The Journal of Symbolic Logic
Volume 55, Number 2, June 1990

TAXONOMIES OF MODEL-THEORETICALLY DEFINED TOPOLOGICAL PROPERTIES

PAUL BANKSTON

Abstract. A *topological classification scheme* consists of two ingredients: (1) an abstract class $\mathscr{K}$ of topological spaces; and (2) a "taxonomy", i.e. a list of first order sentences, together with a way of assigning an abstract class of spaces to each sentence of the list so that logically equivalent sentences are assigned the same class. $\mathscr{K}$ is then endowed with an equivalence relation, two spaces belonging to the same equivalence class if and only if they lie in the same classes prescribed by the taxonomy. A space X in $\mathscr{K}$ is *characterized* within the classification scheme if whenever $Y \in \mathscr{K}$ and Y is equivalent to X, then Y is homeomorphic to X. As prime example, the *closed set taxonomy* assigns to each sentence in the first order language of bounded lattices the class of topological spaces whose lattices of closed sets satisfy that sentence. It turns out that every compact two-complex is characterized via this taxonomy in the class of metrizable spaces, but that no infinite discrete space is so characterized. We investigate various natural classification schemes, compare them, and look into the question of which spaces can and cannot be characterized within them.

§0. Introduction. By a "taxonomy of topological properties", we mean a set $\{P_i: i \in I\}$ of topological properties defined and indexed in some particularly well-organized way. What we have in mind here is that each P_i should be specified via the mechanisms of first order logic. As a prototype, let Φ_F be the set of first order sentences in the alphabet $\{\vee, \wedge, \bot, \top\}$ of bounded lattices. For each $\varphi \in \Phi_F$, let a topological space X have property P_φ just in case the lattice $F(X)$ of closed subsets of X satisfies φ in the usual sense of model theory [6]. The set $\{P_\varphi: \varphi \in \Phi_F\}$ is a "taxonomy" in our sense of the word, namely the *closed set taxonomy* $\mathbf{T}_F$.

More formally, we define a *taxonomy* to be a triple $\mathbf{T} = \langle \Phi, R, \models^* \rangle$, where:

(i) Φ is a set of first order sentences over an alphabet $L = L_\Phi$ of finitary relation and operation symbols;

(ii) R is a *first order representation* (see [2], [3], [4]), assigning to each topological space X an L-structure $R(X)$ in such a way that $R(X)$ and $R(Y)$ are isomorphic structures whenever X and Y are homeomorphic spaces; and

(iii) $\models^*$ is a "satisfaction relation" between L-structures and members of Φ. (We do not seek to axiomatize satisfaction relations here; however we would certainly

Received June 10, 1988; revised April 10, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03C15, 03C20, 06D99, 54D30, 54F15, 54F25.

want isomorphic structures to satisfy the same sentences, and logically equivalent sentences to have the same models.)

By a *classification scheme* we mean a pair $\langle \mathscr{K}, \mathbf{T} \rangle$, where $\mathscr{K}$ is an abstract "object class" of topological spaces and $\mathbf{T}$ is a taxonomy. $\mathbf{T}$ then classifies the members of $\mathscr{K}$ into $\mathbf{T}$-*taxa*, and we write $X \equiv_{\mathbf{T}} Y$ to mean that $R(X) \models^* \varphi$ if and only if $R(Y) \models^* \varphi$ for all $\varphi \in \Phi$. (In our example above, $\equiv_{\mathbf{T}_F}$ is just the usual elementary equivalence of closed set lattices.) We say $X \in \mathscr{K}$ is *characterized by* $\mathbf{T}$ *in* $\mathscr{K}$ if the $\mathbf{T}$-taxon of X in $\mathscr{K}$ is precisely the homeomorphism type of X. X is *finitely characterized by* $\mathbf{T}$ *in* $\mathscr{K}$ if there is a finite subset Φ_0 of Φ such that X is characterized by $\mathbf{T}_0 = \langle \Phi_0, R, \models^* \rangle$ in $\mathscr{K}$. For example, any finite discrete space is finitely characterized by $\mathbf{T}_F$ in the class $\{T_1\}$ of all spaces satisfying the T_1 separation axiom.

We identify the following general issues as central to our investigation.

(**I1**) Every finite $X \in \mathscr{K}$ is (finitely) characterized by $\mathbf{T}$ in $\mathscr{K}$.

(**I2**) There are infinite spaces $X \in \mathscr{K}$ that are (finitely) characterized by $\mathbf{T}$ in $\mathscr{K}$.

(**I3**) We can determine the cardinality of the set of $\mathbf{T}$-taxa in $\mathscr{K}$. (This number is bounded above by $\exp(\aleph_0 \cdot |L|)$.)

(**I4**) $\mathbf{T}'$ *is finer than* $\mathbf{T}$ (*relative to* $\mathscr{K}$). That is, every $\mathbf{T}$-taxon in $\mathscr{K}$ is a union of $\mathbf{T}'$-taxa in $\mathscr{K}$. (So any $X \in \mathscr{K}$ that is characterized by $\mathbf{T}$ in $\mathscr{K}$ is also characterized by $\mathbf{T}'$ in $\mathscr{K}$.)

(**I5**) If $X \in \mathscr{K}$ is characterized by $\mathbf{T}$ in $\mathscr{K}$, then X is characterized by $\mathbf{T}$ in $\mathscr{K}'$, where $\mathscr{K}' \supseteq \mathscr{K}$. (This automatically happens when, but not necessarily when, $\mathscr{K}$ is a union of $\mathbf{T}$-taxa in $\mathscr{K}'$.)

(**I6**) $\mathscr{K}$ *is dense in* $\mathscr{K}'$ (*relative to* $\mathbf{T}$), i.e. every $\mathbf{T}$-taxon in $\mathscr{K}'$ intersects $\mathscr{K}$, where $\mathscr{K} \subseteq \mathscr{K}'$. (So no space in $\mathscr{K}' \backslash \mathscr{K}$ is characterized by $\mathbf{T}$ in $\mathscr{K}'$.)

There are two taxonomies of particular interest here, and our research revolves around them. The first is the closed set taxonomy $\mathbf{T}_F$ introduced above; the second we call the *Banach space taxonomy* $\mathbf{T}_C$ (to be introduced in §2). Other taxonomies are important and do come into play, but these two are the ones of greatest importance to us. Only the second half of the paper, from Theorem 2.3 onward, contains any new material. The first half, including all results concerning the closed set taxonomy, is meant to give a partial survey of the relevant literature and to provide an appropriate setting for what is genuinely new.

§1. The closed set taxonomy. This taxonomy, as mentioned earlier, is $\mathbf{T}_F = \langle \Phi_F, F, \models \rangle$, where Φ_F is the set of first order sentences over the alphabet of bounded lattices, $F(X)$ is the lattice of closed subsets of a space X, and $\models$ is the usual notion of satisfaction from mathematical logic. The first serious study of just how much one can say about a space using only first order properties of $F(X)$ (and related structures, including the lattice $Z(X)$ of zero sets, the lattice $B(X)$ of clopen sets, and the unital ring $C(X)$ of continuous functions into the real line) can be found in [12].

Issue (**I1**) is, in a sense, a minimal condition we impose on a classification scheme (homology groups, say, fail to meet this condition), and every finite T_1 space is easily seen to be finitely characterized by $\mathbf{T}_F$ in the class $\{T_1\}$. As for (**I2**), it is an open question, raised in [12], whether there are any infinite T_1 spaces that are characterized by $\mathbf{T}_F$ in $\{T_1\}$. The authors of [12] offered the closed unit interval $\mathscr{I} =$

$[0,1]$ as a possible candidate, but added in a note that A. K. Swett [18], using techniques from the monadic theory of orderings [17], found a nonmetrizable σ-compact first countable linearly orderable space X such that $X \equiv_{\mathbf{T}_F} \mathscr{I}$. (Swett also showed that if X is a T_1 space such that $X \equiv_{\mathbf{T}_F} \mathscr{I}$ and, for some $2 \leq n < \omega$, $X^n \equiv_{\mathbf{T}_F} \mathscr{I}^n$, then $X \simeq \mathscr{I}$.)

