To appear in Volume 49 (2017) of
Topology Proceedings

AMALGAMATION-TYPE PROPERTIES OF ARCS AND PSEUDO-ARCS

PAUL BANKSTON

Abstract. A continuum X is **base** if it satisfies the following "dual amalgamation" condition: whenever $f : Y \to X$ and $g : Z \to X$ are continuous maps from continua onto X, there is a continuum W and continuous surjections $\varphi : W \to Y$, $\gamma : W \to Z$ such that $f \circ \varphi = g \circ \gamma$. A metrizable continuum is **base metrizable** if it satisfies the condition above, relativized to the subclass of metrizable continua. It is easy to show that simple closed curves are neither base nor base metrizable; however metrizable continua of span zero are known to be base metrizable. Furthermore, co-existentially closed continua are known to be base. The arc and the pseudo-arc are span zero; but, of the two, only the pseudo-arc is co-existentially closed. Hence the pseudo-arc is base metrizable for being span zero and base for being co-existentially closed. Here we show that: (i) there is a base metrizable continuum which is not span zero; and (ii) any metrizable continuum is base if and only if it is base metrizable.

1. Introduction

In algebra and model theory a structure A in a certain class $\mathfrak{C}$ is referred to as an **amalgamation base** for $\mathfrak{C}$ if whenever A sits as a substructure of two members of $\mathfrak{C}$, there is a third member of $\mathfrak{C}$ which contains all of them. There are many natural variations on this theme; the one we consider here has continua instead of relational structures and quotients

2010 *Mathematics Subject Classification.* 54F15 (54C10, 54F20, 54F50, 54B99, 03C20).

Key words and phrases. arc, pseudo-arc, base continuum, base metrizable continuum, span zero, hereditarily indecomposable, ultracopower, co-existential map, co-elementary map, co-existentially closed continuum.

The author would like to thank the anonymous referee for a careful reading of this paper, as well as for suggesting a stronger form of Corollaries 4.3 and 4.7.

instead of substructures. More precisely: if $\mathfrak{C}$ is a class of continua (i.e., connected compact Hausdorff spaces), define a member X of $\mathfrak{C}$ to be **base** for $\mathfrak{C}$ if whenever we have continua Y and Z in $\mathfrak{C}$ and continuous surjections $f : Y \to X$, $g : Z \to X$, there is some $W \in \mathfrak{C}$ and continuous surjections $\varphi : W \to Y$, $\gamma : W \to Z$, such that $f \circ \varphi = g \circ \gamma$. In this note we restrict our attention to where $\mathfrak{C}$ is either the class of all continua or its subclass of metrizable continua. We will then refer to X as being **base** or **base metrizable** according to whether it is base for the corresponding class $\mathfrak{C}$.

It is easy to show that simple closed curves are neither base nor base metrizable. In [14], J. Krasinkiewicz proves that arcs are base continua; moreover the techniques of [17] may be used to show that any span zero metrizable continuum is base metrizable. From [5, Theorem 2.2] we know that every co-existentially closed continuum is base, and recent work [9] tells us that pseudo-arcs are co-existentially closed continua. (Hence pseudo-arcs are base metrizable on account of being span zero and are base on account of being co-existentially closed.) Our aim here is to prove the following:

(i) *There is a base metrizable continuum which is not span zero.*
(ii) *A metrizable continuum is base if and only if it is base metrizable.*

Our main technique uses the ultracopower construction for **compacta**; i.e., compact Hausdorff spaces, along with some classical model theory of bounded lattices.

2. A Preliminary Example

For the purposes of this note, a quintuple $\langle X, Y, Z, f, g\rangle$ of compacta and continuous surjections is a **wedge** if f and g map Y and Z, respectively, onto X, or *vice versa*. In the first case the wedge is **inward**; in the second it is **outward**. If $Y \xrightarrow{f} X \xleftarrow{g} Z$ is an inward wedge and $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$ is an outward wedge, the latter is a **completion** of the former if the resulting mapping square commutes; i.e., if $f \circ \varphi = g \circ \gamma$.

