theorem 4.4. *Suppose X_i is a nondegenerate decomposable continuum for $i \in I$. Then $\vec{X}_{\mathcal{D}}$ is a decomposable continuum and:*

(i) *If each X_i has three composants and $X_i = \kappa(X_i, z_i)$, then $\vec{X}_{\mathcal{D}}$ has three composants as well, and $\vec{X}_{\mathcal{D}} = \kappa(\vec{X}_{\mathcal{D}}, \vec{z}_{\mathcal{D}})$.*
(ii) *If each X_i equals the same locally connected continuum X with just one composant, then $X_{\mathcal{D}}$ has just one composant also.*

Proof. *Ad (i)*: From Lemma 4.3 (i), each X_i is irreducible. By Proposition 3.11 and Proposition 4.2, $\vec{X}_{\mathcal{D}}$ is a decomposable irreducible continuum. Now apply Lemma 4.3 (ii) to conclude that $\vec{X}_{\mathcal{D}}$ has three composants. Suppose each X_i is irreducible about $\{x_i, y_i\}$, and that z_i is a point whose composant is X_i. Then there are proper subcontinua K_i and M_i such that $\{x_i, z_i\} \subseteq K_i$ and $\{z_i, y_i\} \subseteq M_i$. Then $\{K_i, M_i\}$ is a decomposition of X_i with $z_i \in K_i \cap M_i$, and hence $\{\vec{K}_{\mathcal{D}}, \vec{M}_{\mathcal{D}}\}$ is a decomposition of $\vec{X}_{\mathcal{D}}$ with $\vec{z}_{\mathcal{D}} \in \vec{K}_{\mathcal{D}} \cap \vec{M}_{\mathcal{D}}$. Thus $\kappa(\vec{X}_{\mathcal{D}}, \vec{z}_{\mathcal{D}}) = \vec{X}_{\mathcal{D}}$.

Ad (ii): Assuming X to be locally connected, we still know that $X_{\mathcal{D}}$ is decomposable. Apply Theorem 3.14 and Lemma 4.3 (i,ii). □

Question 4.5. If each X_i is a nondegenerate continuum and $z_i \in X_i$, $i \in I$, when can we be sure that $(\prod_{\mathcal{D}} \kappa(X_i, z_i))^\sharp = \kappa(\vec{X}_{\mathcal{D}}, \vec{z}_{\mathcal{D}})$?

Remark 4.6. Let $\mathcal{D}$ be a nonprincipal ultrafilter on ω. As is proved in [22] (see also [18, Corollary 2.10]), the layers of the ultra-arc $\mathbb{I}_{\mathcal{D}}$ form an upper semicontinuous partition into subcontinua, the quotient of which is a generalized arc (i.e., a totally ordered continuum). As a consequence of the study of layers, bracket-regular intervals define their sets of bracket points; i.e., if $[\mu, \nu] = [\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}]$, then $\{\mu, \nu\} = \{\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}\}$. From this we may conclude that $\kappa(\mathbb{I}_{\mathcal{D}}, t_{\mathcal{D}}) = \mathbb{I}_{\mathcal{D}} \setminus \{(1-t)_{\mathcal{D}}\} = (\kappa(\mathbb{I}, t)^{\mathcal{D}})^\sharp$, for $t \in \{0, 1\}$, and we have a partial answer to Question 4.5.

We now turn our attention to the analysis of composants of ultracopower continua that are indecomposable. We first remark that the metrizability assumption in Lemma 4.3 (iii) is essential: D. Bellamy [12] has produced indecomposable continua, of weight $\aleph_1$, which have one and two composants.

The following is a continuum-theoretic consequence of some deep results in model theory.

Theorem 4.7. *Every nondegenerate indecomposable continuum has an ultracopower with at least $\mathfrak{c}$ composants.*

Proof. Let X be a nondegenerate indecomposable continuum, with $\mathcal{A}$ a lattice base for X. $\mathcal{A}$ is an infinite Wallman lattice; hence, by the Löwenheim-Skolem Theorem (e.g., [16, Theorem 3.1.6]), there is a countably infinite Wallman lattice A_0 elementarily equivalent to $\mathcal{A}$. By the Shelah Ultrapower Theorem [16, Theorem 6.1.15], there is a countably incomplete ultrafilter $\mathcal{D}$ such that the ultrapower lattices $\mathcal{A}^{\mathcal{D}}$ and $A_0^{\mathcal{D}}$ are isomorphic. If B is any Wallman lattice, let $\sigma(B)$ denote its "maximal spectrum space;" i.e., the compactum consisting of the maximal filters of B (topologized by taking sets of the form $\{\mu \in \sigma(B) : b \in \mu\}$, $b \in B$, as a closed-set base). Then–with "$\simeq$" denoting homeomorphism–$X \simeq \sigma(\mathcal{A})$ and $X_{\mathcal{D}} \simeq \sigma(\mathcal{A}^{\mathcal{D}})$. Since $\mathcal{A}^{\mathcal{D}}$ and $A_0^{\mathcal{D}}$ are isomorphic lattices, we have $X_{\mathcal{D}} \simeq \sigma(A_0^{\mathcal{D}}) \simeq Y_{\mathcal{D}}$, where $Y = \sigma(A_0)$.

The continuum Y is indecomposable, by Proposition 4.2. Also, since it has a countable base, it is metrizable. Using Lemma 4.3 (iii), let $S \subseteq Y$ be a subset of cardinality $\mathfrak{c}$, such that Y is irreducible about any two points of S. By Proposition 3.11, $Y_{\mathcal{D}}$ is irreducible about any two points of $S^{\mathcal{D}}$. Since $\mathcal{D}$ is countably incomplete, the cardinalities $|S^{\mathcal{D}}|$ and $|S^{\mathcal{D}}|^{\aleph_0}$ are equal [16, Proposition 4.3.9]. Since $|S| = \mathfrak{c}$ (all we need is that S is infinite), we have $|S^{\mathcal{D}}| \geq \mathfrak{c}$. Thus $Y_{\mathcal{D}}$–and hence $X_{\mathcal{D}}$–has at least $\mathfrak{c}$ composants. □

Remarks 4.8.