When one restricts the object class $\mathscr{K}$ to be {metrizable}, more positive answers are available. Recall that a *continuum* is a connected compact Hausdorff space; a *Peano continuum* is a locally connected metrizable continuum; an *arc* is any space homeomorphic to $\mathscr{I}$; a *simple closed curve* is any space homeomorphic to the unit circle $\mathscr{S}$; a *2-cell* is a space homeomorphic to the unit square $\mathscr{I}^2$; a *2-sphere* is a space homeomorphic to the standard 2-sphere $\mathscr{S}^2$ in $\mathbf{R}^3$; and a *simple triod* consists of three arcs joined at a common endpoint. We collect some classical characterization results in the following:

1.1. THEOREM. (a) (R. L. MOORE [21, p. 206]) *X is an arc if and only if X is a metrizable continuum with exactly two noncut points.*

(b) (R. L. MOORE [21, p. 207]) *X is a simple closed curve if and only if X is a nondegenerate metrizable continuum such that each two-point subset separates X.*

(c) (R. L. MOORE [16, p. 218]) *If X is a nondegenerate Peano continuum containing no simple triod, then X is either an arc or a simple closed curve.*

(d) (L. ZIPPIN [20, p. 119]) *X is a 2-cell if and only if X is a Peano continuum containing a subcontinuum J such that*:

(i) *J is a simple closed curve*;

(ii) *X contains an arc spanning J*;

(iii) *every arc spanning J separates X*; *and*

(iv) *no closed proper subset of an arc spanning J separates X.*

(e) (L. ZIPPIN [20, p. 114]) *X is a 2-sphere if and only if X is a Peano continuum satisfying*:

(i) *X contains a simple closed curve*;

(ii) *every simple closed curve separates X*; *and*

(iii) *no subarc of a simple closed curve separates X.*

(f) (K. KURATOWSKI [14, p. 531]) *X is a 2-sphere if and only if X is a nondegenerate Peano continuum satisfying*:

(i) *X has no cut points*; *and*

(ii) *if C_0, C_1 are two subcontinua such that $C_0 \cap C_1$ is disconnected, then $C_0 \cup C_1$ separates X.*

It is easy to see that connectedness and local connectedness are first order properties of the closed set lattice of a space. Thus, there are sentences φ_K, $K \in \{\mathscr{I}, \mathscr{I}^2, \mathscr{S}, \mathscr{S}^2\}$ such that for any compact metrizable X, $F(X) \models \varphi_K$ if and only if $X \simeq K$. The authors of [12] went beyond this easy application of 1.1 (addressing issue (**I5**)).

1.2. THEOREM (Henson et al. [12]). *The spaces $\mathscr{I}$, $\mathscr{I}^2$, $\mathscr{S}$, $\mathscr{S}^2$ are all finitely characterized by $\mathbf{T}_F$ in {metrizable}.*

The main ingredient used to obtain 1.2 is the following. Let ψ be the sentence in the language of bounded lattices that expresses the Hausdorff axiom, as well as the assertion that each closed discrete subset is contained in a subarc. If X is metrizable, then $F(X) \models \psi$ if and only if X is compact metrizable, and each finite subset of X is contained in a subarc of X. Every Peano continuum fulfills these criteria (just use the well-known proof [21] that Peano continua are arcwise connected); consequently we observe the following.

1.3. PROPOSITION. *If X is any Peano continuum (finitely) characterized by $\mathbf{T}_F$ in $\{$compact metrizable$\}$, then X is (finitely) characterized by $\mathbf{T}_F$ in $\{$metrizable$\}$.*

1.4. REMARK. *$\{$compact metrizable$\}$ is not a union of $\mathbf{T}_F$-taxa in $\{$metrizable$\}$. In [12] it is shown that if X is the one-point compactification of a countable discrete space and Y is the disjoint union of X with a countable discrete space, then $X \equiv_{\mathbf{T}_F} Y$.*

One can easily extend 1.2 as follows.

1.5. THEOREM. *Every compact 2-complex is finitely characterized by $\mathbf{T}_F$ in $\{$metrizable$\}$.*

PROOF. A compact 2-complex consists of finitely many closed subsets, the designated simplices. A 2-simplex, conceived as a triangular region, has three bounding arcs. For convenience, no loops are allowed; and each ingredient has its relativized first order description, by 1.2. The simple set-theoretic incidence relations that occur among these three types of simplex completely determine the topological structure of the 2-complex. ■

Since compact 2-manifolds are triangulable [15], 1.3 applies to these spaces as well. A question raised in [12] is whether every σ-compact n-manifold is characterized by $\mathbf{T}_F$ in $\{\sigma$-compact n-manifolds$\}$.

The problem of finding nonhomeomorphic $\mathbf{T}_F$-equivalent T_1 spaces is more difficult than one might expect at first. Although it is relatively easy to prove that any two infinite discrete spaces are $\mathbf{T}_F$-equivalent (and indeed have $\mathbf{T}_F$-equivalent one-point compactifications) [12], the problem becomes much harder once we impose conditions on the spaces. For example, the authors of [12] use techniques from the monadic second order theory of orderings to construct two nonhomeomorphic $\mathbf{T}_F$-equivalent countable Boolean (= zero-dimensional compact Hausdorff) spaces. They also ask whether one can get examples that are zero-dimensional separable metric without isolated points, and we ask the same question for Peano continua. Conceivably every Peano continuum is $\mathbf{T}_F$-characterized in $\{$Peano continua$\}$ (and hence in $\{$metrizable$\}$, by 1.3).

For issues (I3) and (I6) ((I4) is irrelevant here), we continue to have more questions than answers. One interesting problem is to find the number of $\mathbf{T}_F$-taxa in $\{$Boolean$\}$. We believe it is $c = \exp(\aleph_0)$. (The number of $\mathbf{T}_F$-taxa in $\{$Peano continua$\}$ is c; this follows from a stronger result that we prove in §2.)

Finally, we would like to find interesting classes $\mathscr{K}$ that are dense in $\{T_1\}$ (relative to $\mathbf{T}_F$). (An *uninteresting* class would result by letting $\mathscr{K}$ be the complement in $\{T_1\}$ of the homeomorphism type of a space X that is not characterized by $\mathbf{T}_F$ in $\{T_1\}$.) An intriguing question along these lines concerns so-called "Löwenheim numbers". Let $I(\cdot)$ be a cardinal invariant of a space (i.e., $I(X)$ is a cardinal number; $I(X) = I(Y)$ whenever $X \simeq Y$). Cardinality and weight are the most popular cardinal invariants, and there is a host of others of interest to topologists. (The *weight* $w(X)$, defined to be $\aleph_0$ + the smallest cardinality of a possible basis for the topology of X, is in many ways the "right" topological analogue to the cardinality of a relational structure. In both the Stone duality between Boolean spaces and Boolean lattices, and the Pontryagin duality between compact abelian groups and discrete abelian groups, weight is dual to cardinality). Is there a smallest cardinal κ such that every T_1 space

X is $\mathbf{T}_F$-equivalent to a T_1 space Y with $I(Y) = \kappa$ (or perhaps $\leq \kappa$)? And if so, can we determine κ? There is a nice discussion of this issue in [12], and what answers there are are somewhat disappointing. Of course there is always a smallest κ such that each X is $\mathbf{T}_F$-equivalent to a space of I-invariant $\leq \kappa$, for there are at most c $\mathbf{T}_F$-equivalence classes in $\{T_1\}$ (exactly c, in fact). κ can be taken to be the supremum of all the cardinals $\kappa(X) = \min\{\lambda\colon I(Y) = \lambda,\ Y \equiv_{\mathbf{T}_F} X\}$. In the case of weight, let X be any regular T_1 space that is not normal. Since any space $Y \equiv_{\mathbf{T}_F} X$ is also non-normal, we know by the Urysohn metrization theorem that $w(Y) > \aleph_0$. Thus the Löwenheim number for weight is uncountable. A more sophisticated argument in [12] shows that this Löwenheim number is at least $\aleph_\omega$. We will see in the next section that the corresponding situation for the Banach space taxonomy is much more satisfactory.

§2. The Banach space taxonomy. This taxonomy is much weaker than $\mathbf{T}_F$, and it is a surprise that so much can be done with it by way of characterizing spaces. The *Banach space taxonomy* is the triple $\mathbf{T}_C = \langle \Phi_C, C, \models^* \rangle$, where:

(i) Φ_C is the set of "positive-bounded" sentences [11] over the alphabet L of Banach spaces. (To elaborate, L consists of a binary operation $+$ for vector addition, a constant 0 for the zero vector, and a unary operation of scalar multiplication for each rational scalar; also L has two unary predicate symbols P and Q. Px says that a vector x has length ≤ 1; Qx says that x has length ≥ 1. Formulas are built up from equations and atomic expressions involving P and Q, applied to vector polynomials. The building process uses only disjunction, conjunction, and the bounded quantifiers that restrict quantification to vectors of length ≤ 1.)

(ii) For any space X, $C(X)$ is the Banach space of bounded continuous real-valued functions on X. (That is, one defines the vector operations pointwise, and the length of a vector is simply the supremum of its real values. $C(X)$ is then easily seen to be an L-structure.)

(iii) For any L-structure $\mathscr{A}$ and positive-bounded sentence φ, $\mathscr{A} \models^* \varphi$ is the relation of "approximate" satisfaction; $\mathscr{A}$ really satisfies every "approximation" φ_m of φ, $m = 1, 2, \ldots$. (More specifically, given any positive-bounded formula φ, φ_m is defined by induction: If φ is $x = y$ then φ_m is $Pm(x - y)$; if φ is Px then φ_m is $P(1 - 1/m)x$; if φ is Qx then φ_m is $Q(1 + 1/m)x$; $(\varphi \vee \psi)_m$ is $\varphi_m \vee \psi_m$; $(\varphi \wedge \psi)_m$ is $\varphi_m \wedge \psi_m$; $(\exists x(Px \wedge \varphi))_m$ is $\exists x(Px \wedge \varphi_m)$; and $(\forall x(Px \to \varphi))_m$ is $\forall x(Px \to \varphi_m)$.)

