SEMICONTINUITY OF BETWEENNESS FUNCTIONS

PAUL BANKSTON, AISLING MCCLUSKEY, AND RICHARD J. SMITH

Abstract. A ternary relational structure $\langle X, [\cdot,\cdot,\cdot]\rangle$, interpreting a notion of betweenness, gives rise to the family of intervals, with interval $[a, b]$ being defined as the set of elements of X between a and b. Under very reasonable circumstances, X is also equipped with some topological structure, in such a way that each interval is a closed nonempty subset of X. The question then arises as to the continuity behavior–within the hyperspace context–of the betweenness function $\{x, y\} \mapsto [x, y]$. We investigate two broad scenarios: the first involves metric spaces and Menger's betweenness interpretation; the second deals with continua and the subcontinuum interpretation.

1. Introduction and Preliminaries

Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be a **ternary structure**; i.e., X is a set and $[\cdot,\cdot,\cdot] \subseteq X^3$ is a ternary relation on X. The relation is intended to convey a notion of inclusive betweenness, so we assume it to be **basic**; i.e., it satisfies the conditions that $[a, a, b]$ and $[a, b, b]$ always hold (*inclusivity*), that $[a, c, b]$ implies $[b, c, a]$ (*symmetry*), and that $[a, c, a]$ implies $a = c$ (*uniqueness*).

For each $a, b \in X$, we define the **interval** $[a, b]$ to be the set $\{x \in X : [a, x, b]\}$. Then, in interval terms, the three basic criteria above become $[a, b] \supseteq \{a, b\}$, $[a, b] = [b, a]$, and $[a, a] = \{a\}$, respectively. There is a unique smallest basic relation, namely the one where $[a, b] = \{a, b\}$ identically. This we refer to here as the **minimal** ternary relation on X.

The points a and b are **bracket points** (and $\{a, b\}$ a **bracket pair**) for the interval $[a, b]$. If I is an interval, its **bracket set** is defined to be $\{\{a, b\} : [a, b] = I\}$.

The assignment $\{x, y\} \mapsto [x, y]$ is the **betweenness function** associated with $[\cdot,\cdot,\cdot]$, and is denoted throughout the text by $[\cdot,\cdot]$. Hence the bracket set for interval I is just the fiber over I with respect to this function.

The present paper is a continuation of the project initiated in [2] (see also [3, 4]); here we are interested in the issue of when nearby bracket pairs give rise to nearby intervals. The best way to make sense of this is to give X some topological structure, and inquire into whether the betweenness function is continuous in the context of hyperspaces [15].

We consider two broad case studies: the first is where X is a metric space, and $[a, c, b]$ means that c lies between a and b in the sense of Menger [14]; the second is where X is a continuum, and $[a, c, b]$ means that c lies in every subcontinuum of X that contains $\{a, b\}$. In the first study it is both the topology and the geometry of

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metric spaces that dictate the continuity of the betweenness function; in the second it is the topology alone of (not necessarily metrizable) continua.

For a topological space X, we denote by 2^X (resp., $\mathcal{K}(X)$) its *hyperspace* of all nonempty closed (resp., nonempty closed connected) subsets. If U is an open set in X, U^+ (resp., U^-) denotes the set $\{C \in 2^X : C \subseteq U\}$ (resp., $\{C \in 2^X : C \cap U \neq \emptyset\}$). The **upper** (resp., **lower**) **Vietoris topology** on 2^X is subbasically generated by sets of the form U^+ (resp., U^-), as U ranges over the open subsets of X. The join of these two topologies is the **Vietoris topology** on 2^X, and we view $\mathcal{K}(X)$ as inheriting this topology.

We let $\omega := \{0, 1, 2, \dots\}$ denote the set of finite ordinals. It will be convenient to eliminate zero at times, so we use the symbol $\mathbb{N}$ to denote $\omega \setminus \{0\}$.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n(X)$ denote the n**-fold symmetric power** of X, the hyperspace consisting of those $C \in 2^X$ with at most n elements (also equipped with the inherited Vietoris topology). When X is a T_1 space, the function $x \mapsto \{x\}$ defines a homeomorphism from X onto $\mathcal{F}_1(X)$ (where the inherited upper and lower Vietoris topologies coincide); when X is Hausdorff, each $\mathcal{F}_n(X)$ is a closed subspace of 2^X. If X is also normal, then $\mathcal{K}(X)$ is closed in 2^X as well. Of the hyperspaces $\mathcal{F}_n(X)$, we will be interested only in the case $n = 2$ from here on.

The following is a simple, but useful, result (see, e.g., [15]).

Lemma 1.1. *The Vietoris topology on 2^X is basically generated by sets of the form $[\![U_1, \dots, U_n]\!] := \{C \in 2^X : C \subseteq U_1 \cup \dots \cup U_n$ and $C \cap U_i \neq \emptyset$ for $1 \leq i \leq n\}$, where $n \in \mathbb{N}$ and $\langle U_1, \dots, U_n \rangle$ ranges over all n-tuples of open subsets of X.*

Proof. This is a direct consequence of the following identities: $U^+ \cap V^+ = [\![U \cap V]\!]$, $U^- \cap V^- = [\![X, U, V]\!]$, $U^+ \cap V^- = [\![U, U \cap V]\!]$, and $[\![U_1, \dots, U_n]\!] = (\bigcup_{i=1}^n U_i)^+ \cap (\bigcap_{i=1}^n U_i^-)$. □

Unless specified otherwise, the default topology on the hyperspaces defined above is the Vietoris topology. It is a basic fact about this topology (see [15, §4]) that X is compact Hausdorff (resp., compact metrizable) if and only if the same is true for any of these hyperspaces.

If X and Y are two topological spaces, a function $\varphi : Y \to 2^X$ is **upper** (resp., **lower**) **semicontinuous** (**usc** and **lsc**, respectively) at $a \in Y$ if it is continuous at a in the usual sense for the upper (resp., lower) Vietoris topology on 2^X. So φ is continuous at a if and only if it is both usc and lsc at a. And when we unpack the definitions, we see that φ is usc (resp., lsc) at a just in case for any open $U \subseteq X$ such that $\varphi(a) \subseteq U$ (resp., $\varphi(a) \cap U \neq \emptyset$), there is an open neighborhood V of a in Y such that $\varphi(x) \subseteq U$ (resp., $\varphi(x) \cap U \neq \emptyset$) for all $x \in V$.

Recall that a subset of a topological space is **residual** if it contains the intersection of countably many dense open sets, and a **Baire space** is a topological space in which all residual sets are dense. So while residual sets can even be empty in general, they form a countably complete filter of subsets in a Baire space.

By the Baire Category Theorem, all topologically complete metric spaces, as well as all locally compact Hausdorff spaces, are Baire spaces.

Let us say that a certain localized property holds **at almost every** point of a space Y if the set of points at which the property holds is a dense residual subset of Y.

The following result of M. K. Fort [11] (strengthening earlier work of K. Kuratowski [13, §43,VII, Corollary 1]) gives an important link between the two kinds of semicontinuity under consideration here.

Lemma 1.2. *Let X and Y be topological spaces, with X metrizable and Y a Baire space, and suppose $\varphi : Y \to 2^X$ is such that $\varphi(y)$ is compact for each $y \in Y$. If φ is usc (resp., lsc) at every point of Y, then φ is also lsc (resp., usc) at almost every point of Y.*

In the sequel, all of our basic ternary structures $\langle X, [\cdot,\cdot,\cdot]\rangle$ will be **closed**; i.e., X is equipped with a Hausdorff topology for which all intervals are closed subsets. In this way the betweenness function will have domain $\mathcal{F}_2(X)$ and codomain 2^X.

Remark 1.3. In applications of Lemma 1.2, the space Y will be $\mathcal{F}_2(X)$, where X is a topologically complete metric space. In that case X^2 is topologically complete as well, and hence Baire. The function $\langle x, y\rangle \to \{x, y\}$ defines a continuous open map from X^2 onto $\mathcal{F}_2(X)$, and it is an easy exercise to show that the Baire property is thus preserved.

In a slight abuse of language below, we refer to the members of $\mathcal{F}_2(X)$ generically as **pairs**, using the terms *singleton* (resp., *doubleton*) to specify that the pair has cardinality one (resp., two). Typical basic Vietoris-open sets for $\mathcal{F}_2(X)$ may be written as $[\![U, V]\!]_2 := [\![U, V]\!] \cap \mathcal{F}_2(X)$, where U, V are open in X.

The following result concerning semicontinuity is trivial, but worth recording for later reference.

Proposition 1.4. *Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be a closed basic ternary structure, with $a, b \in X$. Then $[\cdot,\cdot]$ is usc at $\{a, b\}$ if $[a, b] = X$, and is lsc at $\{a, b\}$ if $a = b$.*

2. Menger Betweenness in Metric Spaces

Given a metric space $X = \langle X, \varrho\rangle$ and $a, b, c \in X$, we say c **lies between** a **and** b **in the Menger sense** (in symbols, $[a, c, b]_{\mathsf{M}}$ or $c \in [a, b]_{\mathsf{M}}$) if $\varrho(a, b) = \varrho(a, c) + \varrho(c, b)$ (see [14]). We call this relation **M-betweenness**, and the associated intervals **M-intervals**. When there is no confusion over betweenness interpretation, we drop subscripts–i.e., $[a, b] := [a, b]_{\mathsf{M}}$, etc.

Proposition 2.1. M*-betweenness is a closed basic ternary relation. Indeed, each* M*-interval is bounded, as well as closed.*

Proof. M-betweenness is clearly a basic ternary relation, so fix $a, b \in X$ and define $f : X \to \mathbb{R}$ by $f(x) = \varrho(a, x) + \varrho(x, b) - \varrho(a, b)$. Then f is continuous and $[a, b] = f^{\leftarrow}[\{0\}]$, which is closed in X.

To show boundedness, we prove that the diameter of $[a, b]$ is $\varrho(a, b)$. For if $c, d \in [a, b]$, then $\varrho(a, c) + \varrho(c, b) = \varrho(a, b) = \varrho(a, d) + \varrho(d, b)$. From the triangle inequality, we have $\varrho(c, d) \leq \varrho(c, a) + \varrho(a, d)$ and $\varrho(c, d) \leq \varrho(c, b) + \varrho(b, d)$ both holding. Hence

$$2\varrho(c, d) \leq (\varrho(a, c) + \varrho(c, b)) + (\varrho(a, d) + \varrho(d, b)) = 2\varrho(a, b),$$

and therefore $\varrho(c, d) \leq \varrho(a, b)$. □

A metric space (or metric) is **proper** (resp., **M-proper**) if each of its closed bounded subsets (resp., M-intervals) is compact. The metric space is **M-minimal** if its M-betweenness relation is minimal. M-minimal metrics are obviously M-proper. We define a metric space $\langle X, \varrho \rangle$ to be **topologically proper** (resp., **topologically** M**-proper, topologically** M**-minimal**) in exact analogy with how one defines topological completeness; i.e., there is a proper (resp., M-proper, M-minimal) metric on X that is equivalent to ϱ. Every proper metric is M-proper, by Proposition 2.1; proper metrics are easily seen to be complete.

While being topologically proper is an interesting metric space property, the topological modifications of M-proper and M-minimal are not.

Proposition 2.2. *Every metric space is topologically* M*-minimal.*

Proof. Let $f : [0, \infty) \to [0, \infty)$ be the square root function $x \mapsto \sqrt{x}$. Then f is a strictly increasing homeomorphism, satisfying the condition that $f(a) + f(b) > f(a + b)$, for $a, b > 0$. Thus composing any metric with f results in an equivalent metric that is M-minimal. □

Remark 2.3. Our original proof of Proposition 2.2 involved the needlessly sophisticated process of embedding a given metric space into the unit sphere of a Hilbert space. We are grateful to D. Anderson [1] for suggesting the simple argument above.

Any two-valued metric on an infinite set is complete and M-minimal, without being proper. The following shows that an M-minimal metric space with no isolated points can also fail to be topologically proper.

Example 2.4. Let X be the set of rational points on the unit circle (i.e., $X = \{\langle x, y \rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cap (\mathbb{Q} \times \mathbb{Q})$), with ϱ the inherited euclidean metric. Then $\langle X, \varrho \rangle$ is easily seen to be M-minimal. However, this space is countable with no isolated points, so it is not a Baire space and thus not topologically complete. Consequently, it is not topologically proper.

Theorem 2.5. *For a metric space that is either proper or* M*-minimal,* $[\cdot, \cdot]$ *is usc at all pairs.*

Proof. If the metric space is M-minimal, then $[\cdot, \cdot]$ is the inclusion map, and is hence continuous. Suppose we have a proper metric space $\langle X, \varrho \rangle$ such that usc fails for some $\{a, b\} \in \mathcal{F}_2(X)$. Then there is an open subset U of X such that: (1) $[a, b] \subseteq U$; and (2) for each $n \in \mathbb{N}$, there are a_n, b_n, c_n, where $\varrho(a, a_n), \varrho(b, b_n) \leq \frac{1}{n}$ and $c_n \in [a_n, b_n] \setminus U$. Then $\varrho(c_n, a) \leq \varrho(c_n, a_n) + \varrho(a_n, a) \leq \varrho(b_n, a_n) + \frac{1}{n} \leq (\frac{2}{n} + \varrho(b, a)) + \frac{1}{n} = \frac{3}{n} + \varrho(b, a)$, implying that the sequence $\langle c_n \rangle$ is bounded. A metric's being proper is clearly equivalent to bounded sequences having convergent subsequences, so we know there is a subsequence of $\langle c_n \rangle$ that converges. We lose no generality in assuming that $c_n \to c$ for some $c \in X$.