(i) Let us call a nondegenerate indecomposable continuum with just one composant a **Bellamy continuum**. (Bellamy continua are not all that rare; every continuum embeds as a retract of one of them [12, 25].) Regarding Question 4.5: the sets $(\prod_{\mathcal{D}} \kappa(X_i, z_i))^{\sharp}$ need not be composants at all, even if $\kappa(X_i, z_i) = X_i$ for all $i \in I$. Indeed, if X is any Bellamy continuum, X is a composant of itself. Theorem 4.7 gives us an ultracopower $X_{\mathcal{D}}$ with many composants, all disjoint from one another. Hence $X_{\mathcal{D}}$ is not a composant of itself. This example also shows that Question 3.12 has a negative answer in general, but we do not know whether the ultracoproduct construction reflects irreducibility for families of decomposable (or locally connected) continua.

(ii) Continuing with our Bellamy continuum X, let $X_{\mathcal{D}}$ be an ultracopower with many composants. Proposition 4.1 tells us that $X^{\mathcal{D}} \subseteq \kappa(X_{\mathcal{D}}, \vec{x}_{\mathcal{D}})$ for any regular point $x_{\mathcal{D}}$. Since the composants of $X_{\mathcal{D}}$ form a partition, this says that $\kappa(X_{\mathcal{D}}, \vec{x}_{\mathcal{D}}) = \kappa(X_{\mathcal{D}}, \vec{y}_{\mathcal{D}})$ for any two regular points $\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}} \in X_{\mathcal{D}}$. Thus only one composant is semiregular. If C is any of the irregular composants, with $K \subseteq C$ a subcontinuum, then the regular hull $R(K)$ is all of $X_{\mathcal{D}}$, and therefore a proper superset of K (see Theorem 3.2 for contrast).

(iii) While the semiregularity hypothesis in Theorem 3.2 cannot be discarded altogether, it is not strictly necessary: using a Martin's Axiom argument (see [18, Proposition 7.3 and Theorem 8.3]), one can show that a point hull (layer) $R(\mu)$ of $\mathbb{I}_{\mathcal{D}}$ can equal $\{\mu\}$ for an irregular point μ.

(iv) It is worthy of note that while $\mathbb{H}^*$ is well known to be indecomposable, the number of its composants is contingent upon the ambient set theory: if the CH holds this number is $2^{\mathfrak{c}}$; if the (equally consistent with ZFC) Near Coherence of Filters axiom holds, this number is exactly one, and $\mathbb{H}^*$ is a Bellamy continuum (see, e.g., [14, 18] for details).

5. Relative Composants

The notion of *relative composant* is important for the discussion of noncut points in the next section.

Let X be a continuum, with K a nonempty subcontinuum of X and $A \subseteq X$. Then the **composant of X at K relative to A** is the union of all subcontinua of $X \setminus A$ that contain K, and is denoted $\kappa(X, K; A)$. In degenerate cases we simplify notation in the obvious way; so, e.g., $\kappa(X, \{x\}; \emptyset) = \kappa(X, x)$, consistent with the usual composant notation.

If $K \neq X$, then "boundary bumping" [24] tells us that $\kappa(X, K)$ is dense in X. And while it is true that $\kappa(X, K; y)$ always contains y in its closure, it can easily fail to be dense in X. The question of when relative composants are dense was first addressed by R. H. Bing [13]; the following is an immediate corollary of the proof of Theorem 5 in that paper.

Theorem 5.1. *Let K be a proper subcontinuum of a metrizable continuum X. Then there exists a point $y \in X$ with $\kappa(X, K; y)$ dense in X.*

D. Anderson [1] defines a continuum X to be **coastal at** $x \in X$ if $\kappa(X, x; y)$ is dense in X for some $y \in X$. Theorem 5.1 says that a metrizable continuum is not only coastal at each of its points, but coastal at each of its proper subcontinua (in the obvious broader sense). In the interests of extending this result to all continua, the following "reduction" theorem is an immediate consequence of the techniques developed in [1], and is a minor improvement on [1, Corollary 4.16].

Theorem 5.2. *If all indecomposable continua are coastal at their points, than all continua are coastal at their proper subcontinua.*

Remarks 5.3.

(i) Obviously an indecomposable continuum with more than one composant is coastal at all its proper subcontinua; so in order to apply Theorem 5.2, we need only concentrate on Bellamy continua.

(ii) Continuing the discussion in Remark 4.8 (iv), $\mathbb{H}^*$ is a Bellamy continuum if and only if the Near Coherence of Filters (NCF) axiom holds; hence $\mathbb{H}^*$ is coastal at each of its proper subcontinua if NCF does not hold. On the other hand, Anderson has recently shown [2, Theorem 3.11] that $\mathbb{H}^*$ fails to be coastal at any of its proper subcontinua if NCF holds. So the question of whether Bing's Theorem 5.1 can be extended to all continua has a conditional negative answer.

We prove the following as we did Proposition 4.1.

Proposition 5.4. *For each $i \in I$, let K_i be a subcontinuum of continuum X_i, with $y_i \in X_i$. Then*

$$\prod_{\mathcal{D}} \kappa(X_i, K_i; y_i) \subseteq \kappa(\vec{X}_{\mathcal{D}}, \vec{K}_{\mathcal{D}}; \vec{y}_{\mathcal{D}}) \subseteq (\prod_{\mathcal{D}} \kappa(X_i, K_i; y_i))^{\sharp}.$$

Theorem 5.5. *For each $i \in I$ assume continuum X_i is coastal at each of its proper subcontinua. Then $\vec{X}_{\mathcal{D}}$ is coastal at each of its points and at each of its proper semiregular subcontinua.*

Proof. If K is a proper semiregular subcontinuum of $\vec{X}_{\mathcal{D}}$, then use Theorem 3.2 to obtain a proper regular subcontinuum $\vec{M}_{\mathcal{D}} \supseteq K$. For $\mathcal{D}$-almost every $i \in I$ we have $y_i \in X_i$ with $\kappa(X_i, M_i; y_i)$ dense in X_i. Then $\prod_{\mathcal{D}} \kappa(X_i, M_i; y_i)$ is dense in $\vec{X}_{\mathcal{D}}$. By Proposition 5.4, we have the density of $\kappa(\vec{X}_{\mathcal{D}}, \vec{M}_{\mathcal{D}}; \vec{y}_{\mathcal{D}})$, which is contained in $\kappa(\vec{X}_{\mathcal{D}}, K; \vec{y}_{\mathcal{D}})$. Thus $\vec{X}_{\mathcal{D}}$ is coastal at K.