Given an inward wedge $Y \xrightarrow{f} X \xleftarrow{g} Z$, we let $P(f, g)$ denote the **pullback** of f and g, defined to be the compactum $\{\langle y, z\rangle \in Y \times Z : f(y) = g(z)\}$. With p and q denoting the coordinate projection maps from $P(f, g)$ to Y and Z, respectively, we see that $Y \xleftarrow{p} P(f, g) \xrightarrow{q} Z$ is an outward wedge which is a completion of the original. (Commutativity is immediate; surjectivity of p (resp., q) follows from that of g (resp., f).)

The signal feature of the pullback $P(f, g)$ is the following universal mapping property: given any completion $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$ for $Y \xrightarrow{f} X \xleftarrow{g} Z$, there is a unique continuous map $\lambda : W \to P(f,g)$–given by $\lambda(w) = \langle \varphi(w), \gamma(w) \rangle$–such that $p \circ \lambda = \varphi$ and $q \circ \lambda = \gamma$.

Proposition 2.1. *An inward wedge $Y \xrightarrow{f} X \xleftarrow{g} Z$ of continua has a completion $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$, where W is a continuum, if and only if there is a component C of $P(f,g)$ such that both restrictions $p|_C$ and $q|_C$ are surjective.*

Proof. Suppose $Y \xrightarrow{f} X \xleftarrow{g} Z$ is an inward wedge of continua, and there is a component $C \subseteq P(f,g)$ such that $p|_C$ and $q|_C$ are surjective. Then $Y \xleftarrow{p|_C} C \xrightarrow{q|_C} Z$ is a suitable completion.

Given a completion $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$ for $Y \xrightarrow{f} X \xleftarrow{g} Z$, where W is a continuum, let C be the component of $P(f,g)$ containing the continuum $\lambda[W]$. □

Up to homeomorphism, there is just one metrizable continuum which has exactly two noncut points; this continuum is known as the **arc**. The following example, suggested by L. C. Hoehn [11], illustrates how a one-dimensional continuum can fail to be base (metrizable).

Example 2.2. Let X be a metrizable simple closed curve, as represented by the unit circle in the complex plane; let Y be an arc, as represented by the interval $[0, 2\pi]$ in the real line. Let $f : Y \to X$ be given by $t \mapsto \cos(t) + i \sin(t)$, and let $g = -f$. Then $P(f,g)$ consists of two disjoint line segments, resulting from the intersection of the square $[0, 2\pi] \times [0, 2\pi]$ with the graphs of the lines $y = x \pm \pi$. Since neither of these line segments projects onto Y, we infer from Proposition 2.1 that no suitable completion for $Y \xrightarrow{f} X \xleftarrow{g} Y$ can exist. Since Y is a metrizable continuum, this tells us that X is neither base nor base metrizable.

3. Span Zero

Given a set X, let $p, q : X \times X \to X$ be the first and second coordinate projections, respectively, and let $\Delta := \{\langle x, x \rangle : x \in X\}$ denote the diagonal in $X \times X$. A continuum X is **span zero** if whenever Z is a subcontinuum of $X \times X$ and $p[Z] = q[Z]$, then $Z \cap \Delta \neq \emptyset$. X is **chainable** if it has the property that each of its open covers refines to a finite open cover $\{U_1, \ldots, U_n\}$, where $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Finally, X is **indecomposable** if it is not the union of two of its proper subcontinua, and is **hereditarily indecomposable** if each of its subcontinua is indecomposable.

Up to homeomorphism, there is just one nondegenerate metrizable continuum which is both hereditarily indecomposable and chainable; this continuum is known as the **pseudo-arc**.

Remarks 3.1.

(i) It is well known [15] that every metrizable continuum is span zero if it is chainable. And after lying open for almost fifty years, the problem whether the converse holds was finally settled by L. Hoehn in the negative [12].

(ii) Apparently stronger than span zero is **semispan zero**, where the condition of equality of $p[Z]$ and $q[Z]$ is weakened to where one of the two is contained in the other. A main result of [8], however, is that both notions agree for metrizable continua.

(iii) L. Hoehn [11] has informed us that the techniques of [17] may be used to prove that any span zero metrizable continuum is base metrizable. This, of course, includes both arcs and pseudo-arcs. Hoehn also conjectured that span zero is equivalent to being base metrizable; but, as we show in the sequel, there are base metrizable continua which are not span zero.