Note that $\varrho(a_n, b_n) = \varrho(a_n, c_n) + \varrho(c_n, b_n)$ for $n \in \mathbb{N}$, and that $a_n \to a$, $b_n \to b$, $c_n \to c$. We may thus use continuity of the metric function to infer that $\varrho(a, b) = \varrho(a, c) + \varrho(c, b)$. Hence $c \in [a, b] \subseteq U$. This implies that some c_n is contained in U, a contradiction. □

Remark 2.6. Theorem 2.5 no longer holds for metric spaces that are merely complete. (See Proposition 4.15 and Example 4.19 below.)

Question 2.7. Does Theorem 2.5 still hold if the metric is complete and M-proper?

The following example shows that Theorem 2.5 need not hold, even for metric spaces that are both topologically proper and M-proper.

Example 2.8. Let X be the "deleted" harmonic fan[1] in the euclidean plane, namely the union $D \cup \bigcup_{n=1}^{\infty} S_n$, where $D = ([0,1] \times \{0\}) \setminus \{\langle \frac{1}{2}, 0\rangle\}$ and $S_n = \{\langle t, \frac{t}{n}\rangle : 0 \leq t \leq 1\}$. Let ϱ be the euclidean metric on $\mathbb{R}^2$, restricted to X. This metric is not complete, hence not proper either. However, X is both locally compact and separable, and is hence topologically proper [12, Theorem 5.3].

Let $a = \langle 0,0\rangle$, $b = \langle 1,0\rangle$, with $b_n = \langle 1, \frac{1}{n}\rangle$, $n \in \mathbb{N}$. Given $x \in X \setminus \{b\}$, we see that $[x,b]$ is a simple sequence (i.e., homeomorphic to the ordinal space $\omega + 1$) if $x \neq a$ and is $\{a,b\}$ if $x = a$. If $x, y \in X \setminus \{b\}$, then $[x,y]$ is either finite or a closed line segment. In any case, M-intervals in X are compact.

Now each $[a, b_n]$ is the closed line segment S_n, and hence connected, while $[a,b]$ is the disconnected set $\{a,b\}$. Let U and V be disjoint open sets, where $a \in U$ and $b \in V$. Then $[a,b] \subseteq U \cup V$. However, the sequence $\langle \{a, b_n\}\rangle$ converges to $\{a,b\}$ in $\mathcal{F}_2(X)$ and $[a,b_n] \not\subseteq U \cup V$ for $n \in \mathbb{N}$. This shows $[\cdot,\cdot]$ not to be usc at $\{a,b\}$.

If X is a complete metric space, then $\mathcal{F}_2(X)$ is a Baire space (see Remark 1.3). Since proper metrics are complete, we may combine Theorem 2.5 and Lemma 1.2 to obtain the following.

Corollary 2.9. *For a proper metric space,* $[\cdot,\cdot]$ *is lsc–and hence continuous–at almost every pair.*

The assumption that the metric is proper is unnecessary in Theorem 2.5 if we limit our attention to singletons.

Proposition 2.10. *For any metric space,* $[\cdot,\cdot]$ *is continuous at each singleton.*

Proof. By Proposition 1.4, we need only prove upper semicontinuity. Let $a \in X$ and let $U \subseteq X$ be open, such that $[a,a] = \{a\} \subseteq U$. Fix $r > 0$ such that the open $3r$-ball $B(a; 3r)$, centered at a, is contained in U. If $\{a', b'\} \in [\![B(a,r), B(a,r)]\!]_2$ and $c \in [a', b']$, then $\varrho(a,c) \leq \varrho(a,a') + \varrho(a',c) \leq \varrho(a,a') + \varrho(a',b') \leq 2\varrho(a,a') + \varrho(a,b') < 3r$; hence $[a',b'] \subseteq U$. □

The question arises whether Corollary 2.9 may be extended to lsc at all pairs, and hence to continuity itself, but that is not possible, even for compact metric spaces.

Example 2.11. Let X be the unit circle with ϱ the intrinsic (i.e., "shortest arc") metric on X. Then $[\cdot,\cdot]$ fails to be lsc precisely at the antipodal pairs: Without loss of generality, let $a = \langle 0, -1\rangle$ and $b = \langle 0, 1\rangle$, with U equal to X intersected with the open right half-plane. If $b' \in X$ is any point with negative first coordinate, then $[a,b'] \cap U = \emptyset$, while $[a,b] = X$, and hence intersects U. It is easy to show that $[\cdot,\cdot]$ is lsc at any nonantipodal pair, and that the set of such pairs is a dense open subset of $\mathcal{F}_2(X)$ (more than just a dense residual set, as guaranteed by Corollary 2.9) .

The following four betweenness notions will prove useful in subsequent discussions. The first two are "transitivities" (in the sense of [18, 19]).

A basic ternary structure is τ**-basic** (resp., κ**-basic**) if it satisfies the *transitivity* (resp., *convexity*) axiom, namely that $[a,c] \subseteq [a,b]$ (resp., $[c,d] \subseteq [a,b]$) for all $c, d \in [a,b]$. Clearly every κ-basic structure is τ-basic. Menger proves in [14, Erste Untersuchungen, §2] that (the M-betweenness structures of) metric spaces are always τ-basic; he also provides an *ad hoc* example of a finite metric space that is not κ-basic. Here is one that is a bit more geometric.

[1] Corrected from the published version.

Example 2.12. Let X be the unit circle from Example 2.11, and let $Y = X \cup H$, where $H = [-1, 1] \times \{0\}$ and the "shortest arc" metric is extended in the obvious way. To show X is not κ-basic, we choose the points a and b as before, and we set $c = \langle -1, 0\rangle$, $d = \langle 1, 0\rangle$, and $e = \langle 0, 0\rangle$. Then $\{c, d\} \subseteq [a, b] = X$ and $e \in [c, d] = H$, but $e \notin [a, b]$.

The second two notions are as follows: a basic ternary structure is **weakly disjunctive** if $[a, b] = [a, c] \cup [c, b]$ whenever $c \in [a, b]$; it is **antisymmetric** if intervals $[a, b]$ and $[a, c]$ are unequal whenever $b \neq c$.

Proposition 2.13. *Every weakly disjunctive τ-basic structure is κ-basic.*

Proof. Given $c, d \in [a, b]$, let $x \in [c, d]$ be arbitrary. By weak disjunctivity, we have either $d \in [a, c]$ or $d \in [c, b]$. In each instance, two applications of transitivity give us $x \in [a, b]$. ☐

In a basic ternary structure $\langle X, [\cdot, \cdot, \cdot]\rangle$, fix $a \in X$ and define the binary relation $\leq_a$ by setting $x \leq_a y$ to mean $x \in [a, y]$. Then transitivity in the betweenness sense is equivalent to saying each $\leq_a$ is transitive in the order sense, and hence a pre-order. And for τ-basic structures, antisymmetry in the betweenness sense is equivalent to antisymmetry in the order sense, so that each $\leq_a$ becomes a partial order. For any $a, b \in X$, define $\leq_{a,b}$ to be the restriction of $\leq_a$ to $[a, b]$. The following is an amalgamation of Propositions 5.0.4 and 5.0.5 of [2], and will be of use later on.

Lemma 2.14. *For any τ-basic structure that is both antisymmetric and weakly disjunctive, each $\leq_a$ is a tree order with least element (root) a, and each $\leq_{a,b}$ is a total order with least element a and greatest element b.*

It is easy to see that metric spaces are always antisymmetric; the space X in Example 2.11 is κ-basic without being weakly disjunctive.

The following result, also of use in the sequel, is about betweenness functions, and is an immediate consequence of [2, Theorem 5.0.6].

Lemma 2.15. *For a τ-basic weakly disjunctive ternary structure, antisymmetry is equivalent to injectivity of the betweenness function.*

3. Menger Betweenness in Geodesic Spaces

Our main aim in this section is to remove the word *almost* from the conclusion of Corollary 2.9. We show that this can be done if we add to the hypothesis the condition that the metric space is *unique-geodesic*, meaning (roughly) that between any two points, there is–up to reparameterization–a unique path whose length is the distance between those points.

In light of the fact that there is considerable terminological variation in the metric geometry literature ([7, 8, 16] are good modern sources), we beg the reader's indulgence and carefully lay out the elementary notions we use.

A connected compact Hausdorff topological space is called a **continuum**; a **subcontinuum** of a space is a subset that is a continuum in its subspace topology. A continuum–or any topological space–is **nondegenerate** if it has at least two points. A **Peano continuum** is a metrizable continuum that is also locally connected.

Let $\langle X, \varrho\rangle$ be a metric space, with $a, b \in X$. A **path from a to b** is a continuous map $p : [\alpha, \beta] \to X$, where $[\alpha, \beta] \subseteq \mathbb{R}$ is a closed bounded interval, $p(\alpha) = a$, and $p(\beta) = b$. The interval $[\alpha, \beta]$ is the **parameterization interval**, a is the

initial point, and b is the **terminal point** of the path. The image of p, a Peano subcontinuum of X, is called the **support** of p, and is denoted $\lfloor p \rfloor$.

If p is a path from a to b, any path q from b to a is said to be **oppositely oriented** to p. As a prime example of this, we have the **reverse** path $\overleftarrow{p} : [\alpha, \beta] \to X$, defined by $\overleftarrow{p}(s) := p(\alpha + \beta - s)$. Clearly $\lfloor \overleftarrow{p} \rfloor = \lfloor p \rfloor$.

We define the *length* $\Lambda(p)$ of a path $p : [\alpha, \beta] \to X$ in the classical way. First define a **subdivision** of $[\alpha, \beta]$ to be a finite sequence $\langle s_0, \dots, s_n \rangle$, where $\alpha = s_0 \leq s_1 \leq \cdots \leq s_n = \beta$. Given subdivision $\Sigma = \langle s_0, \dots, s_n \rangle$, we denote by $\Lambda(p, \Sigma)$ the sum $\sum_{i=0}^{n-1} \varrho(p(s_i), p(s_{i+1}))$. Then the **length** $\Lambda(p)$ of p is the (possibly infinite) supremum of the set of real numbers $\Lambda(p, \Sigma)$, as Σ ranges over all subdivisions of $[\alpha, \beta]$.

The length of a path is largely–but not entirely–independent of its parameterization or orientation, as we delineate next.

Given paths $p : [\alpha, \beta] \to X$ and $q : [\gamma, \delta] \to X$ from a to b, write $p \preceq q$ to mean that there is a weakly increasing surjection $\mu : [\alpha, \beta] \to [\gamma, \delta]$ such that $p = q \circ \mu$. This relation between paths from a to b is clearly reflexive and transitive, and we define $\simeq$ to be the smallest equivalence relation containing $\preceq$.

The following is well known and an easy exercise.

Lemma 3.1. *If $p : [\alpha, \beta] \to X$ is any path from a to b, then $\Lambda(\overleftarrow{p}) = \Lambda(p)$. Also if $q : [\gamma, \delta] \to X$ is any path from a to b such that $q \simeq p$, then $\Lambda(q) = \Lambda(p)$.*

We next come to the important notion of path concatenation. Suppose $p : [\alpha, \gamma] \to X$ and $q : [\gamma, \beta] \to X$ are paths, where $\alpha \leq \gamma \leq \beta$ and $p(\gamma) = q(\gamma)$. Then the **concatenation** $pq : [\alpha, \beta] \to X$ is given by the rule:

$$(pq)(t) := \begin{cases} p(t) & \text{if } \alpha \leq t \leq \gamma \\ q(t) & \text{if } \gamma \leq t \leq \beta. \end{cases}$$

We leave the straightforward proof of the following to the reader.

Lemma 3.2. *Under the assumptions above, $\Lambda(pq) = \Lambda(p) + \Lambda(q)$.*

The metric space X is **intrinsic** if for each $a, b \in X$, $\varrho(a, b)$ equals the infimum of the lengths $\Lambda(p)$ as p ranges over all paths from a to b. If p is a path whose length is $\varrho(a, b)$, then we call p a **geodesic** from a to b. From Lemma 3.1, $\overleftarrow{p}$ is a geodesic from b to a if and only if p is a geodesic from a to b. Also, if p and q are paths from a to b and $p \simeq q$, then p is a geodesic if and only if q is one too.

If each pair of points of X can be joined by a geodesic, we call X a **geodesic space**. Clearly every geodesic space is intrinsic, but the converse is not true: equipped with the euclidean metric, the punctured plane $\mathbb{R}^2 \setminus \{\langle 0, 0 \rangle\}$ is a (locally compact) intrinsic metric space, but no two points $\langle a, b \rangle$ and $\langle -a, -b \rangle$ can be joined by a geodesic. It is well known [16] that a locally compact intrinsic metric space is a geodesic space if its metric is complete.

Lemma 3.3. *Let X be an intrinsic metric space, with $p : [\alpha, \beta] \to X$ a geodesic from a to b and $\alpha \leq \gamma \leq \delta \leq \beta$. If $c = p(\gamma)$ and $d = p(\delta)$, then $p|_{[\gamma, \delta]}$, the restriction of p to $[\gamma, \delta]$, is a geodesic from c to d.*

Proof. Assume the contrary. Then there is a path $q : [\gamma, \delta] \to X$ from c to d such that $\Lambda(q) < \Lambda(p|_{[\gamma, \delta]})$. But then we have the concatenation $r = (p|_{[\alpha, \gamma]})q(p|_{[\delta, \beta]})$, a path from a to b; and, by Lemma 3.2, $\Lambda(r) = \Lambda(p_{[\alpha, \gamma]}) + \Lambda(q) + \Lambda(p_{[\delta, \beta]}) <$

$\Lambda(p_{[\alpha,\gamma]}) + \Lambda(p|_{[\gamma,\delta]}) + \Lambda(p_{[\delta,\beta]}) = \Lambda(p)$. This contradicts the assumption that p is a geodesic. $\square$

We now bring Menger betweenness into the discussion.