A main result of [11] is that two Banach spaces $\mathscr{A}$ and $\mathscr{B}$ approximately satisfy the same positive-bounded sentences if and only if $\mathscr{A}$ and $\mathscr{B}$ have isometrically isomorphic Banach ultrapowers. (Briefly, one forms the Banach ultraproduct $\prod_{\mathscr{D}}^{B} \mathscr{A}_i$ by first taking the usual ultraproduct, then removing the elements of infinite norm, and finally identifying two elements if they are infinitely close.) In the case where the Banach spaces are of the form $C(X)$ for X a Tichonov ($=$ completely regular T_1) space, the classical Banach-Stone theorem allows one to recover the topological structure of the Stone-Čech compactification $\beta(X)$ from the Banach space structure of $C(X)$.

In [11] Henson shows that the Banach ultrapower $\prod_{\mathscr{D}}^{B} C(X)$ is of the form $C(\hat{X})$, where $\hat{X}$ is a compact Hausdorff space. In [2] and [3] we study the ultracoproduct construction $\sum_{\mathscr{D}} X_i$ for Tichonov spaces, show this is naturally homeomorphic to

$\sum_{\mathscr{D}} \beta(X_i)$, and further that $\sum_{\mathscr{D}} X$ is precisely Henson's $\hat{X}$. In general, $C(\sum_{\mathscr{D}} X_i)$ is isometrically isomorphic to $\prod_{\mathscr{D}}^{B} C(X_i)$. Putting these facts together yields immediately the following.

2.1. THEOREM. *Let X and Y be Tichonov spaces. The following are equivalent*:

(a) $X \equiv_{\mathbf{T}_C} Y$.

(b) *$C(X)$ and $C(Y)$ have isometrically isomorphic Banach ultrapowers.*

(c) *X and Y have homeomorphic ultracopowers.*

If condition (c) above holds for Tichonov spaces X and Y, then we say the spaces are *co-elementarily equivalent* and write $X \equiv Y$. In [2] we proved that if $X \equiv_{\mathbf{T}_F} Y$, then $X \equiv Y$ (so $\mathbf{T}_F$ is finer than $\mathbf{T}_C$ (relative to {Tichonov})). The converse is false, as many different counterexamples witness [3]. For the class of strongly zero-dimensional spaces (i.e., those with Boolean Stone-Čech compactifications) co-elementary equivalence is the same as $\mathbf{T}_B$-equivalence, where $\mathbf{T}_B$ is the clopen set taxonomy. (This follows from the felicitous isomorphism $B(\sum_{\mathscr{D}} X_i) \cong \prod_{\mathscr{D}} B(X_i)$. Another consequence of this isomorphism is that both connectedness and strong zero-dimensionality are preserved and reflected by ultracoproducts.) [10] treats questions concerning the model-theoretic relationship between $\mathbf{T}_B$ and $\mathbf{T}_C$ for the object class {Boolean}. Our treatment follows along the lines of [2], [3], [4], [5] and [9], differing in approach from [10] and [11] in that we place no emphasis on the model theory of Banach spaces. In particular, we do not address questions of finite characterization by $\mathbf{T}_C$. (This is due to lack of techniques, not lack of interest.)

Every finite T_1 space is characterized by $\mathbf{T}_C$ in {Tichonov}, but no infinite space is since ultracopowers of infinite spaces can be arbitrarily large. Also {compact Hausdorff} is dense in {Tichonov} (relative to $\mathbf{T}_C$) since $X \equiv \beta(X)$ always holds. In [9] (and, subsequently, in a slightly stronger form in [4]) it is proved that {compact metrizable} is dense in {compact Hausdorff}. Thus {Tichonov} has Löwenheim number $\aleph_0$ relative to $\mathbf{T}_C$. For the remainder of this section, we restrict our attention to object classes $\mathscr{K} \subseteq$ {compact Hausdorff}. (Again, this is not due to lack of interesting questions.)

The positive characterization results so far are modest; what we know appears in [3] and [5]. Specifically we have

2.2. THEOREM. (i) *If $\mathscr{B}$ is any countable Boolean lattice whose first order theory is $\aleph_0$-categorical, then the Stone space $\omega(\mathscr{B})$ is characterized by $\mathbf{T}_C$ in {compact metrizable}. (Examples of such spaces include the disjoint union of the Cantor discontinuum with any finite discrete space.)*

(ii) *Any finite disjoint union of arcs and simple closed curves is characterized by $\mathbf{T}_C$ in {locally connected compact metrizable}.*

Like Theorems 1.2 and 1.5, 2.2(ii) uses the classical results of 1.1, namely part (c). The other parts of 1.1, so useful in questions involving $\mathbf{T}_F$-characterizability, have so far defied application in the context of $\mathbf{T}_C$.

In order to pave the way for our new results, we follow a lattice-theoretic approach to the ultracoproduct construction, à la [9] and [13]. Let $\mathscr{A} = \langle A, \vee, \wedge, \bot, \top \rangle$ be a bounded distributive lattice. Denote by $\omega(\mathscr{A})$ the set of all maximal proper filters in $\mathscr{A}$. For each $a \in A$, set $a^{\#} = \{p \in \omega(\mathscr{A}) \colon a \in p\}$. Then $\bot^{\#} = \varnothing$; $\top^{\#} = \omega(\mathscr{A})$; $(a \wedge b)^{\#} = a^{\#} \cap b^{\#}$; and $(a \vee b)^{\#} = a^{\#} \cup b^{\#}$. Let $A^{\#} = \{a^{\#} \colon a \in A\}$, and $\mathscr{A}^{\#} = \langle A^{\#}, \cup, \cap, \varnothing, \omega(\mathscr{A}) \rangle$. Then $A^{\#}$ forms a closed set basis for a compact T_1 topology on $\omega(\mathscr{A})$ (the *Stone space* of $\mathscr{A}$), and the mapping $a \mapsto a^\#$ is a lattice homomorphism of $\mathscr{A}$ onto $\mathscr{A}^\#$.

A bounded distributive lattice $\mathscr{A}$ is *normal* if, given $a, b \in A$ with $a \wedge b = \perp$, there are $a', b' \in A$ with $a \wedge a' = b \wedge b' = \perp$ and $a' \vee b' = \top$. If $\mathscr{A}$ is normal, then its Stone space is compact Hausdorff. Define $\mathscr{A}$ to be *separated* if whenever $a, b \in A$ are distinct, then either there is some $a' \leq a$ with $a' \neq \perp$ and $a' \wedge b = \perp$, or there is some $b' \leq b$ with $b' \neq \perp$ and $b' \wedge a = \perp$. It is easy to see that if $\mathscr{A}$ is separated, then the homomorphism $a \mapsto a^\#$ is an isomorphism beteen $\mathscr{A}$ and $\mathscr{A}^\#$.

A bounded distributive lattice is a *Wallman lattice* if it is both normal and separated. If X is any topological space, $F(X)$ is a normal lattice if and only if X is a normal space. If X is a T_1 space, $F(X)$ is separated.

Let X be a topological space. A *Wallman basis* for X is a normal sublattice $\mathscr{B}$ of $F(X)$ satisfying the condition that if $x \in X$ and $C \in F(X)$ does not contain x, then there are $A, B \in \mathscr{B}$ with $x \in A$, $C \subseteq B$, and $A \cap B = \varnothing$. It is well known [19] that a space X is Tichonov if and only if it has a Wallman basis. Clearly if $\mathscr{B} \subseteq F(X)$ is a closed set basis that is also a Wallman lattice, then $\mathscr{B}$ is a Wallman basis for X. It is also easy to check the converse: if $\mathscr{B}$ is a Wallman basis for X, then $\mathscr{B}$ is a Wallman lattice. (The lattice $Z(X)$ of zero sets is always a Wallman basis for a Tichonov space X.) Finally, if X is compact Hausdorff, then every closed set basis for X that is a sublattice of $F(X)$ is also a Wallman basis for X.