Now suppose $\mu \in \vec{X}_{\mathcal{D}}$. If it happens that there is some $\nu \in \vec{X}_{\mathcal{D}} \setminus \kappa(\vec{X}_{\mathcal{D}}, \mu)$, then $\kappa(\vec{X}_{\mathcal{D}}, \mu; \nu) = \kappa(\vec{X}_{\mathcal{D}}, \mu)$, which is dense in the ultracoproduct. Hence assume that $\kappa(\vec{X}_{\mathcal{D}}, \mu) = \vec{X}_{\mathcal{D}}$, and fix a regular point $\vec{x}_{\mathcal{D}}$. Then there is a proper subcontinuum $K \subseteq \vec{X}_{\mathcal{D}}$ containing both μ and $\vec{x}_{\mathcal{D}}$. K is semiregular; so by the previous paragraph, we have $\kappa(\vec{X}_{\mathcal{D}}, K; \vec{y}_{\mathcal{D}})$ dense in $\vec{X}_{\mathcal{D}}$, for some regular point $\vec{y}_{\mathcal{D}}$. Thus $\kappa(\vec{X}_{\mathcal{D}}, \mu; \vec{y}_{\mathcal{D}}) \supseteq \kappa(\vec{X}_{\mathcal{D}}, K; \vec{y}_{\mathcal{D}})$ is also dense therein. $\square$

When we add in Bing's Theorem 5.1, we obtain the following.

Corollary 5.6. *An ultracoproduct of metrizable continua is coastal at each of its points and each of its proper semiregular subcontinua.*

Question 5.7. Can we remove "semiregular" from the conclusions of Theorem 5.5 and Corollary 5.6?

6. Varieties of Noncut Point

A topological space is **continuumwise connected** if any two of its points are contained in a subcontinuum. Each space is partitioned into its maximal continuumwise connected subsets, called the **continuum components** of the space. A point c in a connected space X is a **weak cut point** of X if $X \setminus \{c\}$ is not a continuumwise connected set. (So c is a weak cut point if and only if $c \in [a, b]$ for some $a, b \in X \setminus \{c\}$.) Clearly being a cut point implies being a weak cut point; so we say that a point is a **strong noncut point** if it is not a weak cut point.

The existence of at least two noncut points in nondegenerate metrizable continua was first proved by R. L. Moore [23], and significantly improved by G. T. Whyburn [28].

Theorem 6.1. ([24, Corollary 6.7]) *Every compact connected* T_1 *space is irreducible about its set of noncut points.*

It is well known that continua need not contain strong noncut points; indeed, any indecomposable continuum with more than one composant serves as an example. However, if the continuum is **aposyndetic**; i.e., if for each pair of its points there is a subcontinuum containing one of them in its interior and excluding the other (clearly a condition weaker than local connectedness), then weak cut points and cut points are the same. This fact is expressed as the Cut Point Equivalence Theorem in F. B. Jones' survey [19], where it is stated for metrizable continua and attributed to Whyburn [27]. The proof does not rely essentially on metric notions, however.

Theorem 6.2. *Every noncut point of an aposyndetic continuum is a strong noncut point.*

As mentioned above, an indecomposable continuum with more than one composant is evidently devoid of strong noncut points. However, in the case of Bellamy continua, the situation is a bit less clear. It is known [18] that $\mathbb{H}^*$ is an indecomposable continuum, but its number of composants can be one or many, depending on the set theory. Nevertheless, no strong noncut points exist in this continuum.

Theorem 6.3. ([2, Theorem 3.1]) *Every point of* $\mathbb{H}^*$ *is a weak cut point. Indeed, if* $z \in \mathbb{H}^*$ *is any given point, there are* $x, y \in \kappa(\mathbb{H}^*, z) \setminus \{z\}$ *with* $z \in [x, y]$.

Returning to the role of cut points and their kin to ultracoproducts, we first address the question of how the connectedness of a regular set $(\vec{S}^{\mathcal{D}})^\sharp$ relates to that of its factor sets. We know the answer if the sets S_i are closed, and the following tells what we know then they are open.

Lemma 6.4. *Suppose $U_i \subseteq X_i$ is open for $i \in I$.*

(i) *If $\{i \in I : U_i$ is disconnected$\} \in \mathcal{D}$, then $(\vec{U}^{\mathcal{D}})^\sharp$ is disconnected.*
(ii) *If $\{i \in I : U_i$ is continuumwise connected$\} \in \mathcal{D}$, then $(\vec{U}^{\mathcal{D}})^\sharp$ is connected.*

Proof. Ad (i): If for $\mathcal{D}$-almost every $i \in I$ we have a disconnection $U_i = V_i \cup W_i$, then we have a disconnection $(\vec{U}^{\mathcal{D}})^\sharp = (\vec{V}^{\mathcal{D}})^\sharp \cup (\vec{W}^{\mathcal{D}})^\sharp$ because the ultraproduct formation commutes with finite Boolean operations.