Proposition 3.4. *If X is an intrinsic metric space and p is a geodesic from a to b, then $\lfloor p \rfloor \subseteq [a, b]$.*

Proof. Suppose $p : [\alpha, \beta] \to X$ is a geodesic from a to b, and pick $\gamma \in [\alpha, \beta]$, with $c = p(\gamma)$. Then, by Lemmas 3.2 and 3.3, $\varrho(a, b) = \Lambda(p) = \Lambda(p|_{[\alpha,\gamma]}) + \Lambda(p|_{[\gamma,\beta]}) = \varrho(a, c) + \varrho(c, b)$. Hence $c \in [a, b]$. $\square$

Proposition 3.5. *Let X be an intrinsic metric space, with $a, b, c \in X$ such that $c \in [a, b]$. If $p : [\alpha, \gamma] \to X$ is a geodesic from a to c and $q : [\gamma, \beta] \to X$ is a geodesic from c to b, then pq is a geodesic from a to b.*

Proof. By Lemma 3.2, $\Lambda(pq) = \Lambda(p) + \Lambda(q) = \varrho(a, c) + \varrho(c, b)$ since p and q are geodesics. The right-hand side is $\varrho(a, b)$ since $c \in [a, b]$; thus pq is a geodesic from a to b. $\square$

Proposition 3.6. *Let X be a geodesic space. Then for any $a, b \in X$, $[a, b] = \bigcup\{\lfloor p \rfloor : p$ is a geodesic from a to $b\}$. In particular, M-intervals are connected closed bounded sets, and $[\cdot, \cdot]$ maps $\mathcal{F}_2(X)$ to $\mathcal{K}(X)$.*

Proof. By Proposition 3.4, the left-hand side contains the right. Now suppose $c \in [a, b]$. Then there are geodesics $q : [\alpha, \gamma] \to X$, from a to c, and $r : [\gamma, \beta] \to X$, from c to b. By Lemma 3.5, $p = qr$ is a geodesic from a to b. Thus $c \in \lfloor p \rfloor$ and we infer that the right-hand side contains the left.

Each support is a Peano continuum in X. Since $[a, b]$ is a union of a family of connected sets containing the point a, it too must be connected. It is closed and bounded, by Proposition 2.1. $\square$

We next set about showing that the supports of geodesics are arcs. Recall that a point a of a connected topological space X is a **cut** point if $X \setminus \{a\}$ is disconnected; a **noncut** point otherwise. It is well known [17] that every nondegenerate continuum has at least two noncut points; arcs are homeomorphic copies of $[0, 1] \subseteq \mathbb{R}$, and are characterized as being those metrizable continua possessing precisely two. If X is a continuum and $A \subseteq X$, we say X is **irreducible about** A if no proper subcontinuum of X contains A. Every continuum is irreducible about its set of noncut points. The following will prove useful in achieving the main aim of this section.

Proposition 3.7. *Let X be an intrinsic metric space. If $p : [\alpha, \beta] \to X$ is a geodesic from a to b, then p is a monotone map; hence $\lfloor p \rfloor$ is either degenerate or an arc with noncut points a and b.*

Proof. If p is nonmonotone, then there are γ, δ, with $\alpha \leq \gamma < \delta \leq \beta$, such that $p(\gamma) = p(\delta)$, but $p|_{[\gamma,\delta]}$ is nonconstant. By Lemma 3.3, we know $p|_{[\gamma,\delta]}$ is a geodesic from a point to itself. It is immediate from the definition that this cannot happen, that any geodesic from a point to itself must be constant.

Thus p must be a monotone map. Assuming $a \neq b$ in $\lfloor p \rfloor$ and $c \in \lfloor p \rfloor \setminus \{a, b\}$, it is easy to show that the monotonicity of p implies that c is a cut point of $\lfloor p \rfloor$. Since every nondegenerate continuum possesses at least two noncut points, we know that $\lfloor p \rfloor$ is an arc with noncut points a and b. $\square$

The intrinsic metric space X is **unique-geodesic** at the pair $\{a, b\}$ if: (1) there is a geodesic p from a to b; and (2) for any geodesic q from a to b (or *vice versa*), we have $\lfloor q \rfloor = \lfloor p \rfloor$. The space is **unique-geodesic** if it is unique-geodesic at each pair.

Theorem 3.8.

(i) *Every unique-geodesic space is weakly disjunctive.*
(ii) *Every* M*-proper weakly disjunctive geodesic space is unique-geodesic.*

Proof. Let X be any geodesic space, with $a, b, c \in X$ such that $c \in [a, b]$. Proposition 3.6 provides us with a geodesic $p : [\alpha, \beta] \to X$ from a to b, where $c \in \lfloor p \rfloor$. Suppose $\alpha \leq \gamma \leq \beta$ is such that $c = p(\gamma)$. By Lemma 3.3, $q = p|_{[\alpha,\gamma]}$ (resp., $r = p|_{[\gamma,\beta]}$) is a geodesic from a to c (resp., from c to b), and by Proposition 3.4, we have $\lfloor q \rfloor \subseteq [a, c]$ (resp., $\lfloor r \rfloor \subseteq [c, b]$).

Given any $d \in [a, b]$, uniqueness of geodesic provides us with some $\alpha \leq \delta \leq \beta$ such that $d = p(\delta)$. If $\delta \leq \gamma$, we have $d \in [a, c]$, by the argument in the last paragraph; if $\gamma \leq \delta$, we have $d \in [c, b]$. Hence X is weakly disjunctive.

Now suppose X is an M-proper weakly disjunctive geodesic space. Then by Lemma 2.14, each binary relation $\leq_{a,b}$ is a total order on $[a, b]$, with least element a and greatest element b.

Thus $[a, b]$ has both an order topology induced by $\leq_{a,b}$ and a subspace topology induced by ϱ. We first claim that every order-closed subset of $[a, b]$ is subspace-closed: given $x, y \in [a, b]$ with $x \leq_{a,b} y$, let $[x, y]_{a,b} := \{z \in [a, b] : x \leq_{a,b} z \leq_{a,b} y\}$. Then a closed-set subbase for the order-closed sets consists of order-intervals of the form $[a, y]_{a,b}$ and $[x, b]_{a,b}$, $x, y \in [a, b]$. Then it is straightforward from the definition of $\leq_{a,b}$ that $[a, y]_{a,b} = [a, y]$ and $[x, b]_{a,b} = [x, b]$ always, so by Proposition 2.1 each of these order-intervals is subspace-closed. This proves our claim.

Because the metric is M-proper, we know that the subspace topology on $[a, b]$ is compact. Hence, so is the order topology. Since the smaller topology is also Hausdorff, the two topologies must coincide. Since M-intervals are also connected, by Proposition 3.6, this makes $[a, b]$ a totally ordered continuum with end points a and b. Therefore $[a, b]$ is an arc with noncut points a and b, showing that X is unique-geodesic. □

Question 3.9. Can the hypothesis of being M-proper be removed from Theorem 3.8 (ii)?

The space in Example 2.11 is a proper (indeed compact) geodesic space that fails to be unique-geodesic at some (i.e., the antipodal) pairs. We next show that this condition is necessary to have failure of lsc at a pair.

Theorem 3.10. *For a proper geodesic space, being unique-geodesic at a pair of points implies that* $[\cdot, \cdot]$ *is lsc (and hence continuous) at that pair.*

Proof. By Theorem 2.5, all we need to concentrate on is lower semicontinuity.

Let X be a proper geodesic space. Then, by Proposition 3.6, every M-interval in X is a subcontinuum.

Fix $\{a, b\} \in \mathcal{F}_2(X)$ so that there is just one geodesic from a to b. Assuming failure of lsc at $\{a, b\}$, we have an open $U \subseteq X$ such that: (1) $[a, b] \cap U \neq \emptyset$; and (2) for each $n \in \mathbb{N}$, there are a_n, b_n, where $\varrho(a, a_n), \varrho(b, b_n) \leq \frac{1}{n}$ and $[a_n, b_n] \cap U = \emptyset$.

Suppose $c \in \bigcup_{n=1}^{\infty} [a_n, b_n]$. Then, arguing as in the proof of Theorem 2.5, we infer that $\varrho(c, a) \leq 3 + \varrho(b, a)$; hence that $\bigcup_{n=1}^{\infty} [a_n, b_n]$ is bounded in X. Let

$Y = \overline{\bigcup_{n=1}^{\infty}[a_n, b_n]}$. Then Y, being both closed and bounded in X, is compact. Consequently 2^Y is a compact metrizable subspace of 2^X, and the sequence $\langle [a_n, b_n]\rangle$ has a subsequence that converges to something in 2^Y. Without loss of generality, we may assume the sequence itself converges, say, to $A \in 2^Y$.

To show $a \in A$, suppose otherwise. Invoking regularity, we have open $V \subseteq X$ with $a \in V \subseteq \overline{V} \subseteq X \setminus A$. But then $A \in [\![X \setminus \overline{V}]\!]$, and hence for all but finitely many $n \in \mathbb{N}$, we have $[a_n, b_n] \in [\![X \setminus \overline{V}]\!]$. This says that $[a_n, b_n] \cap V = \emptyset$ for all but finitely many n, and hence that $a_n \not\to a$. Similarly we show $b \in A$.

If A were not connected, we could invoke normality to find disjoint open sets V, W such that $A \in [\![V, W]\!]$. But then $[a_n, b_n] \in [\![V, W]\!]$ for some $n \in \mathbb{N}$, contradicting connectedness in M-intervals. Now we know A is a subcontinuum of X containing $\{a, b\}$.

Next we show $A \subseteq [a, b]$. Indeed, fix $c \in A$; and for $n \in \mathbb{N}$, let $U_n = B(c; \frac{1}{n})$. Using the facts that $[a_n, b_n] \to A$ and that $[\![U_1, X]\!]$ is a Vietoris-open neighborhood of A, let $n_1 \in \mathbb{N}$ be least such that $[a_n, b_n] \cap U_1 \neq \emptyset$ for all $n \geq n_1$. Pick $c_{n_1} \in [a_{n_1}, b_{n_1}] \cap U_1$. For our inductive hypothesis, assume we have $n_1 < \cdots < n_k$, points $c_{n_i} \in [a_{n_i}, b_{n_i}] \cap U_{n_i}$, $1 \leq i \leq k$, and that $[a_n, b_n] \cap U_{n_k} \neq \emptyset$ for all $n \geq n_k$. Then pick $n_{k+1} > n_k$ to be least such that $[a_n, b_n] \cap U_{n_{k+1}} \neq \emptyset$ for all $n \geq n_{k+1}$. Fix $c_{n_{k+1}} \in [a_{n_{k+1}}, b_{n_{k+1}}] \cap U_{n_{k+1}}$. This gives us a sequence $\langle c_{n_i}\rangle$ converging to c. Since $\varrho(a_{n_k}, c_{n_k}) + \varrho(c_{n_k}, b_{n_k}) = \varrho(a_{n_k}, b_{n_k})$, $k \in \mathbb{N}$, $a_{n_k} \to a$, and $b_{n_k} \to b$, we know $\varrho(a, c) + \varrho(c, b) = \varrho(a, b)$. Hence $c \in [a, b]$, and we infer $A \subseteq [a, b]$.

Finally, by Proposition 3.7, and since there is just one geodesic from a to b, we know that $[a, b]$ is an arc with noncut points a, b. A is a subcontinuum of $[a, b]$ containing the noncut points of $[a, b]$; hence $A = [a, b]$, thanks to irreducibility. Thus $A \cap U \neq \emptyset$, and we may conclude–as above–that $[a_n, b_n] \cap U \neq \emptyset$ for some n, which is a contradiction. □

We can now put Theorems 2.5 and 3.10 together to fulfill the stated aim of this section, to remove the word *almost* from Corollary 2.9.

Corollary 3.11. *For a proper unique-geodesic space,* $[\cdot, \cdot]$ *is continuous at every pair.*

Because M-betweenness is always antisymmetric, we may combine Lemma 2.15, Proposition 3.6, Theorem 3.8, and Corollary 3.11 to obtain the following.

Corollary 3.12. *For a proper unique-geodesic space* X, $[\cdot, \cdot]$ *is a continuous injection from* $\mathcal{F}_2(X)$ *to* $\mathcal{K}(X)$*. It is a topological embedding if* X *is compact.*

4. Menger Betweenness in Normed Vector Spaces

In this section we consider Menger betweenness in geodesic spaces arising from linear algebra. Here we take a **normed (vector) space** to be a pair $\langle X, \|\cdot\|\rangle$, where X is a vector space over the field $\mathbb{R}$ of real numbers and $\|\cdot\|$ is a norm. (As usual, we abuse notation slightly, writing X for $\langle X, \|\cdot\|\rangle$ when there is no possible ambiguity.) We define the **(closed) unit ball** and **unit sphere** of X by $B_X := \{x \in X : \|x\| \leq 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$, respectively. A norm on a vector space naturally gives rise to a metric ϱ, defined by $\varrho(x, y) := \|x - y\|$, which we refer to as the **norm metric**. We obtain geodesics in normed spaces in the simplest possible way: for any $a, b \in X$, define the **standard straight path** $L_{a,b} : [0, 1] \to X$ by $L_{a,b}(s) := (1 - s)a + sb$. The support $[a, b]_{\mathsf{L}} := \lfloor L_{a,b} \rfloor$ is, of course, the closed line segment with end points a, b.