Let $\mathscr{A}$ and $\mathscr{B}$ be bounded distributive lattices, with $\alpha\colon \mathscr{A} \to \mathscr{B}$ a homomorphism. Let $q \in \omega(\mathscr{B})$ and set $\alpha^\omega(q) = \{a \in A\colon \alpha(a) \in q\}$. Then $\alpha^\omega(q)$ is a proper prime filter in $\mathscr{A}$. If $\mathscr{A}$ is normal, then prime filters extend to unique maximal filters, so we may regard $\alpha^\omega(q)$ as a point in $\omega(\mathscr{A})$. α^ω is then continuous; $(\alpha^\omega)^{-1}[a^\#] = (\alpha(a))^\#$. If α is one-one, α^ω is onto (and $\alpha^\omega[(\alpha(a))^\#] = a^\#$). Moreover, if α is an embedding that is "weakly separating", i.e., whenever $b_1 \wedge b_2 = \perp$ in $\mathscr{B}$ then there is some $a \in \mathscr{A}$ such that either $b_1 \leq \alpha(a)$ and $\alpha(a) \wedge b_2 = \perp$, or $b_2 \leq \alpha(a)$ and $b_1 \wedge \alpha(a) = \perp$, then α^ω is a homeomorphism. (It suffices to show α^ω is one-one. Let $q_1, q_2 \in \omega(\mathscr{B})$ be distinct, say $b_1 \in q_1$, $b_2 \in q_2$ and $b_1 \wedge b_2 = \perp$. Suppose $a \in \mathscr{A}$ satisfies $b_1 \leq \alpha(a)$ and $\alpha(a) \wedge b_2 = \perp$. Then $\alpha(a) \in q_1$, so $a \in \alpha^\omega(q_1)$. But $\alpha^\omega(a) \wedge b_2 = \perp$ implies $\alpha(a) \notin q_2$; i.e., $a \notin \alpha^\omega(q_2)$.)

If $\mathscr{A}$ is a sublattice of $\mathscr{B}$, we say $\mathscr{A}$ is *weakly separating* if the inclusion map from $\mathscr{A}$ to $\mathscr{B}$ is weakly separating in the sense above. Clearly, if X is compact Hausdorff, every closed set basis that is a sublattice of $F(X)$ is a Wallman basis that is weakly separating. Thus $\omega(\mathscr{B}) \simeq X$ for any such basis $\mathscr{B}$.

Now let I be an index set, $\mathscr{D}$ an ultrafilter on I, and $\langle X_i\colon i \in I\rangle$ an indexed family of compact Hausdorff spaces. Then $\sum_{\mathscr{D}} X_i$, the topological ultracoproduct, is, by definition [3], the Stone space $\omega(\prod_{\mathscr{D}} Z(X_i))$. (There are several other descriptions of $\sum_{\mathscr{D}} X_i$, but this is the most useful for us.). If $\mathscr{B}_i$ is any Wallman basis for X_i, $i \in I$, then $\prod_{\mathscr{D}} \mathscr{B}_i$ is a Wallman basis for the topological ultraproduct $\prod_{\mathscr{D}} X_i$ [1]; moreover $\prod_{\mathscr{D}} \mathscr{B}_i$ is weakly separating for $\prod_{\mathscr{D}} F(X_i)$. Consequently, $\sum_{\mathscr{D}} X_i \simeq \omega(\prod_{\mathscr{D}} \mathscr{B}_i)$.

Define two compact Hausdorff spaces X and Y to be *elementarily tolerant* (in symbols $X \doteqdot Y$) if there are Wallman bases $\mathscr{B}$ and $\mathscr{C}$ for X and Y respectively such that $\mathscr{B}$ and $\mathscr{C}$ are elementarily equivalent lattices. This relation is plainly symmetric and reflexive, but we do not know whether it is transitive in general. (Such relations are commonly called "tolerance relations".)

2.3. THEOREM. *Let X and Y be compact Hausdorff. Then $X \equiv Y$ if and only if there is a compact Hausdorff space Z such that $X \doteqdot Z$ and $Z \doteqdot Y$.*

PROOF. Suppose $X \equiv Y$, and let $\sum_{\mathscr{D}} X$ and $\sum_{\mathscr{E}} Y$ be homeomorphic ultracopowers. (We may take $\mathscr{D}$ and $\mathscr{E}$ to be the same ultrafilter, if we like [3].) Let $\mathscr{B}$ be any Wallman basis for X. Then $\mathscr{B}$ and $\prod_{\mathscr{D}} \mathscr{B}$ are elementarily equivalent, $(\prod_{\mathscr{D}} \mathscr{B})^{\#}$ is a Wallman basis for $\omega(\prod_{\mathscr{D}} \mathscr{B}) \simeq \sum_{\mathscr{D}} X$, and $\prod_{\mathscr{D}} \mathscr{B}$ is isomorphic to $(\prod_{\mathscr{D}} \mathscr{B})^{\#}$. Consequently, $X \doteqdot \sum_{\mathscr{D}} X$. Set $Z = \sum_{\mathscr{D}} X$. Then $Z \doteqdot Y$ also (via a possibly different Wallman basis for Z).

Conversely, if $X \doteqdot Z$, as witnessed by elementarily equivalent Wallman bases $\mathscr{B} \subseteq F(X)$ and $\mathscr{C} \subseteq F(Z)$, then there are isomorphic ultrapowers $\prod_{\mathscr{D}} \mathscr{B}$ and $\prod_{\mathscr{E}} \mathscr{C}$. Consequently, their Stone spaces $\sum_{\mathscr{D}} X$ and $\sum_{\mathscr{E}} Y$ are homeomorphic, so $X \equiv Z$. Thus, if $X \doteqdot Z$ and $Z \doteqdot Y$, then $X \equiv Y$. ∎

2.4. COROLLARY. *The transitive closure of the tolerance relation $\doteqdot$ is the relational composition $\doteqdot \circ \doteqdot$.*

Putting 2.3 with 2.1, we have

2.5. COROLLARY. *Let X and Y be compact Hausdorff spaces. The following are equivalent:*

(a) $X \equiv_{\mathbf{T}_C} Y$.

(b) *$C(X)$ and $C(Y)$ have isometrically isomorphic Banach ultrapowers.*

(c) $X \equiv Y$.

(d) *$X \doteqdot Z$ and $Z \doteqdot Y$ for some compact Hausdorff space Z.*

2.6. REMARK. For Boolean spaces X and Y, we know $X \equiv Y$ if and only if $X \equiv_B Y$. Thus elementary tolerance is an equivalence relation when restricted to Boolean spaces. We do not know whether elementary tolerance is transitive in general, but it is very close to being so, by 2.4.

2.7. COROLLARY. *$\mathbf{T}_F$ is finer than $\mathbf{T}_C$ (relative to* {compact Hausdorff }). *Moreover, if X is any Peano continuum that is characterized by $\mathbf{T}_C$ in {Peano continua}, then X is characterized by $\mathbf{T}_F$ in {metrizable}.*

PROOF. The first statement is immediate from 2.5. Suppose X is characterized by $\mathbf{T}_C$ in {Peano continua}. Then X is so characterized by $\mathbf{T}_F$. If Y is metrizable and $Y \equiv_{\mathbf{T}_F} X$, then Y is compact, by the remarks in the paragraph following 1.2. Y is then shown to be connected and locally connected, since these are first order properties of the closed set lattice. Therefore, Y is a Peano continuum and thus homeomorphic to X. ∎

A very powerful technique, initiated by R. Gurevič [9] for obtaining co-elementarily equivalent spaces, is to combine ultracopowers with the Löwenheim-Skolem theorem. Suppose $\mathscr{A}$ and $\mathscr{B}$ are Wallman lattices, and suppose $\varepsilon: \mathscr{A} \to \mathscr{B}$ is an elementary embedding. Then $\varepsilon^{\omega}: \omega(\mathscr{B}) \to \omega(\mathscr{A})$ is a continuous surjection as we saw earlier, but much more is true. By the ultrapower theorem, there are ultrafilters $\mathscr{D}$ and $\mathscr{E}$ and an isomorphism $\eta: \prod_{\mathscr{D}} \mathscr{A} \to \prod_{\mathscr{E}} \mathscr{B}$ such that $\eta \circ \Delta = \Delta \circ \varepsilon$, where Δ denotes the appropriate elementary embedding of a relational structure into its ultrapower. By functoriality of the Stone space operation for Wallman lattices, we obtain continuous surjections ε^{ω}, Δ^{ω} and η^{ω} such that η^{ω} is a homeomorphism and $\varepsilon^{\omega} \circ \Delta^{\omega} = \Delta^{\omega} \circ \eta^{\omega}$. Now the space $\omega(\mathscr{A})$ has Wallman basis $\mathscr{A}^{\#} \cong \mathscr{A}$. Thus $\omega(\prod_{\mathscr{D}} \mathscr{A})$ is naturally homeomorphic to $\sum_{\mathscr{D}} \omega(\mathscr{A})$. The mapping Δ^{ω} is referred to as the natural *codiagonal map* $\nabla: \sum_{\mathscr{D}} X \to X$ in [3]; $x = \nabla(p)$ if and only if for each open neighborhood U of x, the ultrapower $\prod_{\mathscr{D}} U$ contains a member of p.