(iv) It has recently been shown by Hoehn and L. Oversteegen [13] that span zero does imply chainable for hereditarily indecomposable metrizable continua. Thus the pseudo-arc may now be characterized as being the only (up to homeomorphism) nondegenerate metrizable continuum that is both hereditarily indecomposable and span zero.

(v) A slight rewording of the so-called "mountain climbing problem" asks under what circumstances an inner wedge $Y \xrightarrow{f} X \xleftarrow{g} Z$ has a completion $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$, where $X = Y = Z = W$. This is of special interest when X is an arc, and both f and g satisfy further conditions, such as fixing both end points and being piecewise monotone (see [18]).

4. Ultracopowers and Co-existential Maps

The topological ultracopower–more generally, ultracoproduct–construction for compacta was initiated in [1]; also, independently (in the case of arcs), by J. Mioduszewski in [16]. We start with a compactum X and an infinite set I, viewed as a discrete topological space. With $p : X \times I \to X$ and $q : X \times I \to I$ the coordinate projection maps, we apply the Stone-Čech functor $\beta(\cdot)$ to obtain the outward wedge

$$X = \beta(X) \xleftarrow{p^\beta} \beta(X \times I) \xrightarrow{q^\beta} \beta(I).$$

Regarding an ultrafilter $\mathcal{D}$ on the set I as a point in $\beta(I)$, we form the **$\mathcal{D}$-ultracopower** $X_\mathcal{D}$ as the pre-image $(q^\beta)^{-1}[\{\mathcal{D}\}]$. It is a basic fact about this construction that $X_\mathcal{D}$ is a continuum if and only if X is a continuum. (Many more important properties of compacta are preserved–and reflected–by ultracopowers; the reader is directed to either [2] or [4] for a detailed account of this construction and its connections with the ultrapower construction in model theory.)

The restriction $p_{X,\mathcal{D}} = p^{\beta}|_{X_{\mathcal{D}}}$ is a continuous surjection from $X_{\mathcal{D}}$ onto X, known as the canonical **codiagonal map**. A continuous map $f : Y \to X$ between compacta is **co-existential** if there is an ultrafilter $\mathcal{D}$ and a continuous surjection $g : X_{\mathcal{D}} \to Y$ such that $f \circ g = p_{X,\mathcal{D}}$. The continuum X is **co-existentially closed** if every continuous map from a continuum onto X is co-existential.

Recall that the **weight** $w(X)$ of a space X is the smallest infinite cardinal κ such that X has an open-set base of cardinality $\leq \kappa$. (So a compactum X is metrizable if and only if $w(X) = \aleph_0$.)

Our first background result states that there are "enough" co-existentially closed continua, and is based on the analogous model-theoretic idea that ensures the existence of "enough" existentially closed models of a theory (see, e.g., [7, Lemma 3.5.7]).

Theorem 4.1. ([3, Theorem 6.1]) *Every continuum is a continuous image of a co-existentially closed continuum of equal weight.*

Recall that a continuum X is of **covering dimension one** if it is nondegenerate, and each of its open covers refines to a finite open cover, no three of whose members has nonempty intersection.

The second background result was proved in several stages over the years, and takes the following present form.

Theorem 4.2. ([6, Corollary 4.13]) *Every co-existentially closed continuum is hereditarily indecomposable, and of covering dimension one.*

Theorems 4.1 and 4.2 may be combined with known results to obtain the following.

Corollary 4.3. *There are uncountably many topologically distinct metrizable continua which are co-existentially closed, but not span zero.*

Proof. By Theorem 4.1, every metrizable continuum is a continuous image of a metrizable continuum which is co-existentially closed. Since a countable product of metrizable continua is a metrizable continuum, and no one metrizable continuum continuously surjects onto every metrizable continuum [19], there must be an uncountable number of pairwise nonhomeomorphic metrizable co-existentially closed continua. By Theorem 4.2, all are hereditarily indecomposable; and, by [13, Theorem 1], only one of them can be span zero. $\square$

Remark 4.4. By the Main Theorem of [9], the pseudo-arc is a span zero co-existentially closed continuum.