A normed space X is called **strictly convex** if for any $a, b \in S_X$ distinct, $[a,b]_{\mathsf{L}} \cap S_X = \{a,b\}$. Among the strictly convex normed spaces are those whose norms arise from an inner product (e.g., Hilbert spaces, those inner product spaces whose norm metrics are complete). There are many characterizations of strict convexity; the one of most relevance here is the following.

Proposition 4.1 ([7, Proposition I.1.6])**.** *In any normed space, $L_{a,b}$ is a geodesic from a to b. Moreover, a normed space is strictly convex if and only if it is unique-geodesic.*

Consequently, by Proposition 3.6, $[a,b] \supseteq [a,b]_{\mathsf{L}}$ whenever a and b are in a normed space X, and equality always holds if and only if X is strictly convex. Thus, in this context, we may use the terms *strictly convex* and *unique-geodesic* interchangeably. Below, in Theorem 4.10, we provide a complete geometric description of the intervals $[a,b]$, in all cases.

The following result provides analogues–but not consequences–of Theorem 3.10 and Corollary 3.11. As is well known, norm metrics are proper exactly when the vector space dimension is finite, and that is not assumed here. Note that the usc component of the result below easily follows from later results (i.e., Theorems 4.10 and 4.21)–the additional proof given below is included as it is simple and direct.

Theorem 4.2. *Let X be a normed space and let $a, b \in X$. If $[a,b] = [a,b]_{\mathsf{L}}$, then $[\cdot,\cdot]$ is lsc at $\{a,b\}$. Moreover, if X is unique-geodesic, then $[\cdot,\cdot]$ is usc (and consequently continuous) at every pair.*

Proof. Let $a, b \in X$ and $r > 0$. Note that if $\|a' - a\|, \|b' - b\| \le r$, then

$$\|((1-s)a' + sb') - ((1-s)a + sb)\| = \|(1-s)(a'-a) + s(b'-b)\| \le r,$$

whenever $0 \le s \le 1$. Thus each point in $[a',b']_{\mathsf{L}}$ is r-close to some point in $[a,b]_{\mathsf{L}}$.

It follows that if $[a,b] = [a,b]_{\mathsf{L}}$, then $[\cdot,\cdot]$ is lsc at $\{a,b\}$. Indeed, if U is open and $[a,b] \cap U = [a,b]_{\mathsf{L}} \cap U \neq \emptyset$, then using the observation above, $[a',b'] \cap U \supseteq [a',b']_{\mathsf{L}} \cap U \neq \emptyset$ for a sufficiently small $r > 0$.

Now assume that X is unique-geodesic. For upper semicontinuity, we simply remark that, given an open set U such that $[a,b] \subseteq U$, the compactness of $[a,b] = [a,b]_{\mathsf{L}}$ ensures that, for a sufficiently small $r > 0$, we know that $y \in U$ whenever y is r-close to a point in $[a,b]$. Hence $[a',b'] = [a',b']_{\mathsf{L}} \subseteq U$ for such an r. □

We next consider how being unique-geodesic can be cast in terms of convexity. A subset K of a normed space X is **M-convex** (resp., **linearly convex**) if whenever $a, b \in K$, we also have $[a,b] \subseteq K$ (resp., $[a,b]_{\mathsf{L}} \subseteq K$). Linear convexity is the usual notion of convexity from linear algebra, and relies solely on the vector space structure of X. On the other hand, the clearly stronger notion of M-convexity is a special case of something that makes sense for any basic ternary structure. (Indeed, the ternary structure associated with a metric space is κ-basic precisely when its M-intervals are M-convex.)

The following is to be expected of M-betweenness in normed spaces.

Proposition 4.3. *The M-intervals of a normed space are linearly convex.*

Proof. Given points $a, b, c, d \in X$, with $c, d \in [a,b]$, and $e = (1-s)c + sd \in [c,d]_{\mathsf{L}}$, it suffices to show that $\|e - a\| + \|b - e\| \le \|b - a\|$.

We are already given that $\|c-a\|+\|b-c\| = \|b-a\| = \|d-a\|+\|b-d\|$. Hence

$$\begin{aligned}\|e-a\|+\|b-e\| &= \|((1-s)c+sd)-a\|+\|b-((1-s)c+sd)\|\\ &= \|(1-s)(c-a)+s(d-a)\|+\|(1-s)(b-c)+s(b-d)\|\\ &\le (1-s)(\|c-a\|+\|b-c\|)+s(\|d-a\|+\|b-d\|) = \|b-a\|,\end{aligned}$$

as desired. □

Remark 4.4. As Example 2.12 shows, one cannot immediately generalize Proposition 4.3 to geodesic spaces, with $[c,d]_{\mathsf{L}}$ being replaced with even a unique-geodesic from c to d.

In the sequel we will refer to linearly convex sets simply as convex. While the unit ball (or any open or closed ball) in a normed space is convex, it is not necessarily M-convex, as Example 4.5 below shows.

Example 4.5. Let $X = \mathbb{R}^n_p := \langle \mathbb{R}^n, \|\cdot\|_p\rangle$, where the ***p*-norm** ($p>1$) of $\vec{x} = \langle \vec{x}(1),\ldots,\vec{x}(n)\rangle$ is given by $\|\vec{x}\|_p := \left(\sum_{i=1}^n |\vec{x}(i)|^p\right)^{\frac{1}{p}}$. (So the usual euclidean norm is just $\|\cdot\|_2$.) The **∞-norm** is given by $\|\vec{x}\|_\infty := \max\{|\vec{x}(1)|,\ldots,|\vec{x}(n)|\}$, which in turn equals $\lim_{p\to\infty}\|\vec{x}\|_p$. (See, e.g., [7, I.1], [9, III.1].)

For $X = \mathbb{R}^2_1$, the unit ball B_X is the square with corners $\langle \pm 1, 0\rangle$ and $\langle 0,\pm 1\rangle$, and the M-intervals are rectangles with sides parallel to the coordinate axes. In particular, when $\vec{a}\in B_X$ is in the first quadrant, $[\vec{0},\vec{a}]$ is the rectangle with lower-left corner $\vec{0}$ and upper-right corner $\vec{a}$. All M-intervals of the form $[\vec{0},\vec{a}]$, where $\vec{a}\in B_X$, lie in B_X; however if $\vec{a} = \langle 0,1\rangle$ and $\vec{b} = \langle 1,0\rangle$, then $[\vec{a},\vec{b}] = [0,1]\times[0,1] \not\subseteq B_X$. Hence B_X is not M-convex. The case with $p=\infty$ is similar because the normed spaces $\mathbb{R}^2_1$ and $\mathbb{R}^2_\infty$ are isometrically isomorphic, via the linear transformation $\langle x,y\rangle \mapsto \langle x-y, x+y\rangle$.

Finally, when $1<p<\infty$, the unit sphere contains no nondegenerate closed line segments; consequently $\mathbb{R}^2_p$ is strictly convex; and, by Proposition 4.1, unique-geodesic.

The following simple example shows that normed vector spaces need not be κ-basic, and complements Example 2.12.

Example 4.6. $\mathbb{R}^3_\infty$ is not κ-basic. To see this, let $\vec{a} = \langle 0,0,0\rangle$, $\vec{b} = \langle 1,0,0\rangle$, $\vec{c} = \langle \frac{1}{2},\frac{1}{2},0\rangle$, $\vec{d} = \langle \frac{1}{2},-\frac{1}{2},0\rangle$, and $\vec{e} = \langle 0,0,\frac{1}{2}\rangle$. Then: (1) $\|\vec{a}-\vec{b}\| = 1$, $\|\vec{a}-\vec{c}\| + \|\vec{c}-\vec{b}\| = \|\vec{a}-\vec{d}\|+\|\vec{d}-\vec{b}\| = \frac{1}{2}+\frac{1}{2} = 1$, so $\{\vec{c},\vec{d}\}\subseteq[\vec{a},\vec{b}]$; (2) $\|\vec{c}-\vec{d}\| = 1$, $\|\vec{c}-\vec{e}\|+\|\vec{e}-\vec{d}\| = \frac{1}{2}+\frac{1}{2} = 1$, so $e\in[c,d]$; but (3) $\|\vec{a}-\vec{e}\|+\|\vec{e}-\vec{b}\| = \frac{1}{2}+1 = \frac{3}{2} > \|\vec{a}-\vec{b}\|$, so $e\notin[\vec{a},\vec{b}]$.

Question 4.7. Is every two-dimensional normed vector space κ-basic?

The rest of this section is devoted to a further exploration of conditions that allow (or disallow) semicontinuity of the M-betweenness function *at a given pair*, offering a complete characterization in the usc case. In order to do this–as well as to address the problem of upper semicontinuity–we will need some vocabulary from convexity theory.

Let X be a vector space, let $K\subseteq X$ be convex, and let $a\in K$. We define the **facet** of a in K to be the set

$$F(a) = \{b\in K : s^{-1}a + (1-s^{-1})b \in K \text{ for some } s\in(0,1)\}.$$

Remark 4.8. This definition is equivalent to the one given in Bourbaki [6, TVS II.87]. The set $F(a)$ is always convex, however, as is noted in Bourbaki, if X is in addition a topological vector space (for example, a normed vector space), then $F(a)$ is not always closed. While Example 4.18 below is provided principally for another purpose, it happens to be another example of this phenomenon.

It is clear that $a \in F(a)$, and if $b \in F(a)$ and

$$c := s^{-1}a + (1 - s^{-1})b \in K,$$

where $s \in (0, 1)$, then $a = (1 - s)b + sc$ and, by considering $1 - s$, we have $c \in F(a)$ as well. Recall that $a \in K$ is an **extreme point** of K if no closed line segment containing a in its interior lies entirely in K. Let $\mathrm{ext}(K)$ denote the (possibly empty) set of extreme points of K. Evidently, $a \in \mathrm{ext}(K)$ if and only if $F(a) = \{a\}$.

Suppose that X is a normed vector space, $K \subseteq X$ is closed, bounded and convex, $a \in K$ and $b \in F(a)$. We define

$$\sigma_a(b) = \inf\{s > 0 : s^{-1}a + (1 - s^{-1})b \in K\}.$$

Clearly $\sigma_a(a) = 0$ and $\sigma_a(b) < 1$ in general. The boundedness of K and the inequality

$$\|s^{-1}a + (1 - s^{-1})b\| \geq s^{-1}\|a - b\| - \|b\|,$$

demonstrates that if $b \neq a$, then $\sigma_a(b) > 0$. Moreover, as K is closed, the infimum in the definition of $\sigma_a(b)$ is attained whenever $b \neq a$. Furthermore, regardless of the value of $\sigma_a(b)$, by the convexity of K, we see that $s^{-1}a + (1 - s^{-1})b \in K$ whenever $s \in [\sigma_a(b), 1] \cap (0, 1]$. If a and K are clear from the context, we will write $\sigma(b)$ instead of $\sigma_a(b)$.

The following lemma gives a useful criterion for when a convergent sequence of points in $F(a)$ converges in $F(a)$.

Lemma 4.9. *Fix a bounded, closed convex set K and $a \in K$. Let $\langle b_n \rangle$ be a sequence from $F(a)$ converging to $b \in X$, and assume that $\sigma(b_n) \to s \in [0, 1)$. Then $b \in F(a)$ and $\sigma(b) \leq s$.*

Proof. Suppose that $b_n \neq a$. Then $\sigma(b_n) > 0$ and by the infimum attainment discussed above, we have

$$\sigma(b_n)^{-1}a + (1 - \sigma(b_n)^{-1})b_n \in K,$$

and thus

$$a + (\sigma(b_n) - 1)b_n \in \sigma(b_n)K. \tag{1}$$

Observe that (1) also holds whenever $b_n = a$: if $b_n = a$ then $\sigma(b_n) = 0$ and $\sigma(b_n)K = \{0\}$. By taking limits in (1), and using the fact that K is closed, $b \in K$ and

$$a + (s - 1)b \in sK. \tag{2}$$

If $s = 0$ then we glean from this that $a - b \in 0K = \{0\}$, and so $b = a \in F(a)$, and $\sigma(b) = 0 \leq s$. If $s > 0$ then $s \in (0, 1)$, and it follows from (2) that

$$s^{-1}a + (1 - s^{-1})b \in K,$$

which implies that $b \in F(a)$ and, by definition, $\sigma(b) \leq s$. $\square$

Now assume X has a norm. In the sequel, we will be most concerned with the case $K = B_X$, the (closed) unit ball of X. In this scenario, X is strictly convex if and only if $F(a) = \{a\}$ for all $a \in S_X$, or equivalently, $\mathrm{ext}(B_X) = S_X$. (In Example 4.5, with $1 < p < \infty$, we have $\mathrm{ext}(B_X) = S_X$. When $p = 1$ (resp., $p = \infty$), $\mathrm{ext}(B_X) = \{\langle \pm 1, 0\rangle, \langle 0, \pm 1\rangle\}$ (resp., $\{\langle \pm\sqrt{2}, \pm\sqrt{2}\rangle\}$).) If a and b are distinct points of X, the mapping $x \mapsto \frac{x-a}{\|b-a\|}$ is an affine transformation on X that takes the M-interval $[a, b]$ to the M-interval $[0, \frac{b-a}{\|b-a\|}]$, whose nonzero bracket point lies on S_X. These two intervals are topologically and (with the obvious exception of the scaling factor) geometrically identical. Moreover, $[\mathrm{cot}, \cdot]$ is usc or lsc at $\{a, b\}$ if and only if the same is true at $\{0, \frac{b-a}{\|b-a\|}\}$; hence we may confine our attention to the geometric analysis of M-intervals of the form $[0, a]$, where $\|a\| = 1$.