Ad (ii): Suppose $\vec{x}_{\mathcal{D}}$ and $\vec{y}_{\mathcal{D}}$ are two regular points in $(\vec{U}^{\mathcal{D}})^\sharp$. Then for $\mathcal{D}$-almost every $i \in I$ we have a subcontinuum $K_i \subseteq U_i$ containing both x_i and y_i. But then $(\vec{K}^{\mathcal{D}})^\sharp = \vec{K}_{\mathcal{D}}$ is a subcontinuum of $(\vec{U}^{\mathcal{D}})^\sharp$ containing both $\vec{x}_{\mathcal{D}}$ and $\vec{y}_{\mathcal{D}}$. If we now write $(\vec{U}^{\mathcal{D}})^\sharp$ as a union $V \cup W$ of open subsets of $\vec{X}_{\mathcal{D}}$, we use the density of $\vec{U}^{\mathcal{D}}$ in $(\vec{U}^{\mathcal{D}})^\sharp$ to find regular points $\vec{x}_{\mathcal{D}} \in V$ and $\vec{y}_{\mathcal{D}} \in W$. The existence of the subcontinuum $\vec{K}_{\mathcal{D}}$ as argued above tells us that V and W cannot be disjoint. Hence $(\vec{U}^{\mathcal{D}})^\sharp$ is connected. □

Theorem 6.5. *Assume each X_i is a continuum, with $c_i \in X_i$, $i \in I$.*

(i) *If $\{i \in I : c_i$ is a weak cut point of $X_i\} \in \mathcal{D}$, then $\vec{c}_{\mathcal{D}}$ is a weak cut point of $\vec{X}_{\mathcal{D}}$.*
(ii) *If $\{i \in I : c_i$ is a cut point of $X_i\} \in \mathcal{D}$, then $\vec{c}_{\mathcal{D}}$ is a cut point of $\vec{X}_{\mathcal{D}}$.*
(iii) *If $\{i \in I : c_i$ is a strong noncut point of $X_i\} \in \mathcal{D}$, then $\vec{c}_{\mathcal{D}}$ is a noncut point of $\vec{X}_{\mathcal{D}}$.*

Proof. Ad (i): Assume that for $\mathcal{D}$-almost each $i \in I$ there are points $a_i, b_i \in X_i \setminus \{c_i\}$ such that $c_i \in [a_i, b_i]$. Since $\vec{a}_{\mathcal{D}}$ and $\vec{b}_{\mathcal{D}}$ are both in $\vec{X}_{\mathcal{D}} \setminus \{\vec{c}_{\mathcal{D}}\}$, it suffices to show $\vec{c}_{\mathcal{D}} \in [\vec{a}_{\mathcal{D}}, \vec{b}_{\mathcal{D}}]$. But $\vec{c}_{\mathcal{D}} \in \sum_{\mathcal{D}}[a_i, b_i]$, and this set is contained in $[\vec{a}_{\mathcal{D}}, \vec{b}_{\mathcal{D}}]$, by Theorem 3.3.

Ad (ii): This follows immediately from Lemma 6.4 (i).

Ad (iii): This follows immediately from Lemma 6.4 (ii). □

Combining the above with Theorem 6.2 quickly affords the following.

Corollary 6.6. *Assume each X_i is an aposyndetic continuum, with $c_i \in X_i$, $i \in I$. Then $\vec{c}_{\mathcal{D}}$ is a cut point of $\vec{X}_{\mathcal{D}}$ if and only if $\{i \in I : c_i$ is a cut point of $X_i\} \in \mathcal{D}$.*

We round out this section with a push toward improvements of Theorem 6.5 (iii). The two obvious ones–when we replace "strong noncut" with "noncut," and *vice versa*–are open questions, as far as we know. However there is an interesting condition on points that interpolates between the stated ones.

If $c \in X$, we say c is a **nonblock point** of X if some continuumwise connected subset of $X \setminus \{c\}$ is dense in X. If X is coastal at x, then any $y \in X$ for which $\kappa(X, x; y)$ is dense is a nonblock point; conversely, if $A \subseteq X \setminus \{y\}$ is continuumwise connected and dense in X, then X is coastal at any $x \in A$. So a continuum is coastal at some point if and only if it has a nonblock point. Clearly every strong noncut point is nonblock, and every nonblock point is noncut.

Remarks 6.7.

(i) Nonblock points are first identified in [15] as a direct response to the paper [20] of R. Leonel, in which Theorem 5.1 is used to show the existence of at least two shore points in every nondegenerate metrizable continuum. While the definition in [20] is formulated in hyperspace metric terms, one may also use more topological language: $c \in X$ is a **shore point** if whenever $\mathcal{U}$ is a finite family of nonempty open subsets of X, there is a subcontinuum of $X \setminus \{c\}$ that meets each member of $\mathcal{U}$. (Intuitively, this says that "arbitrarily large" subcontinua of X miss c.) The authors of [15] show the notion of *shore point* to interpolate strictly between those of *nonblock point* and *noncut point*; they then use Bing's Theorem 5.1 to observe that any metrizable continuum is irreducible about its set of nonblock points.

(ii) If X is an indecomposable continuum with more than one composant, then each point is a nonblock point which is also weak cut.

(iii) If X results from the disjoint union of two $\sin(\frac{1}{x})$ curves, with the vertical segments identified in the obvious way, and if c is any point on that common segment, then c is a noncut point which fails to be shore. Hence being nonblock interpolates strictly between being strongly noncut and being shore.

(iv) In addition to the reduction result Theorem 5.2, Anderson [1] also extended Bing's Theorem 5.1 to separable continua. Hence every separable continuum (e.g., $\beta(\mathbb{H})$) is irreducible about its set of nonblock points. The existence of nonblock points in arbitrary continua, however, is not provable in ZFC (see Remark 5.3 (ii)).

Our strengthening of Theorem 6.5 (iii) is now the following easy consequence of Proposition 5.4.

Corollary 6.8. *Assume each X_i is a continuum, with $c_i \in X_i$, $i \in I$. If $\{i \in I : c_i$ is a nonblock point of $X_i\} \in \mathcal{D}$, then $\vec{c}_{\mathcal{D}}$ is a nonblock point of $\vec{X}_{\mathcal{D}}$.*

Proof. For $\mathcal{D}$-almost every $i \in I$, we have the existence of a point $a_i \in X_i$, with $\kappa(X_i, a_i; c_i)$ dense in X_i. By Proposition 5.4, $\kappa(\vec{X}_{\mathcal{D}}, \vec{a}_{\mathcal{D}}; \vec{c}_{\mathcal{D}})$ is dense in $\vec{X}_{\mathcal{D}}$, making $\vec{c}_{\mathcal{D}}$ a nonblock point of the ultracoproduct. □

Now we put Theorem 6.8 together with Proposition 3.11.