The following is a special case of [5, Theorem 2.2]. We include a sketch of the proof for completeness; we note that the second step requires some deep results from model theory, and the interested reader is invited to consult [7] for details.

Theorem 4.5. *Every co-existentially closed continuum is base.*

Proof.

(Step 1) Start with an inward wedge $Y \xrightarrow{f} X \xleftarrow{g} Z$ of continua, where X is co-existentially closed. By definition, there are outward wedges

$$Y \xleftarrow{f'} X_{\mathcal{D}} \xrightarrow{p_{X,\mathcal{D}}} X \text{ and } Z \xleftarrow{g'} X_{\mathcal{E}} \xrightarrow{p_{X,\mathcal{E}}} X,$$

with $f \circ f' = p_{X,\mathcal{D}}$ and $g \circ g' = p_{X,\mathcal{E}}$.

(Step 2) Let $F(X)$ be the relational structure consisting of the closed-set lattice for X, augmented with constant symbols naming the points $x \in X$ (viewed as atoms of $F(X)$). Letting $d_{F(X),\mathcal{D}} : F(X) \to F(X)^{\mathcal{D}}$ be the canonical diagonal embedding into the (model-theoretic) $\mathcal{D}$-ultrapower, we note [7, Theorem 4.1.9] that $d_{F(X),\mathcal{D}}$ is an elementary embedding. Doing the same thing with the ultrafilter $\mathcal{E}$, we see that the two ultrapowers $F(X)^{\mathcal{D}}$ and $F(X)^{\mathcal{E}}$ are elementarily equivalent as lattices with extra constants from X. By the main construction in the proof of the ultrapower theorem [7, Theorem 6.1.15], there is an ultrafilter $\mathcal{F}$, and an isomorphism $e : (F(X)^{\mathcal{E}})^{\mathcal{F}} \to (F(X)^{\mathcal{D}})^{\mathcal{F}}$. Since this isomorphism respects all constants naming elements of X, the obvious mapping pentagon commutes; i.e., $e \circ (d_{F(X)^{\mathcal{E}},\mathcal{F}}) \circ (d_{F(X),\mathcal{E}}) = (d_{F(X)^{\mathcal{D}},\mathcal{F}}) \circ (d_{F(X),\mathcal{D}})$.

Now the points of the ultracopower $X_{\mathcal{D}}$ are the maximal filters from the lattice $F(X)^{\mathcal{D}}$. This "maximal spectrum" operation is contravariantly functorial, converting diagonal embeddings to codiagonal maps and the isomorphism e into a homeomorphism $h : (X_{\mathcal{D}})_{\mathcal{F}} \to (X_{\mathcal{E}})_{\mathcal{F}}$. What is more, the resulting mapping pentagon commutes; i.e., $(p_{X,\mathcal{E}}) \circ (p_{X_{\mathcal{E}},\mathcal{F}}) \circ h = (p_{X,\mathcal{D}}) \circ (p_{X_{\mathcal{D}},\mathcal{F}})$. (What is important for this argument is that h is a continuous surjection which results in a commuting pentagon.)

(Step 3) Applying the ultracopower construction via a single ultrafilter is covariantly functorial. Thus the application of $(\cdot)_{\mathcal{F}}$ produces continuous surjections $f'_{\mathcal{F}} : (X_{\mathcal{D}})_{\mathcal{F}} \to Y_{\mathcal{F}}$ and $g'_{\mathcal{F}} : (X_{\mathcal{E}})_{\mathcal{F}} \to Z_{\mathcal{F}}$, such that the appropriate mapping squares commute; i.e., $(p_{Y,\mathcal{F}}) \circ f'_{\mathcal{F}} = f' \circ (p_{X_{\mathcal{D}},\mathcal{F}})$ and $(p_{Z,\mathcal{F}}) \circ g'_{\mathcal{F}} = g' \circ (p_{X_{\mathcal{E}},\mathcal{F}})$.

(Step 4) Finally, let W be $(X_{\mathcal{D}})_{\mathcal{F}}$, φ be $p_{Y,\mathcal{F}} \circ f'_{\mathcal{F}}$, and γ be $p_{Z,\mathcal{F}} \circ g'_{\mathcal{F}} \circ h$. Then $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$ is the required completion for $Y \xrightarrow{f} X \xleftarrow{g} Z$.