We wish to relate the geometry of the M-betweenness interval $[0, a]$ to that of the facet $F(a)$ of $a \in S_X$. Observe that $F(a) \subseteq S_X$: if $s^{-1}a + (1 - s^{-1})b \in B_X$, then

$$s \geq \|a + (s-1)b\| \geq \|a\| - (1-s)\|b\| = 1 - (1-s)\|b\|,$$

which implies $\|b\| \geq 1$.

From Proposition 3.6, we know that an M-interval is the union of (the supports of) all geodesics joining the points of a bracket pair for the interval. In the next result we show that an M-interval is also a union of closed line segments, all fanning out from one bracket point. In order to determine the other end point of such a line segment, we use the s-functions introduced above.

In the following, $K = B_X$. The following is a strengthening of Proposition 4.1.

Theorem 4.10. *Let X be a normed vector space, with $a \in S_X$. Then*

$$[0, a] = \{\lambda b : b \in F(a) \text{ and } \lambda \in [0, 1 - \sigma(b)]\}.$$

In particular, $[0, a] = [0, a]_{\mathsf{L}}$ if and only if $a \in \mathrm{ext}(B_X)$.

Proof. Let $x \in [0, a]$, so that $\|x\| + \|a - x\| = \|a\| = 1$. If $x = 0$ then set $b = a$ and $\lambda = 0$. Otherwise, let $b = x/\|x\|$ and $\lambda = \|x\| \in (0, 1]$. In either case, $x = \lambda b$. If $b = a$ then $b \in F(a)$ and $\lambda \leq 1 = 1 - \sigma(b)$. Suppose that $b \neq a$. In this case, set $s = 1 - \lambda = 1 - \|x\|$. Since $b \neq a$, we have $x \neq 0$ and $x \neq a$, which implies $s \in (0, 1)$. Then observe that

$$s^{-1}a + (1 - s^{-1})b = \frac{1}{1 - \|x\|}(a - \|x\|b) = \frac{a - x}{1 - \|x\|} = \frac{a - x}{\|a - x\|} \in S_X,$$

given that $\|x\| + \|a - x\| = 1$. Consequently, $b \in F(a)$ and, moreover, $\sigma(b) \leq s = 1 - \lambda$, giving $\lambda \leq 1 - \sigma(b)$.

Conversely, let $x = \lambda b$, where $b \in F(a)$ and $\lambda \in [0, 1 - \sigma(b)]$. If $b = a$ then $\sigma(b) = 0$ and $\|x\| + \|a - x\| = \lambda + (1 - \lambda) = 1 = \|a\|$. If $b \neq a$ then $\sigma(b) > 0$ and, from the discussion about $\sigma(b)$ above, as $1 - \lambda \in [\sigma(b), 1]$, we have $(1-\lambda)^{-1}a + (1 - (1-\lambda)^{-1})b \in B_X$, which implies

$$1 - \|x\| = 1 - \lambda \geq \|a - \lambda b\| = \|a - x\|,$$

and thus $\|a - x\| + \|x\| \leq 1 = \|a\|$. It follows that $x \in [0, a]$ by the triangle inequality. □

Note that Proposition 4.1 is an immediate consequence of Theorem 4.10, which, when combined with Theorem 4.2, gives us the following.

Corollary 4.11. *Let X be a normed vector space, with $a, b \in X$. If either $a = b$ or $\frac{b-a}{\|b-a\|} \in \operatorname{ext}(B_X)$, then $[\cdot,\cdot]$ is lsc at $\{a, b\}$.*

We are now in a position to show that the unique-geodesic assumption implicit in Corollary 4.11 cannot be eliminated. Given a normed space X and a closed, symmetric, convex, bounded subset $K \subseteq X$ that contains 0 as an interior point, the **Minkowski functional** defined by

$$\|x\| = \inf\{t > 0 : t^{-1}x \in K\},$$

is a norm on X equivalent to the original norm and having closed unit ball K [10, Definition 2.9, Lemma 2.11].

Example 4.12. We construct a norm $\|\cdot\|$ on $\mathbb{R}^3$, with respect to which $[\cdot,\cdot]$ is not lsc at all pairs.

First define the function $f : [-1,1]^2 \to [0,1]$ by the assignment $\langle x, y\rangle \mapsto \sqrt{(1-x^2)(1-y^2)}$. Then f satisfies the following conditions:

(1) $f(0,0) = 1$;
(2) $f(x,y) = 0$ whenever $|x| = 1$ or $|y| = 1$;
(3) f is *strictly concave* on $(-1,1)^2$ (i.e., $f_{x,x} < 0$, $f_{y,y} < 0$, and $f_{x,x}f_{y,y} - f_{x,y}^2 > 0$); and
(4) f is *symmetric* (i.e., $f(-x,-y) = f(x,y)$ on $[-1,1]^2$).

Let

$$B := \{\langle x,y,z\rangle \in [-1,1]^3 : |z| \le f(x,y)\}.$$

The set B is symmetric, compact and convex, and the origin is an interior point; so let $\|\cdot\|$ be its Minkowski functional. Then $\langle \mathbb{R}^3, \|\cdot\|\rangle$ is a normed space, with $B_{\langle \mathbb{R}^3,\|\cdot\|\rangle} = B$. Since $B \subseteq [-1,1]^3$, we have $\|\cdot\| \ge \|\cdot\|_\infty$. More to the point: since the intersection of B with the plane $\mathbb{R}^2 \times \{0\}$ is $[-1,1]^2 \times \{0\}$, it follows that $\|\cdot\|$ agrees with $\|\cdot\|_\infty$ on that plane (i.e., $\|\langle x,y,0\rangle\| = \max\{|x|,|y|\}$).

Let S be the unit sphere $S_{\langle \mathbb{R}^3,\|\cdot\|\rangle}$. By the strict concavity of f, we have that

$$\operatorname{ext}(B) = S \setminus \{\langle x,y,z\rangle : (|x| = 1 \text{ and } |y| < 1) \text{ or } (|x| < 1 \text{ and } |y| = 1)\}.$$

We claim that for $\vec{a} \in S$, $[\cdot,\cdot]$ is lsc at $\{\vec{0}, \vec{a}\}$ if and only if $\vec{a} \in \operatorname{ext}(B)$. Indeed, one direction is immediate from Corollary 4.11; as for the other direction, we lose no generality in taking a simplifying case, namely $\vec{a} = \langle 1,0,0\rangle$. Here it is easy to check that $F(\vec{a}) = \{1\} \times [-1,1] \times \{0\}$ and $[\vec{0},\vec{a}]$ is the square with corners $\langle 0,0,0\rangle$, $\langle 1,0,0\rangle$, and $\langle \frac{1}{2}, \pm\frac{1}{2}, 0\rangle$. Let U be the open set $\{\langle x,y,z\rangle : y > \frac{1}{4}\}$. Then $[\vec{0},\vec{a}] \cap U \ne \emptyset$. However, given $r \in (0, \frac{1}{4}]$, there exists $\vec{a}' \in \operatorname{ext}(B)$ such that $\|\vec{a}' - \vec{a}\| \le r$. Thus, by Corollary 4.11, $[\vec{0},\vec{a}']$ is a closed line segment that clearly misses U, and we conclude that $[\cdot,\cdot]$ is not lsc at $\{\vec{0},\vec{a}\}$.

We now show that dimension three is lowest possible for Example 4.12.

Proposition 4.13. *The* M*-betweenness function on any two-dimensional normed space is continuous at all pairs.*

Proof. Because finite-dimensional normed spaces have proper norm metrics, Theorem 2.5 allows us to focus on lower semicontinuity.

Let X be a two-dimensional normed space, with $a \in S_X$. Having in mind the argument in the proof of Theorem 4.2, it is sufficient to demonstrate that, given $r > 0$, there exists $\delta > 0$ such that every point of $[u,v]$ is r-close to some point of $[0,a]$, whenever $\|u\|, \|v-a\| \leq \delta$. The case $a \in \mathrm{ext}(B_X)$ has been covered by Corollary 4.11 and Theorem 4.2, so hereafter we assume that $a \notin \mathrm{ext}(S_X)$.

Let $r > 0$. We will find δ in two steps. First, we show that, given $r > 0$, there exists $\delta' > 0$ such that every point of $[0,x]$ is $\frac{1}{2}r$-close to some point of $[0,a]$ whenever $x \in S_X$ and $\|x-a\| \leq \delta'$. Since X is two-dimensional, it follows that the facet $F(a)$ in B_X is a non-degenerate straight line segment $[p,q]_{\mathsf{L}}$, and $a = u + \alpha(q-p)$, for some $\alpha \in (0,1)$. Using this, given $b = p + \beta(q-p) \in F(a)$ ($\beta \in [0,1]$), it is easy to compute that

$$\sigma_a(b) \;=\; \max\left\{\frac{\beta-\alpha}{\beta}, \frac{\alpha-\beta}{1-\beta}\right\}.$$

In particular, for $x \in S_X$

$$\sup_{b \in F(a)} |\sigma_x(b) - \sigma_a(b)| \to 0,$$

as $\|x-a\| \to 0$ (note that $F(x) = F(a)$ for all x sufficiently close to a). Thus there exists $\delta' \in (0, \min\{1, \frac{1}{2}r\}]$ such that $F(x) = F(a)$ and $\sigma_x(b) \leq \sigma_a(b) + \frac{1}{2}r$ whenever $x \in S_X$ and $\|x-a\| \leq \delta'$. Take such an x, and let $w \in [0,x]$. By Theorem 4.10, we know that $w = \lambda b$ for some $b \in F(x) = F(a)$ and $\lambda \in [0, 1-\sigma_x(b)]$. If $\lambda \leq \sigma_a(b)$, then $w \in [0,a]$, and if not, then $|\lambda - \sigma_a(b)| \leq \frac{1}{2}r$, giving $\|w - \sigma_a(b)b\| \leq \frac{1}{2}r$. Whatever the case, w is $\frac{1}{2}r$-close to a point in $[0,a]$. This completes the first step.

In the second step, we show that there exists $\delta > 0$ such that if $\|u\|, \|v-a\| \leq \delta$ and $x := (v-u)/\|v-u\| \in S_X$, then $\|x-a\| \leq \delta'$ and every point of $[u,v]$ is $\frac{1}{2}r$-close to a point of $[0,x]$. We complete the proof by stitching the two steps together. Set $\delta = \frac{1}{6}\delta' \leq \frac{1}{6}$. Let $\|u\|, \|v-a\| \leq \delta$ and set $x = (v-u)/\|v-u\| \in S_X$. Given $w \in [u,v]$, it is easy to check that $z := (w-u)/\|v-u\| \in [0,x]$. By elementary considerations we have

$$|\|v-u\| - 1| = |\|v-u\| - \|a\|| \leq \|(v-a) - u\| \leq 2\delta,$$

and as $w \in [u,v]$,

$$\|w\| \leq \|w-u\| + \|u\| \leq \|v-u\| + \|u\| \leq 1 + 3\delta \leq \tfrac{3}{2}.$$

Therefore,

$$\|w-z\| = \frac{1}{\|v-u\|}\|(\|v-u\|-1)w - u\| \leq \frac{4\delta}{1-2\delta} \leq 6\delta = \delta' \leq \tfrac{1}{2}r,$$

and

$$\|x-a\| = \frac{1}{\|v-u\|}\|(v-a) + (\|v-u\|-1)a - u\| \leq \frac{4\delta}{1-2\delta} \leq \delta'. \qquad \square$$

The final series of results of this section deals with the question of when the M-betweenness function is (or is not) upper semicontinuous. We first consider pairs whose M-intervals are not compact. We will need the following elementary result.

Lemma 4.14. *Let X be a normed space, with $K \subseteq X$ a closed bounded subset that is not compact. Then there exists an open set U containing K, such that*

$$\inf\{\|w-v\| : w \in K \text{ and } v \in X \setminus U\} = 0.$$

Proof. Since K is not compact, there is a sequence $\langle a_1, a_2, \dots\rangle$ of points of K, with no convergent subsequence. Fix $a \in S_X$ arbitrary, and for $n \in \mathbb{N}$, let

$$\mu_n = \sup\{\mu \geq 0 : a_n + \mu a \in K\}.$$

Since K is bounded, the sequence $\langle \mu_n \rangle$ is bounded in $\mathbb{R}$, and hence has a convergent subsequence. Without loss of generality, we may assume $\langle \mu_n \rangle$ itself is convergent.

Now, for $n \in \mathbb{N}$, set $b_n = a_n + (\mu_n + \frac{1}{n})a$, by definition a point of $X \setminus K$. The sequence μ_n converges, while a_n has no convergent subsequence; hence b_n has no convergent subsequence either. Consequently the set $U = X \setminus \{b_1, b_2, \dots\}$ is open and contains K.

Finally, since K is closed, we have $a_n + \mu_n a \in K$, $n \in \mathbb{N}$, and thus

$$\inf\{\|w - v\| : w \in K \text{ and } v \in X \setminus U\} \leq \|(a_n + \mu_n a) - b_n\| = \frac{1}{n} \to 0. \qquad \square$$

An almost immediate consequence of Lemma 4.14 is the following.