Corollary 6.9. *Assume, for each $i \in I$, that X_i is a continuum that is irreducible about its set of nonblock points. Then so is $\vec{X}_{\mathcal{D}}$.*

With another appeal to Theorem 5.1, we then have the following.

Corollary 6.10. *An ultracoproduct of metrizable continua is irreducible about its set of nonblock points.*

When we add in the fact that each continuum has an ultracopower which is homeomorphic to an ultracopower of a metrizable continuum (see the proof of Theorem 4.7), we can state a weak version of the desired nonblock point existence theorem.

Corollary 6.11. *Every continuum has an ultracopower which is irreducible about its set of nonblock points.*

Question 6.12. Is there a ZFC example of a continuum with no coastal (or nonblock) points? What about the existence of shore points? (Anderson [3] has recently shown that every point of $\mathbb{H}^*$ is a shore point.)

7. Point Hulls

In this section we return to the topic of regular hulls of an ultracoproduct continuum, focusing our attention on point hulls. As mentioned in Remarks 3.6 and 4.6, the point hulls and the layers of $\mathbb{I}_{\mathcal{D}}$ coincide, and partition the ultra-arc into subcontinua in such a way that the resulting quotient is a generalized arc. (As we saw earlier in Theorem 3.10, it is the hereditary n-coherence ($n = 1$) on the part of $\mathbb{I}$ that guarantees the connectedness of regular hulls in $\mathbb{I}_{\mathcal{D}}$, and hence the monotonicity of the associated quotient map.)

At the opposite extreme–as mentioned in Remark 4.8 (ii)–if X is a Bellamy continuum, then there is an ultracopower $X_{\mathcal{D}}$ with many composants, and therefore many points μ with $R(\mu) = X_{\mathcal{D}}$. We would like to investigate just what it takes for continuum ultracoproducts to have point hulls that behave in interesting ways. To do this we introduce an equivalence relation whose equivalence classes partition each point hull.

In any ultracoproduct continuum, we define a **subcontinuum ultraproduct** to be an ultraproduct of the form $\vec{K}^{\mathcal{D}}$, where K_i is a subcontinuum of X_i, $i \in I$. (Note that an I-sequence $\vec{K}$ gives rise to both the subcontinuum ultraproduct $\vec{K}^{\mathcal{D}}$ and its compactification $\vec{K}_{\mathcal{D}} = (\vec{K}^{\mathcal{D}})^{\sharp}$. As a topological space, a subcontinuum ultraproduct is hardly ever compact or connected.)

We recall that members of $\vec{X}_{\mathcal{D}}$ are maximal filters in the bounded lattice of all closed subset ultraproducts. However, no continuum ultraproduct other than $\vec{X}^{\mathcal{D}}$ itself is guaranteed to be a member of any given $\mu \in \vec{X}_{\mathcal{D}}$. Define two points in $\vec{X}_{\mathcal{D}}$ to be **$\mathcal{R}$-equivalent** if they contain the same continuum ultraproducts. Since $\vec{F}^{\mathcal{D}} \in \mu$ if and only if $\mu \in \vec{F}_{\mathcal{D}}$, we see that μ and ν are $\mathcal{R}$-equivalent ($\mu \sim_{\mathcal{R}} \nu$) if and only if $\mathcal{R}(\mu) = \mathcal{R}(\nu)$. From this definition it is plain that each point hull is a union of $\mathcal{R}$-(equivalence) classes; in particular the $\mathcal{R}$-class of a regular point is degenerate. Hence $\sim_{\mathcal{R}}$ partitions a nondegenerate ultracoproduct into many equivalence classes, each having empty interior (because nonempty open sets contain many regular points).

The **regularization map** is the quotient map $r_{\mathcal{D}} := r_{\vec{X},\mathcal{D}} : \vec{X}_{\mathcal{D}} \to \vec{X}_{\mathcal{D}}^{\mathcal{R}}$ from the ultracoproduct to its associated space of $\mathcal{R}$-classes. We refer to $\vec{X}_{\mathcal{D}}^{\mathcal{R}}$ as the **regularized $\mathcal{D}$-ultracoproduct** of $\vec{X}$.

Proposition 7.1. *Every regularized ultracoproduct is a connected compact* T_0 *space.*

Proof. Connectedness and compactness are immediate because $r_{\mathcal{D}}$ is a continuous surjection. Suppose $x, y \in \vec{X}_{\mathcal{D}}^{\mathcal{R}}$ are distinct points, say $x = r_{\mathcal{D}}(\mu)$ and $y = r_{\mathcal{D}}(\nu)$. Then $\mathcal{R}(\mu) \neq \mathcal{R}(\nu)$. Suppose we have $\mathcal{R}(\mu) \not\subseteq \mathcal{R}(\nu)$. Then there is a continuum ultraproduct $\vec{M}^{\mathcal{D}}$ that is a member of any $\mu' \sim_{\mathcal{R}} \mu$ but not of any $\nu' \sim_{\mathcal{R}} \nu$. For each $i \in I$, let $U_i = X_i \setminus M_i$. Then $(\vec{U}^{\mathcal{D}})^{\sharp}$ is an open neighborhood of ν which is **$\mathcal{R}$-saturated**–i.e., a union of $\mathcal{R}$-classes–and which misses μ. Hence $r_{\mathcal{D}}[(\vec{U}^{\mathcal{D}})^{\sharp}]$ is an open neighborhood of $y = r_{\mathcal{D}}(\nu)$ that misses $x = r_{\mathcal{D}}(\mu)$. This shows the regularization to be a T_0 space. □

Remark 7.2. Referring to Remark 4.8 (ii), let X be a Bellamy continuum, with $X_{\mathcal{D}}$ an ultracopower having more than one composant. Let C_R be the composant of $X_{\mathcal{D}}$ containing $X^{\mathcal{D}}$. If $\mu \in X_{\mathcal{D}} \setminus C_R$, then $\mathcal{R}(\mu) = \{X_{\mathcal{D}}\}$; hence the $\mathcal{R}$-class of μ contains $X_{\mathcal{D}} \setminus C_R$. On the other hand, if $\nu \in C_R$, then–see the proof of Theorem

5.5–there is a proper regular subcontinuum containing ν. Hence $X_{\mathcal{D}} \setminus C_R$ is a single $\mathcal{R}$-class, and $X_{\mathcal{D}}^{\mathcal{R}}$ is *a fortiori* not a T_1 space.