□

If we think of the words *metrizable* and *base* linguistically, as adjectives in English, it matters what order they come in: to check that a metrizable continuum is metrizable base, the quantification domain comprises all continua; while to show it to be base metrizable, all quantification is taking place in the restricted metrizable realm. Thus the expressions *metrizable base continuum* and *base metrizable continuum* have ostensibly different denotations.

In the next section we show that there is actually no difference on a deeper semantic level; we begin the process by showing the easy direction of the equivalence.

Proposition 4.6. *Every metrizable base continuum is base metrizable.*

Proof. Assume X is a metrizable base continuum, and suppose $Y \xrightarrow{f} X \xleftarrow{g} Z$ is an inward wedge, with Y and Z metrizable. Since X is base, we invoke Proposition 2.1 to find a component C of $P(f,g)$ such that both $p|_C$ and $q|_C$ are surjective. C is clearly metrizable; hence the outward wedge $Y \xleftarrow{p|_C} C \xrightarrow{q|_C} Z$ witnesses that X is base metrizable. □

Using Corollary 4.3, Theorem 4.5, and Proposition 4.6 we may now satisfy the first of the aims set out in the Introduction (see also Remark 3.1 (iii)).

Corollary 4.7. *There are uncountably many topologically distinct metrizable continua which are base metrizable, but not span zero.*

5. Base Metrizable Implies Metrizable Base

We begin the section with a notion that first appears in [2], and which is–in a substantial sense–dual to that of *elementary embedding* in model theory. Given a map $f : Y \to X$ between compacta, we say f is **co-elementary** if there is a homeomorphism $h : Y_{\mathcal{E}} \to X_{\mathcal{D}}$ between ultracopowers of Y and X, respectively, such that the appropriate mapping square commutes; i.e., $p_{X,\mathcal{D}} \circ h = f \circ p_{Y,\mathcal{E}}$. Co-elementary maps are clearly co-existential, but the converse is far from true; e.g., co-existential maps can lower covering dimension, while co-elementary maps cannot. (See [2, Theorem 2.2.2] and [3, Example 2.12].) The following helpful result is a refinement of the "Löwenheim-Skolem" approach used by R. Gurevič in [10].

Lemma 5.1. ([3, Theorem 3.1]) *Let $f : X \to Y$ be a continuous surjection between compacta, with κ a cardinal such that $w(X) \geq \kappa \geq w(Y)$. Then* *there is a compactum Z of weight κ, and continuous surjections $g : X \to Z$ and $h : Z \to Y$, where g is co-elementary and $f = h \circ g$.*

We are now in a position to prove the converse of Proposition 4.5.

Theorem 5.2. *Every base metrizable continuum is metrizable base.*

Proof.

(Step 1) Start with an inward wedge $Y \xrightarrow{f} X \xleftarrow{g} Z$ of continua, where X is base metrizable. By Lemma 5.1, there are continuous surjections $f'' : Y \to Y'$, $f' : Y' \to X$, $g''Z \to Z'$, $g' : Z' \to X$, where Y' and Z' are metrizable, f'' and g'' are co-elementary maps, and both mapping triangles commute.

(Step 2) Since X is base metrizable, the inward wedge $Y' \xrightarrow{f'} X \xleftarrow{g'} Z'$ has a completion $Y' \xleftarrow{\varphi'} W' \xrightarrow{\gamma'} Z'$, where W' is a metrizable continuum.

(Step 3) By [3, Lemma 2.1], there is a single ultrafilter $\mathcal{D}$ witnessing the co-elementarity of f'' and g''; i.e., there are homeomorphisms $h_Y : Y_{\mathcal{D}} \to Y'_{\mathcal{D}}$, $h_Z : Z_{\mathcal{D}} \to Z'_{\mathcal{D}}$ such that the appropriate mapping squares commute.