The map ε^ω above, and, more generally, any map $\gamma: X \to Y$ for which there exist ultrafilters $\mathscr{D}$, $\mathscr{E}$ and a homeomorphism $\delta: \sum_{\mathscr{D}} X \to \sum_{\mathscr{E}} Y$ such that $\gamma \circ \nabla = \nabla \circ \delta$, is what we call in [3], [4] a *co-elementary map*. (Gurevič independently discovered these maps and used them to good effect in [9].) The existence of a co-elementary map from X to Y says more (but we are not sure just how much more) than merely that X and Y are co-elementarily equivalent and Y is a continuous image of X. A useful result concerning how co-elementary maps behave with subcontinua is the following.

2.8. LEMMA. *Let $\gamma: X \to Y$ be a co-elementary map between compact Hausdorff spaces, with $K \subseteq Y$ a subcontinuum. Then there is a subcontinuum $C \subseteq X$ such that $K = \gamma[C]$, and $\gamma^{-1}[U] \subseteq C$ whenever $U \subseteq K$ is open in Y.*

PROOF. Let $\delta: \sum_{\mathscr{D}} X \to \sum_{\mathscr{E}} Y$ be a homeomorphism such that $\gamma \circ \nabla = \nabla \circ \delta$, witnessing the co-elementarity of γ. We view $\sum_{\mathscr{E}} Y$ as $\omega(\prod_{\mathscr{E}} F(Y))$ and write $\sum_{\mathscr{E}} A_i$ for $(\prod_{\mathscr{E}} A_i)^\#$, where $A_i \subseteq Y$ is closed, $i \in I$. (This is justified since $(\prod_{\mathscr{E}} A_i)^\#$ is naturally homeomorphic, as a subspace, to $\sum_{\mathscr{E}} A_i$.)

Define $C = \nabla[\delta^{-1}[\sum_{\mathscr{E}} K]]$. $\sum_{\mathscr{E}} K$ is a subcontinuum of $\sum_{\mathscr{E}} Y$, so C is a subcontinuum of X. We need to show that $\gamma[C] = K$, and that if $A \subseteq Y$ is closed such that $K \cup A = Y$, then $C \cup \gamma^{-1}[A] = X$.

We first show $\nabla[\sum_{\mathscr{E}} K] = K$. If $p \in \sum_{\mathscr{E}} K$, then $\prod_{\mathscr{E}} K \in p$. $y = \nabla(p)$ if and only if whenever U is an open neighborhood of y then $\prod_{\mathscr{E}} U$ contains a member of p. So pick such a set U. Then $\prod_{\mathscr{E}} U \cap \prod_{\mathscr{E}} K \neq \varnothing$, so $U \cap K \neq \varnothing$; whence $y \in \mathrm{cl}(K) = K$. ($\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ denote closure and interior, respectively.) Therefore $\nabla[\sum_{\mathscr{E}} K] \subseteq K$. For the reverse inclusion, let $y \in K$. Then for each open neighborhood U of y, $\prod_{\mathscr{E}} \mathrm{cl}(U) \cap \prod_{\mathscr{E}} K \neq \varnothing$, so $\mathscr{S} = \{\prod_{\mathscr{E}} K\} \cup \{\prod_{\mathscr{E}} \mathrm{cl}(U): y \in U,\ U \text{ open in } Y\}$ satisfies the finite intersection property. Thus there is some $p \in \sum_{\mathscr{E}} K$, $\mathscr{S} \subseteq p$. If $y \in U$ and U is open, then there is an open neighborhood V of y with $\mathrm{cl}(V) \subseteq U$. Thus $\prod_{\mathscr{E}} \mathrm{cl}(V) \in p$, so $y = \nabla(p)$. Therefore $K \subseteq \nabla[\sum_{\mathscr{E}} K]$.

Next we show $\gamma[C] = K$. Suppose $x \in C$, say $x = \nabla(p)$, where $p \in \delta^{-1}[\sum_{\mathscr{E}} K]$. Then $\gamma(x) = \nabla(\delta(p)) \in \nabla[\sum_{\mathscr{E}} K] = K$. So $\gamma[C] \subseteq K$. Suppose $y \in K$, say $y = \nabla(q)$, where $q \in \sum_{\mathscr{E}} K$. Let $x = \nabla(\delta^{-1}(q))$. Then $\gamma(x) = \nabla(q) = y$, so $K \subseteq \gamma[C]$.

Finally suppose $A \subseteq Y$ is closed, $A \cup K = Y$. Then $\sum_{\mathscr{E}} A \cup \sum_{\mathscr{E}} K = \sum_{\mathscr{E}} Y$, so $\nabla^{-1}[A] \cup \sum_{\mathscr{E}} K = \sum_{\mathscr{E}} Y$. Suppose $x \notin \gamma^{-1}[A]$. If $x = \nabla(p)$ and $\delta(p) \in \nabla^{-1}[A]$, then $\gamma(x) = \nabla(\delta(p)) \in A$. So whenever $x = \nabla(p)$, $\delta(p)$ must be in $\sum_{\mathscr{E}} K$. Thus $p \in \delta^{-1}[\sum_{\mathscr{E}} K]$; i.e., $x \in \nabla[\delta^{-1}[\sum_{\mathscr{E}} K]] = C$. Consequently, $\gamma^{-1}[A] \cup C = X$. ■

It is worth noting that, although in 2.8 we focused on connectedness, we could have substituted any other property, as long as that property is preserved by ultracopowers and continuous images. Thus the same proof works for finiteness. In particular, if $y \in Y$ is an isolated point, then it follows that $\gamma^{-1}[y]$ is a singleton.

Let $\mathscr{K}_\kappa = \{\text{compact Hausdorff, weight} = \kappa\}$. The result we wish to work toward next says that {locally disconnected} $\cap\ \mathscr{K}_\kappa$ is dense in {infinite compact Hausdorff} (relative to $\mathbf{T}_C$) for any cardinal κ. As immediate consequences we infer: (i) that no infinite locally connected compact Hausdorff space of weight κ is characterized by $\mathbf{T}_C$ in $\mathscr{K}_\kappa$; and (ii) that if $\mathscr{K}_\kappa \cup \mathscr{K}_\lambda$ contains any infinite space that is characterized by $\mathbf{T}_C$ in $\mathscr{K}_\kappa \cup \mathscr{K}_\lambda$, then $\kappa = \lambda$. The result for the case $\kappa = \aleph_0$ is due to Gurevič [9]. In our proof below, we essentially repeat some of his arguments for the sake of expository completeness. (It must be confessed that we did not understand entirely the proof given in [9] (specifically the proof of Proposition 15 therein) until we ascertained that some form of 2.8 had been implicitly used.)

Recall that an ultrafilter $\mathscr{D}$ on I is *κ-regular* if there is a subset $\mathscr{E} \subseteq \mathscr{D}$ of cardinality κ such that each $i \in I$ is contained in only finitely many members of $\mathscr{E}$. κ-regular ultrafilters exist in abundance whenever $|I| \geq \kappa$; $\aleph_0$-regular ultrafilters are precisely those that are countably incomplete; and $|\prod_{\mathscr{D}} A| = |A|^{|I|}$ whenever $\mathscr{D}$ is $|I|$-regular and A is infinite [6].

The next result shows that topological ultracoproducts are hardly ever locally connected.

2.9. LEMMA. *Let $\mathscr{D}$ be a countably incomplete ultrafilter on I, and $\langle X_i: i \in I\rangle$ a family of compact Hausdorff spaces such that $\{i$: X_i is infinite$\} \in \mathscr{D}$. Then $\sum_{\mathscr{D}} X_i$ is not locally connected.*

PROOF. *Step* 1. (This is Lemma 13 in [9].) First assume $I = \{1, 2, \ldots\}$; $X_i = \mathscr{I}$ for all $i \in I$. To show $\sum_{\mathscr{D}} \mathscr{I}$ is not locally connected, it suffices to exhibit a point p such that the subcontinua containing p in their interiors do not form a neighborhood basis at p, i.e., $\sum_{\mathscr{D}} \mathscr{I}$ is not "connected *im kleinen*" at p. For any closed subset B of $\mathscr{I}$, let $\mu(B)$ be the standard Lebesgue measure of B. If $\langle t_i: i \in I\rangle$ is an I-sequence of members of $\mathscr{I}$, let $\lim_{\mathscr{D}}(\langle t_i: i \in I\rangle)$ be that $t \in [0, 1]$ such that for all $\varepsilon > 0$, $\{i \in I$: $t_i \in (t - \varepsilon, t + \varepsilon)\} \in \mathscr{D}$. Define

$$\mathscr{S} = \{\textstyle\prod_{\mathscr{D}} B_i \colon B_i \subseteq \mathscr{I} \text{ is closed and } \lim_{\mathscr{D}}(\langle \mu(B_i) \colon i \in I\rangle) = 1\}.$$

Then $\mathscr{S}$ is a proper filter in $\prod_{\mathscr{D}} F(\mathscr{I})$, and hence extends to a $\prod_{\mathscr{D}} F(\mathscr{I})$-ultrafilter $p \in \sum_{\mathscr{D}} \mathscr{I}$.