We next pursue conditions that ensure stronger separation properties for regularized ultracoproducts.

Lemma 7.3. *Let μ and ν be points in the ultracoproduct continuum $\vec{X}_{\mathcal{D}}$. Then the following two conditions are equivalent.*

(i) $R(\mu) \cap R(\nu) \neq \emptyset$.
(ii) *If $\vec{M}^{\mathcal{D}}$ and $\vec{N}^{\mathcal{D}}$ are subcontinuum ultraproducts such that $\vec{M}^{\mathcal{D}} \in \mu$ and $\vec{N}^{\mathcal{D}} \in \nu$, then $\vec{M}^{\mathcal{D}} \cap \vec{N}^{\mathcal{D}} \neq \emptyset$.*

Proof. Suppose (i) holds, with $\pi \in R(\mu) \cap R(\nu)$. If $\vec{M}_{\mathcal{D}} \in \mathcal{R}(\mu)$, and $\vec{N}_{\mathcal{D}} \in \mathcal{R}(\nu)$ are arbitrarily chosen, then $\pi \in \vec{M}_{\mathcal{D}} \cap \vec{N}_{\mathcal{D}}$; and so the corresponding subcontinuum ultraproducts are in the same maximal filter π. Thus $\vec{M}^{\mathcal{D}} \cap \vec{N}^{\mathcal{D}} \neq \emptyset$.

Conversely, if (ii) holds, let $\mathcal{M}$ (resp., $\mathcal{N}$) be the family of all subcontinuum ultraproducts in μ (resp., ν). Then $\mathcal{M} \cup \mathcal{N}$, as a family of elements of the lattice of all closed-set ultraproducts from $\vec{X}$, has the finite meet property, and hence extends to a maximal filter π on that lattice. Clearly $\pi \in R(\mu) \cap R(\nu)$. □

If X is a continuum and $K \subseteq X$ a subcontinuum, we say X is **n-semilocally connected at** K (abbreviated n-SLC at K) if K has arbitrarily small open neighborhoods whose complements have at most n components. (Being n-SLC *at a point* has its obvious meaning.) X is **n-SLC** if it is n-SLC at each of its subcontinua.

Remarks 7.4.

(i) Simple closed curves are 1-SLC; arcs are 2-SLC; simple triods are 3-SLC; any topological graph is n-SLC for some finite $n \geq 1$.
(ii) For infinite cardinals κ, it is more useful to define *κ-SLC* by stipulating *fewer than* –instead of *at most*–κ components. For example, being $\aleph_0$-SLC at $x \in X$ is Whyburn's notion of *semilocal connectedness* (SLC) at the point. In 1941, Jones proved that *a continuum is SLC at each of its points if and only if it is aposyndetic* (see [19, Equivalence Theorem]).
(iii) The *shrinking harmonic fan*, a dendrite in the euclidean plane, given as the union of segments $\{\langle t, \frac{t}{m}\rangle : 0 \leq t \leq \frac{1}{m}\}$, $m = 1, 2, \ldots$, is locally connected and $\aleph_0$-SLC, but not n-SLC at its vertex $\langle 0, 0\rangle$ for any finite n.
(iv) The *harmonic fan*, a dendroid given as the closure in the euclidean plane of the union of segments $\{\langle t, \frac{t}{m}\rangle : 0 \leq t \leq 1\}$, $m = 1, 2, \ldots$, is $\aleph_1$-SLC, but not $\aleph_0$-SLC at its vertex. If we add to this space the vertical line segment $\{1\} \times [0, 1]$, we obtain a 1-SLC continuum which is not locally connected.
(v) A nondegenerate indecomposable continuum fails to be $\aleph_1$-SLC at any of its proper subcontinua.

Theorem 7.5. *Suppose $n \geq 1$ is finite and each continuum X_i is n-SLC. If $\mu, \nu \in \vec{X}_{\mathcal{D}}$, then $\mu \sim_{\mathcal{R}} \nu$ if and only if $R(\mu) \cap R(\nu) \neq \emptyset$. Hence the point hulls of $\vec{X}_{\mathcal{D}}$ coincide with the $\mathcal{R}$-classes, and form a partition into nowhere dense subcompacta. In particular, $\vec{X}_{\mathcal{D}}^{\mathcal{R}}$ is a compact connected T_1 space, and $R(\mu)$ is a semiregular set if and only if μ is a regular point.*

Proof. If $\mu \sim_{\mathcal{R}} \nu$, then $R(\mu) = R(\nu)$; so one direction of the equivalence is trivial. Suppose now that $R(\mu) \cap R(\nu) \neq \emptyset$. If $\vec{M}_{\mathcal{D}} \in \mathcal{R}(\mu)$, then $\vec{M}^{\mathcal{D}}$ is a subcontinuum ultraproduct that is contained in μ. We show it is also contained in ν, by showing that it intersects every member of ν, and then using the fact that ν is a maximal filter. So, for the sake of a contradiction, let $\vec{F}^{\mathcal{D}} \in \nu$ be disjoint from $\vec{M}^{\mathcal{D}}$. Without loss of generality, we may assume $F_i \cap M_i = \emptyset$ for all $i \in I$. Using the fact that each X_i is n-SLC, there is some $1 \leq m \leq n$ such that, for $\mathcal{D}$-almost each i, we have subcontinua $K_{i,1}, \ldots, K_{i,m}$ of X_i, all disjoint from M_i, with $F_i \subseteq K_{i,1} \cup \cdots \cup K_{i,m}$. Then $\prod_{\mathcal{D}} K_{i,1} \cup \cdots \cup \prod_{\mathcal{D}} K_{i,m} \in \nu$; and–again because maximal filters are prime–therefore there is some $1 \leq k \leq m$ with $\prod_{\mathcal{D}} K_{i,k} \in \nu$. But by Lemma 7.3, we have $\vec{M}^{\mathcal{D}} \cap \prod_{\mathcal{D}} K_{i,k} \neq \emptyset$, an impossibility. Hence $\vec{M}^{\mathcal{D}} \in \nu$, and we have $\vec{M}_{\mathcal{D}} \in \mathcal{R}(\nu)$. This gives us the inclusion $\mathcal{R}(\mu) \subseteq \mathcal{R}(\nu)$; by symmetry, the reverse inclusion is also true, and we conclude that $\mu \sim_{\mathcal{R}} \nu$.