Proposition 4.15. *Let X be a normed space, with $a, b \in X$. If $[a, b]$ is not compact, then $[\cdot, \cdot]$ is not usc at $\{a, b\}$.*

Proof. Since $[a, b]$ is closed and bounded, but not compact, we use Lemma 4.14 to produce an open set U, with $[a, b] \subseteq U$, so that

$$\inf\{\|w - v\| : w \in [a, b] \text{ and } v \in X \setminus U\} = 0.$$

Given any $\delta > 0$, find $w \in [a, b]$ and $v \in X \setminus U$ so that $\|w - v\| < \delta$. Then

$$v = w + (v - w) \in (v - w) + [a, b] = [a + (v - w), b + (v - w)],$$

thus $[a + (v - w), b + (v - w)] \not\subseteq U$. $\square$

Remark 4.16. The statements of Lemma 4.14 and Proposition 4.15 make sense in the general metric context, but are generally false. An easy counterexample comes from taking X to be the irrational line $\mathbb{R} \setminus \mathbb{Q}$, with the usual metric (topologically complete). The M-intervals coincide with the closed bounded intervals inherited from $\mathbb{R}$, and so it is easy to check that $[\cdot, \cdot]$ is continuous at every pair. However, no nondegenerate M-interval is compact. (See Remark 4.20 below for a geodesic space example.)

We will see in Example 4.22 below that the compactness of $[a, b]$ is not sufficient for the upper semicontinuity of $[\cdot, \cdot]$ at $\{a, b\}$. In order to fully characterize upper semicontinuity of $[\cdot, \cdot]$, it will be helpful to present the following simple test for compactness of M-intervals. Recall Lemma 4.9 and the discussion before Theorem 4.10.

Proposition 4.17. *The following statements are equivalent.*

1. *The M-interval $[0, a]$ is compact.*
2. *Given a sequence $\langle b_n \rangle$ from $F(a)$, either $\sigma(b_n) \to 1$, or $\langle b_n \rangle$ admits a subsequence that converges in $F(a)$.*
3. *Given a sequence $\langle b_n \rangle$ from $F(a)$, either $\sigma(b_n) \to 1$, or $\langle b_n \rangle$ admits a subsequence that converges in X.*

Proof. First, we show that (1) $\Rightarrow$ (2). Let $[0, a]$ be compact. Let $\langle b_n \rangle$ be a sequence from $F(a)$ such that $\sigma(b_n) \not\to 1$. According to Theorem 4.10, $\lambda_n b_n \in [0, a]$, where $\lambda_n := 1 - \sigma(b_n)$. Using the compactness of $[0, a]$ and the fact that $\sigma(b_n) \not\to 1$, we can assume, by taking a subsequence if necessary, that there exist $s < 1$, $b \in F(a)$ and $\lambda \in [0, 1 - \sigma(b)]$, such that $\sigma(b_n) \leq s$ for all n and $\lambda_n b_n \to \lambda b$. By continuity of the norm, we have $\lambda_n \to \lambda \geq 1 - s > 0$. Hence $b_n \to b$.

The implication (2) $\Rightarrow$ (3) is trivial. We finish with (3) $\Rightarrow$ (1). Assume the conditions of (3), and let $\langle x_n \rangle$ be a sequence from $[0, a]$. By Theorem 4.10, $x_n =$

$\lambda_n b_n$ for some $b_n \in F(a)$ and $\lambda_n \in [0, 1 - \sigma(b_n)]$. By taking a subsequence if necessary, we can assume that $\sigma(b_n) \to s \in [0,1]$ and $\lambda_n \to \lambda \leq 1 - s$. If $\lambda = 0$ then $x_n \to 0 \in [0, a]$. If $\lambda > 0$ then $s < 1$. By (3), and by taking another subsequence if necessary, there exists $b \in X$ such that $b_n \to b$. According to Lemma 4.9, $b \in F(a)$ and $\sigma(b) \leq s$. Thus $x_n \to \lambda b$ and $\lambda \in [0, 1 - \sigma(b)]$, giving $\lambda b \in [0, a]$. $\square$

It is clear from Proposition 4.17 that if $\overline{F(a)}$ is compact, then so is $[0, a]$. However, as the example below demonstrates, the converse of this statement is false. In other words, it is not possible to drop the condition $\sigma(b_n) \to 1$ in Proposition 4.17. Recall that c_0 is the vector space of all real sequences converging to zero. Unless otherwise specified, c_0 is equipped with the usual sup norm; i.e., given $\vec{x} = \langle \vec{x}(1), \vec{x}(2), \dots \rangle$, $\|\vec{x}\|_\infty := \sup\{|\vec{x}(n)| : n \in \mathbb{N}\}$. Let $\vec{e}_n$ denote the nth standard unit vector of c_0, that is, $\vec{e}_n(k) = \delta_{n,k}$, and let e_n^* denote the corresponding nth evaluation functional, i.e., $e_n^*(\vec{x}) = \vec{x}(n)$ whenever $\vec{x} \in c_0$.

Example 4.18. Let B_{c_0} denote the closed unit ball of c_0 (with respect to $\|\cdot\|_\infty$). Define the closed convex set

$$M = \{\vec{x} \in c_0 : \vec{x}(1) = 1 \text{ and } -2^{-n} \leq \vec{x}(n) \leq 1 \text{ for all } n \geq 2\},$$

let K be the closed convex hull of $\frac{1}{2}B_{c_0} \cup M \cup (-M)$, and let $\|\cdot\|$ be the Minkowski functional of K. Then $\|\cdot\|$ is a norm on c_0, equivalent to $\|\cdot\|$, having closed unit ball K, such that $\|\vec{e}_1\| = 1$, $\overline{F(\vec{e}_1)} = M$ is not compact, and $[\vec{0}, \vec{e}_1]$ is compact.

Proof. As $\frac{1}{2}B_{c_0} \subseteq K \subseteq B_{c_0}$, $\|\cdot\|$ is indeed an equivalent norm on c_0 with closed ball K. Observe that $e_1^*(\vec{x}) = 1$ whenever $\vec{x} \in M$ and $e_1^*(\vec{x}) \leq \frac{1}{2}$ whenever $\vec{x} \in \frac{1}{2}B_{c_0} \cup (-M)$, so $e_1^*(\vec{x}) \leq 1$ whenever $\vec{x} \in K$. It follows that if $\vec{x} \in K$ and $e_1^*(\vec{x}) = 1$, then $\|\vec{x}\| = 1$, because $t^{-1}\vec{x} \notin K$ whenever $t \in (0,1)$. In particular, $\|\vec{e}_1\| = 1$.

Next, we show that $\overline{F(\vec{e}_1)} = M$. Notice that $F(\vec{e}_1) \subseteq H$, where

$$H := \{\vec{x} \in c_0 : e_1^*(\vec{x}) = 1\}.$$

Indeed, given $\vec{b} \in F(\vec{e}_1)$, we have, for some $s \in (0,1)$,

$$1 = s^{-1} + (1 - s^{-1}) \leq s^{-1} + (1 - s^{-1})e_1^*(\vec{b}) = s^{-1}e_1^*(e_1) + (1 - s^{-1})e_1^*(\vec{b}) \leq 1,$$

because $1 - s^{-1} < 0$ and $\vec{b}, s^{-1}\vec{e}_1 + (1 - s^{-1})\vec{b} \in K$. The only way that the line above can hold is if $e_1^*(\vec{b}) = 1$, hence the result.

Therefore $F(\vec{e}_1) \subseteq K \cap H$. The next thing we notice is that $K \cap H \subseteq M$. Indeed, let $\vec{x} \in K \cap H$. As $\vec{x} \in K$, there exist $\vec{x}_i \in M$, $\vec{y}_i \in \frac{1}{2}B_{c_0}$, $\vec{z}_i \in -M$, and $\lambda_{i,j} \geq 0$, $1 \leq j \leq 3$, such that

$$\sum_{j=1}^{3} \lambda_{i,j} = 1 \text{ for all } i, \qquad \text{and} \qquad \lambda_{i,1}\vec{x}_i + \lambda_{i,2}\vec{y}_i + \lambda_{i,3}\vec{z}_i \ \to\ \vec{x} \text{ as } i \to \infty.$$

Applying e_1^* to the sequence and limit above yields $\lambda_{i,1} + \lambda_{i,2}\vec{y}_i(1) + \lambda_{i,3}\vec{z}_i(1) \to 1$. On the other hand,

$$\begin{aligned}\lambda_{i,1} + \lambda_{i,2}\vec{y}_i(1) + \lambda_{i,3}\vec{z}_i(1) &\leq \lambda_{i,1} + \tfrac{1}{2}(\lambda_{i,2} + \lambda_{i,3}) \\ &= \lambda_{i,1} + \tfrac{1}{2}(1 - \lambda_{i,1}) = \tfrac{1}{2}(1 + \lambda_{i,1}) \leq 1,\end{aligned}$$

which implies that $\lambda_{i,1} \to 1$ and, consequently, $\lambda_{i,2}, \lambda_{i,3} \to 0$ and $\vec{x}_i \to \vec{x}$. Since M is closed, we conclude that $\vec{x} \in M$. Finally, again as M is closed, we have $\overline{F(\vec{e}_1)} \subseteq M$.

To see the reverse inclusion, we observe that if $\vec{y} \in M$ has *finite support*, that is, there exists $N \in \mathbb{N}$ such that $\vec{y}(n) = 0$ whenever $n > N$, then $\vec{y} \in F(\vec{e}_1)$. Indeed, given such $\vec{y}$ and N, set $s = (1 + 2^{-N})^{-1} \in (0, 1)$. The reader can verify that

$$-2^{-n} \leq (1 - s^{-1})\vec{y}(n) = -2^{-N}\vec{y}(n) \leq 1,$$

whenever $2 \leq n \leq N$. Given the finite support of $\vec{y}$, it follows easily that $s^{-1}\vec{e}_1 + (1 - s^{-1})\vec{y} \in M \subseteq K$, giving $\vec{y} \in F(\vec{e}_1)$. Since the set of finitely supported elements of M is dense in M, we obtain $\overline{F(e_1)} = M$.

By considering the vectors $\vec{e}_1 + \vec{e}_n \in M$, $n \geq 2$, it is easy to see that M is not compact. However, $[\vec{0}, \vec{e}_1]$ is compact. This will follow from Proposition 4.17, once we show that if $\langle \vec{b}_i \rangle$ is a sequence in $F(\vec{e}_1) \subseteq M$ and $\sigma(\vec{b}_i) \not\to 1$, then $\langle \vec{b}_i \rangle$ admits a convergent subsequence. Given such a sequence, by taking a subsequence if necessary, we can assume that there exists $s \in [\frac{1}{2}, 1)$ such that $\sigma(\vec{b}_i) \leq s$ for all i. We claim that $|\vec{b}_i(n)| \leq 2^{-n}s/(1 - s)$ for all i and all $n \geq 2$. Given $n \geq 2$, since $\vec{b}_i \in M$, we have $-2^{-n} \leq \vec{b}_i(n) \leq 1$. If $-2^{-n} \leq \vec{b}_i(n) \leq 0$ then there is nothing to check, as $s \geq \frac{1}{2}$. Instead, assume that $0 \leq \vec{b}_i(n) \leq 1$. From Remark 4.8, we know that $\sigma(\vec{b}_i)^{-1}\vec{e}_1 + (1 - \sigma(\vec{b}_i)^{-1})\vec{b}_i \in K$, which implies that in fact

$$\sigma(\vec{b}_i)^{-1}\vec{e}_1 + (1 - \sigma(\vec{b}_i)^{-1})\vec{b}_i \ \in \ F(\vec{e}_1) \ \subseteq \ M.$$

According to the definition of M, it follows that

$$-2^{-n} \leq (1 - \sigma(\vec{b}_i)^{-1})\vec{b}_i(n) \leq (1 - s^{-1})\vec{b}_i(n),$$

and since $\vec{b}_i(n) \geq 0$ and $1 - s^{-1} < 0$, we deduce that

$$\vec{b}_i(n) \leq \frac{2^{-n}s}{1 - s}.$$

This completes the proof of the claim.

By taking a diagonal subsequence, we can find $\vec{b} \in c_0$ such that, for all n, $\vec{b}_i(n) \to \vec{b}(n)$ as $i \to \infty$. This, coupled with the condition $|\vec{b}_i(n)| \leq 2^{-n}s/(1 - s)$, ensures that $\vec{b}_i \to \vec{b}$ in norm also. $\square$

We also remark that $F(\vec{e}_1)$ above is not closed. Define $\vec{b} \in M = \overline{F(e_1)}$ by $\vec{b}(1) = 1$ and $\vec{b}(n) = 2^{-\frac{n}{2}}$ for $n \geq 2$. If there exists $s \in (0, 1)$ such that $\vec{c} := s^{-1}\vec{e}_1 + (1 - s^{-1})\vec{b} \in K$, then according to Remark 4.8, $\vec{c} \in F(\vec{e}_1) \subseteq M$. Hence for all $n \geq 2$ we have

$$-2^{-n} \leq (1 - s^{-1})2^{-\frac{n}{2}},$$

meaning that $s \geq (1 + 2^{-\frac{n}{2}})^{-1}$, which is impossible. Thus $\vec{b} \in \overline{F(\vec{e}_1)} \setminus F(\vec{e}_1)$.

We can put together Proposition 2.10 and Propositions 4.15 and 4.17 to obtain an example where upper semicontinuity holds only at singletons.

Example 4.19. Consider c_0 with the usual supremum norm. Then no nondegenerate M-interval is compact. Consequently, $[\cdot, \cdot]$ is usc precisely at the singletons.