For $i = 1, 2, \ldots$, set

$$B_i = \bigcup_{j=1}^{i} \left[\frac{4j-3}{4i}, \frac{4j}{4i}\right] \quad \text{and} \quad C_i = \left[0, \frac{2}{4i}\right] \cup \bigcup_{j=2}^{i} \left[\frac{4j-5}{4i}, \frac{4j-2}{4i}\right].$$

Then for all $i \in I$, $\mathscr{I} = \operatorname{int}(B_i) \cup \operatorname{int}(C_i)$. Hence $\sum_{\mathscr{D}} \mathscr{I} = \operatorname{int}(\sum_{\mathscr{D}} B_i) \cup \operatorname{int}(\sum_{\mathscr{D}} C_i)$. Assume $p \in \operatorname{int}(\sum_{\mathscr{D}} B_i)$, and let A be the connected component of p in $\sum_{\mathscr{D}} B_i$. To see that p cannot lie in the interior of any subcontinuum of $\operatorname{int}(\sum_{\mathscr{D}} B_i)$, it suffices to show $p \notin \operatorname{int}(A)$. Suppose contrariwise, that there is some $\prod_{\mathscr{D}} D_i \in \prod_{\mathscr{D}} F(\mathscr{I})$ such that $p \notin \sum_{\mathscr{D}} D_i$ and $A \cup \sum_{\mathscr{D}} D_i = \sum_{\mathscr{D}} \mathscr{I}$. Fix $n \in I$. For each $i \in I$, $\mu(B_i) = 3/4$, and each component of B_i has measure $3/4i$. Thus, if $i \geq 2n$, then B_i is a disjoint union of n closed sets $B_{i,1}, \ldots, B_{i,n}$, each of measure $< 1/n$. Therefore $\sum_{\mathscr{D}} B_i = \bigcup_{j=1}^{n} \sum_{\mathscr{D}} B_{i,j}$, a disjoint union. (Because $\mathscr{D}$ is a free ultrafilter, we have, for $1 \leq j \leq n$, $\{i$: $\mu(B_{i,j}) < 1/n\} \in \mathscr{D}$.) Since A is connected, there is some $k \in \{1, \ldots, n\}$ with $A \subseteq \sum_{\mathscr{D}} B_{i,k}$. Thus $\sum_{\mathscr{D}} \mathscr{I} = \sum_{\mathscr{D}} D_i \cup \sum_{\mathscr{D}} B_{i,k}$. Consequently, $\{i$: $D_i \cup B_{i,k}\} \in \mathscr{D}$, whence $\{i$: $\mu(D_i) \geq 1 - 1/n\} \in \mathscr{D}$. This says $\lim_{\mathscr{D}}(\langle \mu(D_i): i \in I\rangle) \geq 1 - 1/n$. Because n is arbitrary, we have $\prod_{\mathscr{D}} D_i \in \mathscr{S}$, whence $p \in \sum_{\mathscr{D}} D_i$. This gives a contradiction. The case where $p \in \operatorname{int}(\sum_{\mathscr{D}} C_i)$ is handled similarly, and we infer that $\sum_{\mathscr{D}} \mathscr{I}$ is not locally connected.

Step 2. Let I be any infinite set, $\mathscr{D}$ countably incomplete. We show $\sum_{\mathscr{D}} \mathscr{I}$ is not locally connected by mapping it continuously onto $\sum_{\mathscr{E}} \mathscr{I}$, where $\mathscr{E}$ is a free (hence countably incomplete) ultrafilter on $N = \{1, 2, \ldots\}$.

Since $\mathscr{D}$ is countably incomplete, there is a properly decreasing sequence $I = J_1 \supseteq J_2 \supset \cdots$ of members of $\mathscr{D}$, whose intersection is empty. Define $f: I \to N$ by $f^{-1}[n] = J_n \backslash J_{n+1}$, $n \in N$, and set $\mathscr{E} = \{S \subseteq N: f^{-1}[S] \in \mathscr{D}\}$. Then $\mathscr{E}$ is a free ultrafilter on N, so $\sum_{\mathscr{E}}\mathscr{I}$ is not locally connected by Step 1. For $n \in N$, $C_n \subseteq \mathscr{I}$ closed, $i \in I$, let $C_i = C_{f(i)}$. Define $\delta: \sum_{\mathscr{D}}\mathscr{I} \to \sum_{\mathscr{E}}\mathscr{I}$ by $\delta(p) = \{\prod_{\mathscr{E}} C_n: \prod_{\mathscr{D}} C_i \in p\}$. δ is well defined; it is continuous because $\delta^{-1}[\sum_{\mathscr{E}} C_n] = \sum_{\mathscr{D}} C_i$. Finally, if $q \in \sum_{\mathscr{E}}\mathscr{I}$, then $p = \{\prod_{\mathscr{D}} C_i: \prod_{\mathscr{E}} C_n \in q\}$ is a point in $\sum_{\mathscr{D}}\mathscr{I}$ that is sent to q via δ. Thus δ is a continuous surjection.

Step 3. (This is essentially the argument in Corollary 14 in [9].) Assume the general situation: $\langle X_i: i \in I\rangle$ is a family of compact Hausdorff spaces; $\mathscr{D}$ is a countably incomplete ultrafilter on I; and $\{i: X_i \text{ is infinite}\} \in \mathscr{D}$. If the spaces X_i are Boolean, so is $\sum_{\mathscr{D}} X_i$. The ultracoproduct is infinite; hence it cannot be locally connected. On the other hand, suppose we can choose subcontinua $C_i \subseteq X_i$ in such a way that $J = \{i: C_i \text{ is nondegenerate}\} \in \mathscr{D}$. Then for $i \in J$ there is a continuous surjection $\delta_i: C_i \to \mathscr{I}$. δ_i then extends to a continuous surjection $\eta_i: X_i \to \mathscr{I}$; hence the ultracoproduct map $\sum_{\mathscr{D}}\eta_i: \sum_{\mathscr{D}} X_i \to \sum_{\mathscr{D}}\mathscr{I}$ is a continuous surjection. Invoke Step 2. ■

We are ready to prove our advertised generalization of Gurevič's theorem.

2.10. THEOREM. *Let κ be an infinite cardinal, X an infinite compact Hausdorff space. Then there is a locally disconnected compact Hausdorff Y of weight κ that is co-elementarily equivalent to X.*

PROOF. Given κ and X, let $\mathscr{C}$ be an infinite collection of proper closed subsets of X such that each two-element subcollection of $\mathscr{C}$ covers X. Let $\mathscr{D}$ be a κ-regular ultrafilter on a set I. Then $(\prod_{\mathscr{D}}\mathscr{C})^\# \subseteq (\prod_{\mathscr{D}} F(X))^\# \subseteq F(\sum_{\mathscr{D}} X)$, and $(\prod_{\mathscr{D}}\mathscr{C})^\#$ is a collection of at least $\exp(\kappa)$ proper closed subsets of $\sum_{\mathscr{D}} X$ such that each two-element subcollection of $(\prod_{\mathscr{D}}\mathscr{C})^\#$ covers $\sum_{\mathscr{D}} X$.

By 2.9, $\sum_{\mathscr{D}} X$ is not locally connected; so there is a point $p \in \sum_{\mathscr{D}} X$ and an open $U \subseteq \sum_{\mathscr{D}} X$ containing p such that no subcontinuum of U contains p in its interior. Let $\mathscr{B} \subseteq F(\sum_{\mathscr{D}} X)$ be an elementary sublattice of cardinality κ such that $\{p\}$ and $A = (\sum_{\mathscr{D}} X)\backslash U$ are members of $\mathscr{B}$, and $\mathscr{B}$ contains κ members of $(\prod_{\mathscr{D}}\mathscr{C})^\#$. Set $Y = \omega(\mathscr{B})$. Then $w(Y) \leq \kappa$ because $\mathscr{B}^\#$ is a basis of cardinality κ. Also $w(Y) \geq \kappa$ because Y has κ pairwise disjoint nonempty open sets.