That the point hulls of $\vec{X}_{\mathcal{D}}$ form a partition into subcompacta and coincide with the $\mathcal{R}$-classes is now immediate. The point hulls are nowhere dense because they are closed, and coincide with the $\mathcal{R}$-classes (which have empty interior). $\vec{X}_{\mathcal{D}}^{\mathcal{R}}$ is a T_1 space because point pre-images under the regularization map are closed; and the last assertion follows from the fact that $R(\mu) = \{\mu\}$ whenever μ is a regular point. □

Proposition 7.6. *Any semiregular subcontinuum of $\vec{X}_{\mathcal{D}}$ is $\mathcal{R}$-saturated.*

Proof. Suppose K is a semiregular subcontinuum, with $\mu \in K$. if $\nu \in \vec{X}_{\mathcal{D}} \setminus K$, then Theorem 3.2 gives us a regular subcontinuum containing K and missing ν. Thus $\nu \not\sim_{\mathcal{R}} \mu$. □

Corollary 7.7. *Suppose $n \geq 1$ is finite and each continuum X_i is n-SLC. Then every semiregular subcontinuum of $\vec{X}_{\mathcal{D}}$ is a union of point hulls.*

Proof. Add Theorem 7.5 to Proposition 7.6. □

Corollary 7.8. *Let X be locally connected. Then the point pre-images under the codiagonal map $p_{\mathcal{D}} : X_{\mathcal{D}} \to X$ are $\mathcal{R}$-saturated, and there is a unique continuous surjection $f : X_{\mathcal{D}}^{\mathcal{R}} \to X$ such that $f \circ r_{\mathcal{D}} = p_{\mathcal{D}}$ (i.e., $p_{\mathcal{D}}$* factors through *$r_{\mathcal{D}}$). If, in addition, X is n-SLC for some finite $n \geq 1$, then the point pre-images of $p_{\mathcal{D}}$ are unions of point hulls.*

Proof. By Lemma 3.13, $p_{\mathcal{D}}$ is monotone; hence each point pre-image is a semiregular subcontinuum of $X_{\mathcal{D}}$. Now apply Proposition 7.6. For the additional assertion, use Corollary 7.7. □

Remark 7.9. Still weaker than aposyndesis for a continuum (see [11, Theorem 3.2]) is being **antisymmetric**. This means that for any triple $\langle a, b, c\rangle$ of points, with $b \neq c$, there is a subcontinuum containing a and exactly one of $\{b, c\}$. If $\vec{X}_{\mathcal{D}}$ contains a nondegenerate $\mathcal{R}$-class R, let $\langle a, b, c\rangle$ be chosen so that a is regular and $b \neq c$ are both in R. Then any subcontinuum containing a and intersecting $\{b, c\}$ must contain R, by Proposition 7.6. Thus an ultracoproduct continuum cannot be antisymmetric unless its $\mathcal{R}$-equivalence relation coincides with equality.

We now specify conditions sufficient for regularized ultracoproducts to be Hausdorff spaces.

Theorem 7.10. *Suppose $n \geq 1$ is finite and each continuum X_i is n-SLC and locally connected. Then $\vec{X}_{\mathcal{D}}^{\mathcal{R}}$ is a continuum.*

Proof. The Hausdorff condition for $\vec{X}_{\mathcal{D}}^{\mathcal{R}}$ is equivalent to the condition that the partition of $\vec{X}_{\mathcal{D}}$ into $\mathcal{R}$-classes is upper semicontinuous.

Fix point hull R and open set $U \subseteq \vec{X}_{\mathcal{D}}$ such that $R \subseteq U$. We need to find an open set $V \subseteq U$ such that $R \subseteq V$, and $S \subseteq U$ for any point hull S that intersects V. Because of n-semilocal connectedness on the part of each constituent X_i, the point hulls coincide with the $\mathcal{R}$-classes (Theorem 7.5), and hence constitute a partition of $\vec{X}_{\mathcal{D}}$.

By the definition of *point hull*, and the fact that point hulls are compact, there is a regular subcontinuum $\vec{K}_{\mathcal{D}}$ with $R \subseteq \vec{K}_{\mathcal{D}} \subseteq U$. And because $\vec{K}_{\mathcal{D}}$ is compact, there is an I-sequence $\vec{W}$ of open subsets such that $\vec{K}_{\mathcal{D}} \subseteq (\vec{W}^{\mathcal{D}})^\sharp \subseteq U$. Without loss of generality, we may assume $K_i \subseteq W_i$ for each $i \in I$. Because every X_i is locally connected, we may find a connected open set V_i such that $K_i \subseteq V_i \subseteq \mathrm{cl}_{X_i}(V_i) \subseteq W_i$. Let $V = (\vec{V}^{\mathcal{D}})^\sharp$. If S is any point hull intersecting V, then it intersects the regular subcontinuum $\sum_{\mathcal{D}} \mathrm{cl}_{X_i}(V_i)$ as well. By Corollary 7.7, S is contained in $\sum_{\mathcal{D}} \mathrm{cl}_{X_i}(V_i)$, and hence in U. □

We end this article with a partial answer to the question of when $\vec{X}_{\mathcal{D}}$ is guaranteed to have at least some nondegenerate $\mathcal{R}$-classes (and therefore nondegenerate point hulls). Toward that goal, we prove the following generalization of [18, Proposition 2.12] (attributed to Mioduszewski [22]).