Proof. To see this, first note that, by Proposition 2.10, $[\cdot, \cdot]$ is usc at each singleton. Hence it suffices to assume $\vec{a} \in S_{c_0}$ and show $[\vec{0}, \vec{a}]$ is not compact.

Given that $\vec{a} \in c_0$, there exists $k \in \mathbb{N}$ such that $|\vec{a}(n)| < \frac{1}{2}$ whenever $n \geq k$. Given $n \geq k$, define $\vec{b}_n, \vec{c}_n \in S_{c_0}$ by

$$\vec{b}_n(i) = \begin{cases} \vec{a}(i) & \text{if } i \neq n \\ 1 & \text{if } i = n, \end{cases} \qquad \text{and} \qquad \vec{c}_n(i) = \begin{cases} \vec{a}(i) & \text{if } i \neq n \\ -1 & \text{if } i = n. \end{cases}$$

It is easy to check that $s_n^{-1}\vec{a} + (1 - s_n^{-1})\vec{b}_n = \vec{c}_n \in S_{c_0}$, where $s_n := \frac{1}{2}(1 - \vec{a}(n)) \in (0,1)$. Thus $\vec{b}_n \in F(\vec{a})$. Evidently, $\|\vec{b}_m - \vec{b}_n\|_\infty \geq |(\vec{b}_m)(m) - (\vec{b}_n)(m)| > \frac{1}{2}$ whenever $m, n \geq k$ are distinct. Moreover, $\sigma(\vec{b}_n) \leq s_n \to \frac{1}{2}$. That $[\vec{0}, \vec{a}]$ is not compact follows from Proposition 4.17. Consequently, by Proposition 4.15, $[\cdot,\cdot]$ is not usc at $\{\vec{0}, \vec{a}\}$. $\square$

Remark 4.20. Example 4.19 can be used to obtain an even more convincing example of how much the truth of Lemma 4.14 and Proposition 4.15 depends on the normed metric context. Given c_0, let $X = [\vec{a}, \vec{b}]$ be any nondegenerate M-interval. Then X is not compact. On the other hand, relative to the inherited supremum norm, X is still the M-interval bracketed by $\{\vec{a}, \vec{b}\}$. Thus $[\cdot,\cdot]$ is trivially usc at $\{\vec{a}, \vec{b}\}$ (see Proposition 1.4). Note that X is a convex subset of c_0, by Proposition 4.3, and is hence a geodesic space, by Proposition 4.1.

Next we fine tune Proposition 4.15 to obtain a characterization of upper semicontinuity of $[\cdot,\cdot]$ at a pair.

Theorem 4.21. *Let X be a normed space and let $a \in S_X$. The M-betweenness function is usc at $\{0, a\}$ if and only if, given $a_n \in S_X$ converging in norm to a, and points $b_n \in F(a_n)$, either $\sigma_{a_n}(b_n) \to 1$ or a subsequence of $\langle b_n \rangle$ converges in X.*

Proof. We proceed by proof by contraposition. Suppose that $[\cdot,\cdot]$ is not usc at $\{0, a\}$. If $[0, a]$ is not compact, then the conclusion follows immediately from Proposition 4.17, by setting $a_n = a$ for all $n \in \mathbb{N}$. Hereafter, we assume that $[0, a]$ is compact. By the failure of upper semicontinuity, there exists an open set U such that $[0, a] \subseteq U$, and points $u_n \to 0$, $v_n \to a$ and $w_n \in [u_n, v_n]$, $n \in \mathbb{N}$, such that $w_n \in [u_n, v_n] \backslash U$. By the compactness of $[0, a]$, there exists $r > 0$ such that $\varrho(w_n, [0, a]) := \inf\{\|w_n - x\| : x \in [0, a]\} > r$ for all $n \in \mathbb{N}$. Define

$$a_n = \frac{v_n - u_n}{\|v_n - u_n\|} \qquad \text{and} \qquad x_n = \frac{w_n - u_n}{\|v_n - u_n\|} \ \in \ [0, a_n].$$

As $\|x_n - w_n\| \to 0$, it follows that $\varrho(x_n, [0, a]) > \frac{1}{2}r$ for large enough n. By Theorem 4.10, $x_n = \lambda_n b_n$ for some $b_n \in F(a_n)$ and $\lambda_n \in [0, 1 - \sigma_{a_n}(b_n)]$.

Without loss of generality, assume that $\lambda_n \to \lambda$ for some λ. We claim that $\langle b_n \rangle$ has no convergent subsequence. For a contradiction, suppose that it does: let $b_{n_k} \to b$ for some $b \in X$. Then $x_{n_k} \to \lambda b$, and

$$\|\lambda b\| + \|a - \lambda b\| = \lim_{k \to \infty} \|x_{n_k}\| + \|a_{n_k} - x_{n_k}\| = 1,$$

whence $\lambda b \in [0, a]$. However, this contradicts the fact that

$$\|x_{n_k} - \lambda b\| \geq \varrho(x_{n_k}, [0, a]) > \tfrac{1}{2}r,$$

for large enough k. It follows that $\langle b_n \rangle$ has no convergent subsequence, as required. Finally,

$$\tfrac{1}{2}r \ < \ \varrho(x_n, [0, a]) \leq \|x_n\| = \lambda_n \leq 1 - \sigma_{a_n}(b_n),$$

for large enough n implies that $\sigma_{a_n}(b_n) \not\to 1$.

Conversely, assume the existence of a_n converging to a and $b_n \in F(a_n)$, $n \in \mathbb{N}$, such that $\sigma_{a_n}(b_n) \not\to 1$ and no subsequence of $\langle b_n \rangle$ converges. By taking a subsequence if necessary, we can assume that there exists $s < 1$ such that $\sigma_{a_n}(b_n) \to s$. Now set $x_n = \lambda_n b_n$, where $\lambda_n := 1 - \sigma_{a_n}(b_n)$. We have $\lambda_n \to \lambda := 1 - s > 0$.

According to Theorem 4.10, $x_n \in [0, b_n]$ for all $n \in \mathbb{N}$. As $\lambda > 0$ and $\langle b_n \rangle$ has no convergent subsequence, it follows that the sequence $\langle x_n \rangle$ has no convergent subsequence either. There are now two possibilities: either $\varrho(x_n, [0, a]) \to 0$ or not. If $\varrho(x_n, [0, a]) \to 0$, then there exist points $y_n \in [0, a]$ such that $\|y_n - x_n\| \to 0$. It follows that $\langle y_n \rangle$ has no convergent subsequence, and thus $[0, a]$ is not compact. We conclude from Proposition 4.15 that $[\cdot, \cdot]$ is not usc at $\{0, a\}$. Instead, if $\varrho(x_n, [0, a]) \not\to 0$ then, by taking yet another subsequence if necessary, there exists $\delta > 0$ such that $\varrho(x_n, [0, a]) \geq \delta$ for all $n \in \mathbb{N}$. If we set

$$U \ := \ \{v \in X : \varrho(v, [0, a]) < \delta\},$$

then U is open and $[0, a] \subseteq U$, but $a_n \to a$ and $x_n \in [0, a_n] \setminus U$ for all n, so $[\cdot, \cdot]$ is not usc at $\{0, a\}$. □

As mentioned above, the usc part of Theorem 4.2 follows easily from Theorems 4.10 and 4.21.

We end this section with an example illustrating the difference between Proposition 4.15 and Theorem 4.21. The reader is referred to Example 4.18 and the preceding remarks for notation and terminology.

Example 4.22. There exists a norm $\|\cdot\|$ on c_0, equivalent to $\|\cdot\|_\infty$, having unit ball K and $a \in \operatorname{ext}(K)$, such that $[\cdot, \cdot]$ is not upper semicontinuous at $\{0, a\}$.

Proof. Let B_{c_0} denote the unit ball of $\|\cdot\|_\infty$, and let K be the closed convex hull of the symmetric set

$$S := \tfrac{1}{3} B_{c_0} \cup \{s(\vec{e}_1 + n^{-1}\vec{e}_{2n}) + t\vec{e}_{2n+1} : n \in \mathbb{N} \text{ and } s, t \in \{-1, 1\}\}.$$

Then $\frac{1}{3} B_{c_0} \subseteq K \subseteq B_{c_0}$, and the Minkowski functional of K defines an equivalent norm $\|\cdot\|$ on c_0, having closed unit ball K (again see [10, Definition 2.9, Lemma 2.11]). Given $n \in \mathbb{N}$, set

$$\vec{a}_n = \vec{e}_1 + n^{-1}\vec{e}_{2n} = \tfrac{1}{2}((\vec{e}_1 + n^{-1}\vec{e}_{2n} + \vec{e}_{2n+1}) + (\vec{e}_1 + n^{-1}\vec{e}_{2n} - \vec{e}_{2n+1})) \in K.$$

Observe that $e_1^*(\vec{x}) \leq 1$ for all $\vec{x} \in S$, thus $e_1^*(\vec{x}) \leq 1$ for all $x \in K$. It follows that if $\vec{x} \in K$ and $e_1^*(\vec{x}) = 1$, then $\|\vec{x}\| = 1$, because $r^{-1}\vec{x} \notin B_{c_0}$ whenever $r \in (0, 1)$. In particular, $\|\vec{a}_n\| = 1$ for all n. Since $\|\vec{a}_n - \vec{e}_1\| \to 0$, we have $\|\vec{e}_1\| = 1$ as well. Evidently,

$$\vec{a}_n \pm \vec{e}_{2n+1} \in S \subseteq K.$$

Set $\vec{b}_n = \vec{a}_n + \vec{e}_{2n+1}$. Since

$$2\vec{a}_n - \vec{b}_n = \vec{a}_n - \vec{e}_{2n+1} \in K,$$

it follows that $\vec{b}_n \in F(\vec{a}_n)$ and $\sigma_{\vec{a}_n}(\vec{b}_n) \leq \frac{1}{2}$ for all n. Moreover, given distinct $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|\vec{b}_m - \vec{b}_n\| &= \|m^{-1}\vec{e}_{2m} - n^{-1}\vec{e}_{2n} + \vec{e}_{2m+1} - \vec{e}_{2n+1}\| \\ &\geq \|m^{-1}\vec{e}_{2m} - n^{-1}\vec{e}_{2n} + \vec{e}_{2m+1} - \vec{e}_{2n+1}\|_\infty = 1, \end{aligned}$$

meaning that $\langle \vec{b}_n \rangle$ has no convergent subsequence. Therefore, $[\cdot, \cdot]$ is not upper semicontinuous at $\{0, \vec{e}_1\}$ with respect to $\|\cdot\|$, by Theorem 4.21.

On the other hand, we claim that $\vec{e}_1 \in \mathrm{ext}(K)$, meaning that $[\vec{0},\vec{e}_1]$ is the compact straight line segment $[\vec{0},\vec{e}_1]_{\mathsf{L}}$. To prove the claim, we show that there exists $f \in c_0^*$ such that $f(\vec{e}_1) = \|f\| = 1$, yet $f(\vec{y}) < 1$ whenever $\vec{y} \in K \setminus \{\vec{e}_1\}$. In other words, we will show that $\vec{e}_1$ is an *exposed* point of K [10, Definition 7.10]. This certainly implies that $\vec{e}_1$ is an extreme point: if $\vec{e}_1$ is the midpoint of a non-trivial straight line segment in K, then $f(\vec{y}) = 1$ for all points $\vec{y}$ on said line segment [10, Exercise 7.72].

Define $f = e_1^* - \sum_{i=1}^\infty 2^{-i} e_{2i}^*$. It is clear that $f(\vec{e}_1) = 1$, $f(\vec{x}) \leq \frac{2}{3}$ whenever $\vec{x} \in \frac{1}{3}B_{c_0}$, and

$$f(s(\vec{e}_1 + n^{-1}\vec{e}_{2n}) + t\vec{e}_{2n+1}) = s - n^{-1}2^{-n} \;<\; 1,$$

whenever $n \in \mathbb{N}$ and $s,t \in \{-1,1\}$. Hence, $f(\vec{x}) \leq 1$ for all $\vec{x} \in S$, and thus the same holds for all $\vec{x} \in K$. We conclude that $\|f\| = 1$.

It remains to show that $f(\vec{y}) < 1$ whenever $\vec{y} \in K \setminus \{\vec{e}_1\}$. Let $\vec{y} \in K$ and assume that $f(\vec{y}) = 1$. Since K is the closure of the convex hull of S, for each $k \in \mathbb{N}$, we are able to find vectors $\vec{u}_k \in \frac{1}{3}B_{c_0}$, strictly increasing integers n_k, numbers $\lambda_{k,0},\dots,\lambda_{k,n_k} \geq 0$ and signs $s_{k,1},t_{k,1},\dots,s_{k,n_k},t_{k,n_k} \in \{-1,1\}$, such that

$$\sum_{j=0}^{n_k} \lambda_{k,j} = 1 \quad \text{and the vectors} \quad \vec{y}_k \;:=\; \lambda_{k,0}\vec{u}_k + \sum_{j=1}^{n_k} \lambda_{k,j}\big(s_{k,j}(\vec{e}_1 + j^{-1}\vec{e}_{2j}) + t_{k,j}\vec{e}_{2j+1}\big),$$

converge in norm to $\vec{y}$. We will show that $\vec{y}_k \to \vec{e}_1$ in the *weak* topology of c_0. By uniqueness of limits, it will follow that $\vec{y} = \vec{e}_1$. Since the sequence $\langle \vec{y}_k \rangle$ is norm-bounded, it is sufficient to show that, given $i \in \mathbb{N}$, we have $e_i^*(\vec{y}_k - \vec{e}_1) \to 0$ as $k \to \infty$ [10, Exercise 3.33].