Since $Y \doteq \sum_{\mathscr{D}} X$ and $\sum_{\mathscr{D}} X \doteq X$, we know that $Y \equiv X$ by 2.3. It remains to show that Y is locally disconnected. Let $\varepsilon: \mathscr{B} \to F(\sum_{\mathscr{D}} X)$ be the inclusion map, an elementary embedding. Then the induced map $\varepsilon^\omega: \sum_{\mathscr{D}} X \to Y$ is a co-elementary map. Set $q = \varepsilon^\omega(p)$. Then q is the single point of $\{p\}^\#$. Also $\varepsilon^\omega[A] = A^\#$, and $q \notin A^\#$. We claim that no subcontinuum of $Y\backslash A^\#$ contains q in its interior. Assuming the contrary, let $K \subseteq Y$ be a subcontinuum disjoint from $A^\#$ and containing q in its interior. Then there is some $B \in \mathscr{B}$ with $q \notin B^\#$ and $K \cup B^\# = Y$. By 2.8 there is a subcontinuum $C \subseteq \sum_{\mathscr{D}} X$ such that $K = \varepsilon^\omega[C]$ and $C \cup (\varepsilon^\omega)^{-1}[B^\#] = \sum_{\mathscr{D}} X$. Now $(\varepsilon^\omega)^{-1}[B^\#] = B$ and $p \notin B$. Also, $C \cap A \neq \varnothing$, since $\varepsilon^\omega[C] \cap \varepsilon^\omega[A] = K \cap A^\# = \varnothing$. Thus $C \subseteq U$ and $p \in \text{int}(C)$, a contradiction. ■

We now turn to issue (I3), determining the number of $\mathbf{T}_C$-taxa in various subclasses $\mathscr{K}$ of {compact Hausdorff}. This number is well known to be $\aleph_0$ for $\mathscr{K} =$ {Boolean}, by the Tarski invariants theorem [6]. Hence we get no information concerning the number of $\mathbf{T}_F$-taxa in {Boolean}, although we do get as dividend that there is a family of c pairwise nonhomeomorphic Boolean spaces in the same $\mathbf{T}_C$-taxon.

Because $\mathbf{T}_F$ is finer than $\mathbf{T}_C$ (relative to $\mathscr{K}$) for any $\mathscr{K} \subseteq$ {compact Hausdorff}, we infer that there are c $\mathbf{T}_F$-taxa in $\mathcal{K}$ if we can show there are that many $\mathbf{T}_C$-taxa in $\mathcal{K}$. In [3] we showed there are c $\mathbf{T}_C$-taxa in {compact metrizable}; in [4] we showed there are c $\mathbf{T}_C$-taxa in {compact metrizable, dimension $= n$} for $n > 0$ (where "dimension" is Lebesgue covering dimension: in [3] we showed that having dimension n is preserved and reflected by the ultracoproduct construction). In both proofs, the constructions resulted in spaces with infinitely many connected components. Here we show that there are continuously many $\mathbf{T}_C$-taxa in {Peano continua}, using some of the ideas in [3] and [4], but some new ones too.

2.11. THEOREM. *There are continuously many co-elementary equivalence classes in {Peano continua}.*

PROOF. Let $\kappa =$ the number of co-elementary equivalence classes in {Peano continua}. Since the number of $\mathbf{T}_C$-taxa in {compact Hausdorff} is c, we have immediately that $\kappa \leq c$. We now proceed to construct continuously many pairwise co-elementarily inequivalent Peano continua.

Let H be the Hilbert cube represented as the ω-fold Tichonov power of the real line segment $[-1,1]$, and let S be the set of all sequences $\sigma: \{1,2,3,\ldots\} \to \{0,1\}$. For each $\sigma \in S$, we define the "compactified string of beads" $X_\sigma \subseteq H$ to be $\{\langle 0,0,\ldots\rangle\} \cup \bigcup_{n=1}^{\infty} B_{n,\sigma}$, where each $B_{n,\sigma}$ is the standard closed $(2 + n \cdot \sigma(n))$-cell in H of radius

$$\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)$$

and centered at the point

$$\left\langle \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right), 0, 0, \ldots \right\rangle.$$

If c_n is the point of tangency between $B_{n,\sigma}$ and $B_{n+1,\sigma}$, then these tangent points are the cut points of X_σ. Clearly X_σ is a Peano continuum. Now suppose $\sigma, \tau \in S$ are distinct, say $\sigma(k) = 1 \neq \tau(k)$. In order to show $X_\sigma \not\equiv X_\tau$, assume the contrary. We then have ultrafilters $\mathcal{D}$, $\mathcal{E}$ and a homeomorphism $\eta: \sum_{\mathcal{D}} X_\sigma \to \sum_{\mathcal{E}} X_\tau$.

We claim that η takes $\sum_{\mathcal{D}} B_{k,\sigma}$ onto an ultracoproduct of beads in $\sum_{\mathcal{E}} X_\tau$. Once we establish this claim, a contradiction arises from known facts about ultracoproducts and covering dimension $\dim(\cdot)$. In particular, if $\langle Y_i: i \in I\rangle$ is a family of compact Hausdorff spaces and $\mathcal{F}$ is an ultrafilter on I, then $\dim(\sum_{\mathcal{F}} Y_i) = n$ if and only if $\{i: \dim(Y_i) = n\} \in \mathcal{F}$ [3]. Now $\dim(\sum_{\mathcal{D}} B_{k,\sigma}) = 2 + k$, and no ultracoproduct of beads in $\sum_{\mathcal{E}} X_\tau$ can have dimension $2 + k$ since no bead in X_τ has that dimension.

We will thus be finished once we prove the following three lemmas.

LEMMA 1. *An ultracoproduct of cells of dimension at least* 2 *contains no cut points.*

PROOF. For each $i \in I$, let B_i be a cell of dimension at least 2, $\mathcal{F}$ an ultrafilter on I. Suppose further that p is a cut point of $\sum_{\mathcal{F}} B_i$. Then there are nonempty open subsets U and V of $\sum_{\mathcal{F}} B_i$ such that $U \cap V = \varnothing$ and $U \cup \{p\} \cup V = \sum_{\mathcal{F}} B_i$. Now the set of "standard" points $\sum_{\mathcal{F}} x_i = (\prod_{\mathcal{F}}\{x_i\})^{\#}$, where $x_i \in B_i$, is dense in $\sum_{\mathcal{F}} B_i$, so pick $\sum_{\mathcal{F}} x_i \in U$ and $\sum_{\mathcal{F}} y_i \in V$. For each $i \in I$, let α_i, β_i be arcs in B_i, with endpoints x_i, y_i, such that $\alpha_i \cap \beta_i = \{x_i, y_i\}$. Then $\sum_{\mathcal{F}} \alpha_i$ and $\sum_{\mathcal{F}} \beta_i$ are subcontinua of $\sum_{\mathcal{F}} B_i$ such that

$$\textstyle\sum_{\mathcal{F}} \alpha_i \cap \sum_{\mathcal{F}} \beta_i = \sum_{\mathcal{F}} (\alpha_i \cap \beta_i) = \sum_{\mathcal{F}} \{x_i, y_i\} = \{\sum_{\mathcal{F}} x_i, \sum_{\mathcal{F}} y_i\}.$$

Since p is a cut point, however, this is impossible since $p \in \sum_{\mathcal{F}} \alpha_i \cap \sum_{\mathcal{F}} \beta_i$. ∎

Now, back to the main proof. $B_{k,\sigma}$ is a subcontinuum of X_σ, which equals the closure of its own interior in X_σ. The same, therefore, holds for $\sum_{\mathscr{D}} B_{k,\sigma}$ in $\sum_{\mathscr{D}} X_\sigma$. Let $K = \eta[\sum_{\mathscr{D}} B_{k,\sigma}]$. Then K is a subcontinuum of $\sum_{\mathscr{E}} X_\tau$, which equals the closure of its own interior in $\sum_{\mathscr{E}} X_\tau$. Let $\sum_{\mathscr{E}} x_j$ be a standard point of $\sum_{\mathscr{E}} X_\tau$ that is contained in $\mathrm{int}(K)$. $\sum_{\mathscr{E}} x_j$ can be chosen so that each x_j is contained in the interior of a (unique) bead B_j in X_τ. (B_j is some $B_{n,\tau}$, but j ranges over a possibly large index set J.)

LEMMA 2. $K \subseteq \sum_{\mathscr{E}} B_j$.

PROOF. Suppose otherwise. Since $K = \mathrm{cl}(\mathrm{int}(K))$, it suffices to show $\mathrm{int}(K) \subseteq \sum_{\mathscr{E}} B_j$. If this inclusion does not hold, then there is a standard point $\sum_{\mathscr{E}} y_j \in \mathrm{int}(K) \backslash \sum_{\mathscr{E}} B_j$. As above, we choose this point so that $y_j \in \mathrm{int}(B'_j)$, where B'_j is a bead of X_τ. Since K is a continuum intersecting both $\sum_{\mathscr{E}} B_j$ (witnessed by $\sum_{\mathscr{E}} x_j$) and $\sum_{\mathscr{E}} B'_j$ (witnessed by $\sum_{\mathscr{E}} y_j$), and $B'_j \neq B_j$ for all $j \in J$, we know $\sum_{\mathscr{E}} B_j \cap \sum_{\mathscr{E}} B'_j = \sum_{\mathscr{E}} (B_j \cap B'_j)$ is a single standard point $\{\sum_{\mathscr{E}} c_j\}$, where c_j is the point of tangency of B_j and B'_j. $\sum_{\mathscr{E}} c_j$ is a cut point of $\sum_{\mathscr{E}} (B_j \cup B'_j)$; consequently it is a cut point of K. But K is homeomorphic to an ultracoproduct of cells of dimension ≥ 2, and this contradicts Lemma 1. ■

LEMMA 3. $K = \sum_{\mathscr{E}} B_j$.