(Step 4) Using the functoriality of $(\cdot)_{\mathcal{D}}$, we have continuous surjections $\varphi'_{\mathcal{D}} : W'_{\mathcal{D}} \to Y'_{\mathcal{D}}$ and $\gamma'_{\mathcal{D}} : W'_{\mathcal{D}} \to Z'_{\mathcal{D}}$, also making the appropriate mapping squares commute. There are five mapping squares and two mapping triangles by now, all commutative, so set $W = W'_{\mathcal{D}}$, with $\varphi = p_{Y,\mathcal{D}} \circ h_Y^{-1} \circ \varphi'_{\mathcal{D}}$ and $\gamma = p_{Z,\mathcal{D}} \circ h_Z^{-1} \circ \gamma'_{\mathcal{D}}$. Then $Y \xleftarrow{\varphi} W \xrightarrow{\gamma} Z$ is a completion for $Y \xrightarrow{f} X \xleftarrow{g} Z$, showing that X is a base continuum.

□

We end with some questions.

Questions 5.3.

(i) Since *span zero* makes sense in the non-metrizable context, it is natural to ask whether span zero (Hausdorff) continua are base.

(ii) In [15, Corollary, Section 7] A. Lelek proves that a nondegenerate metrizable span zero continuum must have covering dimension one. Does this result still hold in the non-metrizable context?

(iii) Are nondegenerate base continua necessarily of covering dimension one?

(iv) Co-existential maps preserve many interesting properties, including hereditary indecomposability, chainability, and covering dimension one. Do they also preserve span zero? (being base?) (If they preserve span zero–even if they need to be co-elementary to do it–then there is an affirmative answer to (ii) above.)

References

[1] P. Bankston, *Some obstacles to duality in topological algebra*, Can. J. Math. **34**, no. 1 (1982), 80–90.

[2] ———, *Reduced coproducts of compact Hausdorff spaces*, J. Symbolic Logic **52** no. 2 (1987), 404–424.

[3] ———, *Some applications of the ultrapower theorem to the theory of compacta*, Applied Categorical Structures **8** (2000), 45–66.

[4] ———, *A survey of ultraproduct constructions in general topology*, Topology Atlas Invited Contributions **8**, no. 2 (2003), 1–32.

[5] ———, *Mapping properties of co-existentially closed continua*, Houston J. Math. **31** (4) (2005), 1047–1063.

[6] ———, *The Chang-Łóś-Suszko theorem in a topological setting*, Arch. for Math. Log. **45**, no. 1 (2006), 97–112.

[7] C. C. Chang and H. J. Keisler, *Model Theory*, third ed., North Holland, Amsterdam, 1990.

[8] J. F. Davis, *The equivalence of zero span and zero semispan*, Proc. Amer. Math. Soc. **90** (1) (1984), 133–138.

[9] C. J. Eagle, I. Goldbring and A. Vignati, *The pseudo-arc is a co-existentially closed continuum*, Topology Appl. (to appear).

[10] R. Gurevič, *On ultracoproducts of compact Hausdorff spaces*, J. Symbolic Logic **53** (1988), 294–300.

[11] L. C. Hoehn (private communication).

[12] ———, *A non-chainable plane continuum with span zero*, Fund. Math. **211** (2) (2011), 149–174.

[13] L. C. Hoehn and L. G. Oversteegen, *A complete classification of homogeneous plane continua*, Acta Math. (to appear) (arXiv:14096324v1).

[14] J. Krasinkiewicz, *On the fiber product of mappings into the unit interval*, Bull. Polish Acad. Sci. Math. **48** (2000), 369–378.

[15] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214.

[16] J. Mioduszewski, *On composants of* $\beta\mathbb{R}-\mathbb{R}$, Proc. Conf. on Topology and Measure, I (Zinnowitz, 1974), Part 2 (Greifswald), Ernst-Moritz-Arndt Unov., 1978, pp. 257–283.

[17] L. G. Oversteegen and E. D. Tymchatyn, *On span and weakly chainable continua*, Fund. Math. **122** (2) (1984), 159–174.

[18] R. Sikorski and C. Zarankiewicz, *On uniformization of functions (I)*, Fund. Math. **41** (1955), 339–344.

[19] Z. Waraszkiewicz, *Sur un problème de M. H. Hahn*, Fund. Math. **22** (1934), 180–205.

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201-1881, U.S.A.

E-mail address: paulb@mscs.mu.edu