Proposition 7.11. *Let $\mathcal{D}$ be a countably incomplete ultrafilter, with $\vec{X}$ an I-sequence of generalized arcs. Then not all $\mathcal{R}$-classes of $\vec{X}_{\mathcal{D}}$ are degenerate.*

Proof. By Theorem 7.5, the $\mathcal{R}$-classes and the point hulls of generalized arcs are one and the same.

For each $i \in I$, let X_i be totally ordered by $<_i$, with $<$ the ultraproduct order $\prod_{\mathcal{D}} <_i$. As mentioned earlier, $<$ gives rise to the ultraproduct topology on $\vec{X}^{\mathcal{D}}$.

In $\vec{X}^{\mathcal{D}}$, let A be a countably infinite discrete subset, ordered as a strictly $<$-increasing ω-sequence. It is known that an ultracoproduct of compacta via a countably incomplete ultrafilter is an F-space (see [4, Proposition 6.2]), and any countable subset of an F-space is C^*-embedded (see [26, Proposition 1.6.4]). Hence the closure $\overline{A} := \mathrm{cl}_{\vec{X}_{\mathcal{D}}}(A)$ contains a copy of $\beta(\omega)$; in particular $\overline{A}\backslash A$ is uncountable.

Let B consist of all $<$-upper bounds of A in $\vec{X}^{\mathcal{D}}$. For each $\vec{a}_{\mathcal{D}} \in A$ and $\vec{b}_{\mathcal{D}} \in B$, it is clear that $\overline{A} \subseteq A \cup [\vec{a}_{\mathcal{D}}, \vec{b}_{\mathcal{D}}]$; hence if $\mu \in \overline{A}\backslash A$, then $[\vec{a}_{\mathcal{D}}, \vec{b}_{\mathcal{D}}] \in \mathcal{R}(\mu)$ (see Corollary 3.5). Every member of $\mathcal{R}(\mu)$ is an interval of the form $[\vec{x}_{\mathcal{D}}, \vec{y}_{\mathcal{D}}]$; thus we have $\vec{y}_{\mathcal{D}} \in B$ and $\vec{x}_{\mathcal{D}} < \vec{a}_{\mathcal{D}}$ for some $\vec{a}_{\mathcal{D}} \in A$; so $R(\mu) = \bigcap\{[\vec{a}_{\mathcal{D}}, \vec{b}_{\mathcal{D}}] : \vec{a}_{\mathcal{D}} \in A, \vec{b}_{\mathcal{D}} \in B\}$ and contains $\overline{A} \setminus A$. □

Theorem 7.12. *Let $n \geq 1$ be finite, with $\vec{X}$ an I-sequence of hereditarily n-coherent continua, each of which contains a generalized arc. If $\mathcal{D}$ is a countably incomplete ultrafilter on I, then not all $\mathcal{R}$-classes of $\vec{X}_{\mathcal{D}}$ are degenerate.*

Proof. For each $i \in I$, let $A_i \subseteq X_i$ be a generalized arc, with $\vec{A}_{\mathcal{D}}$ the associated "generalized ultra-arc." By Proposition 7.11 there are distinct points $\mu, \nu \in \vec{A}_{\mathcal{D}}$, where μ and ν are $\mathcal{R}$-equivalent, relative to $\vec{A}_{\mathcal{D}}$. By symmetry, it suffices to show that $\mathcal{R}(\mu) \subseteq \mathcal{R}(\nu)$ (in $\vec{X}_{\mathcal{D}}$).

So let $\vec{M}^{\mathcal{D}} \in \mu$ be a subcontinuum ultraproduct. We are done once we show $\vec{M}^{\mathcal{D}} \in \nu$. By assumption, we also have $\vec{A}^{\mathcal{D}} \in \mu$; so using n-coherence in each coordinate X_i, we argue as in the proof of Theorem 3.10 to obtain some $1 \leq m \leq n$ such that for $\mathcal{D}$-almost every $i \in I$, $M_i \cap A_i = P_{i,1} \cup \cdots \cup P_{i,m}$, where each $P_{i,j}$ is a subcontinuum of X_i. Thus $\vec{M}^{\mathcal{D}} \cap \vec{A}^{\mathcal{D}} = \bigcup_{1 \leq j \leq m} \prod_{\mathcal{D}} P_{i,j}$. And since μ is a prime filter, we have $\prod_{\mathcal{D}} P_{i,k} \in \mu$ for some $1 \leq k \leq m$. But μ and ν are $\mathcal{R}$-equivalent relative to A; hence $\prod_{\mathcal{D}} P_{i,k} \in \nu$. Since $P_{i,k} \subseteq M_i$ for each $i \in I$, we have $\vec{M}^{\mathcal{D}} \in \nu$, completing the proof. □

As is well known [24, Theorem 8.23], any nondegenerate metrizable locally connected continuum contains plenty of arcs. So combining Remark 7.9 and Theorem 7.12 gives the following.

Corollary 7.13. *Let $n \geq 1$ be finite. Using a countably incomplete ultrafilter, an ultracoproduct of nondegenerate hereditarily n-coherent locally connected metrizable continua has nondegenerate $\mathcal{R}$-classes, and hence fails to be antisymmetric.*

To summarize the results of Theorems 3.10, 7.5, and 7.10, along with Corollary 7.13, we have the following.

Corollary 7.14. *Let $n \geq 1$ be finite, with $\vec{X}$ an I-sequence of locally connected continua which are n-SLC and hereditarily n-coherent. If $\mathcal{D}$ is an ultrafilter on I, then the point hulls (i.e., the $\mathcal{R}$-classes) form an upper semicontinuous partition of $\vec{X}_{\mathcal{D}}$ into nowhere dense subcontinua. If each X_i is also metrizable and $\mathcal{D}$ is countably incomplete, then some $\mathcal{R}$-classes are nondegenerate, and the ultracoproduct is not antisymmetric.*

Question 7.15. What makes the partition of $\vec{X}_{\mathcal{D}}$ into $\mathcal{R}$-classes more (or less) like that for an ultra-arc? (For example–see [18]–the layers of an ultra-arc are indecomposable subcontinua. They are also *terminal*, in the sense that any subcontinuum intersecting a layer either contains the layer or is contained within it.)

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