PROOF. By Lemma 2, $K \subseteq \sum_{\mathscr{E}} B_j$. Let $L = \eta^{-1}[\sum_{\mathscr{E}} B_j]$. Then $L = \mathrm{cl}(\mathrm{int}(L))$ in $\sum_{\mathscr{D}} X_\sigma$ since $B_j = \mathrm{cl}(\mathrm{int}(B_j))$ in X_τ. By definition of K, we know $L \supseteq \sum_{\mathscr{D}} B_{k,\sigma}$. If this inclusion were proper, we could conclude that L has cut points by an argument similar to that in Lemma 2. But L is homeomorphic to a space with no cut points, by Lemma 1. Thus $L = \sum_{\mathscr{D}} B_{k,\sigma}$; hence $K = \sum_{\mathscr{E}} B_j$. ■

By Lemmas 2 and 3, we have established that the homeomorphism η takes $\sum_{\mathscr{D}} B_{k,\sigma}$ onto the ultracoproduct of beads $\sum_{\mathscr{E}} B_j$. By earlier remarks, this is a contradiction for reasons of dimension; therefore $X_\sigma \not\equiv X_\tau$, as desired. ■

2.12. REMARK. In Lemma 1 above, "cells of dimension at least 2" may be easily replaced by "nondegenerate continua with the property that each two-element subset is the intersection of two subcontinua."

§3. Other taxonomies. So far we have studied only the two taxonomies $\mathbf{T}_F$ and $\mathbf{T}_C$ (and only $\mathbf{T}_C$ to any depth). The other popular taxonomies that use the alphabet of bounded lattices are $\mathbf{T}_B$ and $\mathbf{T}_Z$, differing from $\mathbf{T}_F$ only in the first order representation used: $B(X)$ is the lattice of clopen sets; $Z(X)$ is the lattice of zero sets. Extensive studies on these taxonomies have been carried out in [2], [3], [10], [12], and [18].

Another taxonomy studied by many authors (see [2], [7], [12]) is the *function ring taxonomy* $\mathbf{T}_R = \langle \Phi_R, R, \models \rangle$, where Φ_R is the first order language of unital rings, and $R(X)$ is the ring of continuous real-valued functions on X (alias $C(X)$, but redubbed here for obvious reasons). It is relatively easy to show that $\mathbf{T}_R$ is finer than $\mathbf{T}_Z$ (relative to {Tichonov}), but that $\mathbf{T}_F$ and $\mathbf{T}_R$ are independent relative to {compact Hausdorff} (see [12]). An especially nice result, mentioned at the end of [12], is G. Cherlin's theorem [7] that the real line is $\mathbf{T}_R$-characterized in {Tichonov}.

A taxonomy that deserves mention is what we call the *open set taxonomy* $\mathbf{T}_t = \langle \Phi_t, G, \models \rangle$, where, for any space X with topology $\mathscr{T}$, $G(X)$ is the structure $\langle X \cup \mathscr{T}, \in \rangle$; i.e., elements are of two sorts, "point" and "open set", and $\in$ is membership restricted to $X \times \mathscr{T}$. Φ_t is T. A. McKee's language L_t as defined in [8] and [22] (only without extra relation and function symbols). Very briefly, formulas have two sorts of variables ("point" and "set"); atomic formulas look like $x = y$ and $x \in U$; universally quantified set variables occur positively (each free occurrence of U within the scope of $\forall U$, when the formula is put in negation normal form, lies within the scope of an even number of negation symbols); existentially quantified set variables occur negatively.

S. Garavaglia's ultrapower theorem [8] says that any two spaces are $\mathbf{T}_t$-equivalent if and only if they have homeomorphic topological ultrapowers (in the sense of [1]: ultraproducts of open sets form an open basis). In [1] it is proved that any two regular T_1 spaces without isolated points have homeomorphic topological ultrapowers, and are hence $\mathbf{T}_t$-equivalent. (In particular, nondegenerate continua cannot be distinguished from one another using this taxonomy.) It should not be surprising, then, that $\mathbf{T}_C$ is finer than $\mathbf{T}_t$ (relative to {compact Hausdorff}).

3.1. THEOREM. *Let X and Y be compact Hausdorff spaces. If $X \equiv Y$, then $X \equiv_{\mathbf{T}_t} Y$.*

PROOF. Suppose $X \equiv Y$. By 2.3, there is a compact Hausdorff Z with $X \doteq Z$ and $Z \doteq Y$. Let $\mathscr{B}$ and $\mathscr{C}$ be Wallman bases for X and Z respectively such that $\mathscr{B}$ and $\mathscr{C}$ are elementarily equivalent. Then there are isomorphic lattice ultrapowers $\prod_{\mathscr{D}}\mathscr{B}$ and $\prod_{\mathscr{E}}\mathscr{C}$. These ultrapowers form closed set bases for the corresponding topological ultrapowers, so $\prod_{\mathscr{D}}X \simeq \prod_{\mathscr{E}}Z$. Thus $X \equiv_{\mathbf{T}_t} Z$. Similarly $Z \equiv_{\mathbf{T}_t} Y$, so $X \equiv_{\mathbf{T}_t} Y$. ■

3.2. REMARK. 3.1 answers affirmatively Question 1.12 in [3], namely whether $\sum_{\mathscr{D}}X \simeq \sum_{\mathscr{E}}Y$ implies that for some ultrafilters $\mathscr{F}$ and $\mathscr{G}$, $\prod_{\mathscr{F}}X \simeq \prod_{\mathscr{G}}Y$. A remark following that question showed that, assuming the Continuum Hypothesis, one could not generally take $\mathscr{F}$ and $\mathscr{G}$ to be $\mathscr{D}$ and $\mathscr{E}$ respectively.

REFERENCES

[1] P. BANKSTON, *Ultraproducts in topology*, ***General Topology and its Applications***, vol. 7 (1977), pp. 289–308.

[2] ———, *Expressive power in first order topology*, this JOURNAL, vol. 49 (1984), pp. 478–487.

[3] ———, *Reduced coproducts of compact Hausdorff spaces*, this JOURNAL, vol. 52 (1987), pp. 404–424.

[4] ———, *Co-elementary equivalence for compact Hausdorff spaces and compact abelian groups*, ***Journal of Pure and Applied Algebra*** (to appear).

[5] ———, *Model-theoretic characterizations of arcs and simple closed curves*, ***Proceedings of the American Mathematical Society***, vol. 104 (1988), pp. 898–904.

[6] C. C. CHANG and H. J. KEISLER, ***Model theory***, North-Holland, Amsterdam, 1973.

[7] G. CHERLIN, *Rings of continuous functions: decision problems*, ***Model theory of algebra and arithmetic***, Lecture Notes in Mathematics, vol. 834, Springer-Verlag, Berlin, 1980, pp. 44–91.

[8] J. FLUM and M. ZIEGLER, ***Topological model theory***, Lecture Notes in Mathematics, vol. 769, Springer-Verlag, Berlin, 1980.

[9] R. GUREVIČ, *On ultracoproducts of compact Hausdorff spaces*, this JOURNAL, vol. 53 (1988), pp. 294–300.

[10] S. HEINRICH, C. W. HENSON, and L. C. MOORE, JR., *A note on elementary equivalence of $C(K)$ spaces*, this JOURNAL, vol. 52 (1987), pp. 368–373.

[11] C. W. HENSON, *Nonstandard hulls of Banach spaces*, ***Israel Journal of Mathematics***, vol. 25 (1976), pp. 108–144.

[12] C. W. HENSON et al., *First order topology*, ***Dissertationes Mathematicae/Rozprawy Matematyczne***, vol. 143 (1977).

[13] P. T. JOHNSTONE, ***Stone spaces***, Cambridge University Press, Cambridge, 1982.

[14] K. KURATOWSKI, ***Topology***. Vol. 2, Academic Press, New York, 1968.

[15] E. E. MOISE, ***Geometric topology in dimensions 2 and 3***, Springer-Verlag, Berlin, 1977.

[16] R. L. MOORE, ***Foundations of point set theory***, rev. ed., American Mathematical Society, Providence, Rhode Island, 1962.

[17] S. SHELAH, *The monadic theory of order*, ***Annals of Mathematics***, ser. 2, vol. 102 (1975), pp. 379–419.

[18] A. K. SWETT, *The first order topology of the real line* (unpublished).

[19] W. C. WALKER, ***The Stone-Čech compactification***, Springer-Verlag, Berlin, 1974.

[20] G. T. WHYBURN, ***Analytic topology***, American Mathematical Society, Providence, Rhode Island, 1942; corrected reprint, 1971.

[21] S. WILLARD, ***General topology***, Addison-Wesley, Reading, Massachusetts, 1970.

[22] M. ZIEGLER, *Topological model theory*, ***Model-theoretic logics*** (J. Barwise and S. Feferman, editors), Springer-Verlag, Berlin, 1985, pp. 557–577.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
MARQUETTE UNIVERSITY
MILWAUKEE, WISCONSIN 53233