Since $f(\vec{y}) = 1$, we have $f(\vec{y}_k) \to 1$ as $k \to \infty$. We estimate

$$\begin{aligned} f(\vec{y}_k) &= \lambda_{k,0} f(\vec{u}_k) + \sum_{j=1}^{n_k} \lambda_{k,j}(s_{k,j} - j^{-1}2^{-j}) \qquad (3)\\ &\leq \tfrac{2}{3}\lambda_{k,0} + \sum_{j=1}^{n_k} \lambda_{k,j}(1 - j^{-1}2^{-j}) = 1 - \tfrac{1}{3}\lambda_{k,0} - \sum_{j=1}^{n_k} j^{-1}2^{-j}\lambda_{k,j} \leq 1. \end{aligned}$$

As $f(\vec{y}_k) \to 1$, and the summands under consideration are all non-negative, we conclude that

$$\lambda_{k,0} \;\to\; 0 \qquad \text{and} \qquad \sum_{j=1}^{n_k} j^{-1}2^{-j}\lambda_{k,j} \;\to\; 0, \tag{4}$$

as $k \to \infty$. If we combine (4) and (3), we obtain

$$\begin{aligned} e_1^*(\vec{y}_k) &= \lambda_{k,0} e_1^*(\vec{u}_k) + \sum_{j=1}^{n_k} \lambda_{k,j} s_{k,j} \\ &= \lambda_{k,0} e_1^*(\vec{u}_k) + f(\vec{y}_k) - \lambda_{k,0} f(\vec{u}_k) - \sum_{j=1}^{n_k} j^{-1}2^{-j}\lambda_{k,j} \;\to\; 1, \qquad (5) \end{aligned}$$

as $k \to \infty$.

Moreover, given $i \in \mathbb{N}$, if we choose k large enough to ensure that $i \leq n_k$ (which we can do as the n_k are strictly increasing), then (4) also yields

$$0 \leq \lambda_{k,i} \leq i2^i \sum_{j=1}^{n_k} j^{-1}2^{-j}\lambda_{k,j} \ \to \ 0,$$

as $k \to \infty$. Therefore, for large enough k, we have

$$e^*_{2i}(\vec{y}_k) = \lambda_{k,0}e^*_{2i}(\vec{u}_k) + i^{-1}\lambda_{k,i}s_{k,i} \ \to \ 0, \tag{6}$$

and

$$e^*_{2i+1}(\vec{y}_k) = \lambda_{k,0}e^*_{2i+1}(\vec{u}_k) + \lambda_{k,i}t_{k,i} \ \to \ 0, \tag{7}$$

as $k \to \infty$. Combining (5), (6) and (7) yields the desired weak convergence. □

5. Subcontinuum Betweenness

For our second case study, we shift attention to the **subcontinuum interpretation of betweenness** in a (not necessarily metrizable) continuum X (see [2, 3, 4]). We refer to this interpretation as K**-betweenness**; the K-interval $[a,b]_{\mathsf{K}}$ is the intersection $\bigcap \mathcal{K}_{a,b}$, where $\mathcal{K}_{a,b} = \mathcal{K}_{a,b}(X)$ is the collection of all subcontinua of X that contain $\{a,b\}$. Clearly the resulting ternary relation is both closed and basic; indeed it is κ-basic and **disjunctive** (i.e., $[a,b] \subseteq [a,c] \cup [c,b]$ for *any* third point c, not just one K-between a and b).

Recalling that M-betweenness is automatically antisymmetric but not necessarily weakly disjunctive, we see that quite the opposite is true for K-betweenness, as the antisymmetry axiom can easily fail (see [4]). We will be mainly interested in two extremes: one where certain connectedness conditions hold at the local level; the other where no such conditions occur anywhere.

A topological space X is **connected im kleinen** (abbr. **cik**) at point $a \in X$ if for each open neighborhood U of a, there exists an open neighborhood V of a such that each two points of V are contained in a connected subset of U. For continua, this condition is well known to be equivalent to saying that for each open neighborhood U of a, there is an open set V and a subcontinuum K such that $a \in V \subseteq K \subseteq U$. Being locally connected for a continuum is equivalent to being cik at each of its points (see, e.g., [20]).

A continuum is **unicoherent** if it is not the union of two subcontinua with disconnected intersection. In this paper, the addition of the modifier *hereditarily* to a property of continua confers the property to all nondegenerate subcontinua. So, for example, a continuum is hereditarily unicoherent if and only if the intersection of any two of its subcontinua is connected (possibly empty). This property is equivalent [3, Proposition 2.1] to the condition that each K-interval is connected.

Remarks 5.1. Suppose our continuum X is a geodesic space.

(i) Then usual open balls are (path) connected; hence X is a Peano continuum.
(ii) By Proposition 3.6, we have $[a,b]_{\mathsf{K}} \subseteq \lfloor p \rfloor \subseteq [a,b]_{\mathsf{M}}$ always holding, where p is any geodesic from a to b.
(iii) If p and q are geodesics from a to b with distinct supports, then, by Proposition 3.7, plus the fact that an arc is irreducible about its pair of noncut points, we know that $\lfloor p \rfloor \cap \lfloor q \rfloor$ is disconnected. This makes $[a,b]_{\mathsf{K}}$ disconnected too; hence we may conclude that in a hereditarily unicoherent geodesic continuum, $[a,b]_{\mathsf{K}} = [a,b]_{\mathsf{M}}$ always holds, and the continuum is unique-geodesic.

(iv) Since M-intervals in geodesic spaces are connected (Proposition 3.6), the identity between corresponding K- and M-intervals implies hereditary unicoherence.

(v) The closed unit square $[0,1]^2$ in the euclidean plane is unique-geodesic without being hereditarily unicoherent: all M-intervals are line segments, while the K-betweenness relation is minimal.

In the remainder of this section K-betweenness is the default interpretation of our interval notation.

Theorem 5.2. *If X is a continuum that is cik at each of the two points $a, b \in X$, and if $[a,b]$ is connected, then $[\cdot,\cdot]$ is usc at $\{a,b\}$. So if X is locally connected then $[\cdot,\cdot]$ is usc at each singleton; and if X is also hereditarily unicoherent, then $[\cdot,\cdot]$ is usc at all pairs.*

Proof. Let $a, b \in X$ be points at which X is cik, and such that $[a,b]$ is connected. Let U be open in X such that $[a,b] \subseteq U$. Since $\{a,b\} \subseteq U$ and X is cik at each point, we have open sets V, W and subcontinua K, M, with $a \in V \subseteq K \subseteq U$ and $b \in W \subseteq M \subseteq U$. Now, $\{a,b\} \in [\![V,W]\!]_2$; and if $\{a',b'\} \in [\![V,W]\!]_2$ then $K \cup [a,b] \cup M \in \mathcal{K}_{a',b'}$ and is contained in U. Hence $[a',b'] \subseteq U$.

The second assertion follows easily since local connectedness is equivalent to being cik at each point, and hereditary unicoherence is equivalent to the condition that each K-interval is a subcontinuum. $\square$

Theorem 5.3. *Let X be a continuum that is cik at $a \in X$. Then $[\cdot,\cdot]$ is usc at $\{a\}$. If X is hereditarily unicoherent and $[\cdot,\cdot]$ is usc at $\{a\}$, then X is cik at a.*

Proof. The first assertion follows immediately from Theorem 5.2. Now assume X is hereditarily unicoherent and that $[\cdot,\cdot]$ is usc at $\{a\}$. Pick open $U \subseteq X$ such that $a \in U$. Then $[a,a] = \{a\} \subseteq U$, so there exists an open neighborhood V of a (i.e., $\{a\} \in [\![V,V]\!]_2$) such that for any $\{a',b'\} \in [\![V,V]\!]_2$, $[a',b'] \subseteq U$. So for any two points of V, their K-interval is a connected set contained in U, showing cik at a. $\square$

Corollary 5.4. *For hereditarily unicoherent continua, being locally connected is equivalent to having $[\cdot,\cdot]$ be usc at singletons.*

Example 5.5. The hereditary unicoherence hypothesis in Corollary 5.4 cannot be eliminated: Let $X \subseteq \mathbb{R}^2$ be the planar continuum $H_0 \cup H_1 \cup V_0 \cup \bigcup_{n=1}^{\infty} V_n$, where $H_m = [0,1] \times \{m\}$, $m = 0,1$, $V_0 = \{0\} \times [0,1]$, and, for $n \in \mathbb{N}$, $V_n = \{\frac{1}{n}\} \times [0,1]$. Then X is K-minimal; hence $[\cdot,\cdot]$ is trivially continuous. On the other hand, X is not locally connected.

The connectedness im kleinen assumption in Theorem 5.2 is not necessary; as, by Proposition 1.4, $[\cdot,\cdot]$ is usc at any pair about which the continuum is irreducible.

A continuum is **decomposable** if it is the union of two proper subcontinua, **indecomposable** otherwise. The **composant** of a point a in continuum X is the union $\kappa(a)$ of all proper subcontinua of X that contain a. Composants are always dense and connected; the composants of an indecomposable continuum are pairwise disjoint. Nondegenerate metrizable indecomposable continua have uncountably many composants [17], but it is possible for a nonmetrizable indecomposable continuum to have exactly one composant [5].

Theorem 5.6. *Let X be an indecomposable continuum that has at least two composants. Then $[\cdot,\cdot]$ is usc at $\{a,b\}$ if and only if $\kappa(a) \neq \kappa(b)$. In particular, $[\cdot,\cdot]$ is never usc at a singleton.*

Proof. Suppose $\kappa(a) \neq \kappa(b)$. Then X is irreducible about $\{a,b\}$, and we may use Proposition 1.4 to conclude that $[\cdot,\cdot]$ is usc at $\{a,b\}$.

If $\kappa(a) = \kappa(b) = \kappa$, then $[a,b]$ is a proper closed subset of X. Hence there is a proper open set U with $[a,b] \subseteq U$. Suppose we are given a neighborhood $[\![V,W]\!]_2$ of $\{a,b\}$, say $a \in V$. Then, because there are composants disjoint from κ and each composant is dense in X, there is some $a' \in V$ that lies in a composant disjoint from κ. Thus $[a',b] = X \not\subseteq U$, showing $[\cdot,\cdot]$ not to be usc at $\{a,b\}$. ☐

Examples 5.7.

(i) Let X be the $\sin\frac{1}{x}$-continuum, namely the union in $\mathbb{R}^2$ of $A = \{0\} \times [-1,1]$ and $S = \{\langle t, \sin\frac{1}{t}\rangle : 0 < t \leq 1\}$. For each $0 < t \leq 1$, let $S_t = S \cap ((-\infty,t] \times \mathbb{R})$. If $a \in A$ and $b = \langle t, \sin\frac{1}{t}\rangle$, then the K-interval $[a,b]$ is $A \cup S_t$; all other intervals are arcs. This continuum is hereditarily unicoherent, as well as hereditarily decomposable, but is not cik at any point of A. It is an easy exercise to show that $[\cdot,\cdot]$ is usc at $\{a,b\}$ if and only if either: (1) a and b are the end points of A; (2) $a \in A$ and $b \in S$; or (3) a and b are both in S. In particular, $[\cdot,\cdot]$ is usc at $\{a\}$ if and only if $a \in S$.

(ii) By Theorem 5.3, hereditarily unicoherent continuum X is cik at no point of X if and only if $[\cdot,\cdot]$ is usc at no singleton. It is easy for $[\cdot,\cdot]$ to fail at singletons without the (hereditarily unicoherent) continuum being indecomposable: Let $X = Y \cup Z$, where Y and Z are (hereditarily unicoherent) indecomposable continua–e.g., pseudo-arcs, bucket handles–and $Y \cap Z$ is a singleton. Then X is decomposable (and hereditarily unicoherent), and an argument similar to the proof of Theorem 5.6 shows $[\cdot,\cdot]$ to be usc at no singleton.

From Theorem 5.2 and Proposition 1.4, we obtain the following.

Corollary 5.8. *For any locally connected continuum, $[\cdot,\cdot]$ is continuous at each singleton.*

The following shows that lower semicontinuity is not affected by any of the issues that confound usc in hereditarily unicoherent continua.

Theorem 5.9. *For any hereditarily unicoherent continuum, $[\cdot,\cdot]$ is lsc at all pairs.*

Proof. Suppose X is a hereditarily unicoherent continuum. Then each K-interval is a subcontinuum.

If $[\cdot,\cdot]$ fails to be lsc at $\{a,b\}$, let $U \subseteq X$ be an open set such that $[a,b] \cap U \neq \emptyset$, but $[a',b'] \cap U = \emptyset$ for $\{a',b'\}$ "arbitrarily near" $\{a,b\}$. To make this more precise, let $\Delta = \langle \Delta, \leq \rangle$ be a directed set, with $\{[\![V_\delta, W_\delta]\!]_2 : \delta \in \Delta\}$ an open neighborhood base at $\{a,b\}$, indexed so that $[\![V_\delta, W_\delta]\!]_2 \supseteq [\![V_\epsilon, W_\epsilon]\!]_2$ whenever $\delta \leq \epsilon$ in Δ.

Because U witnesses the failure of lsc at $\{a,b\}$, we have a net $\langle \{a_\delta, b_\delta\} \rangle_{\delta \in \Delta}$, where $\{a_\delta, b_\delta\} \in [\![V_\delta, W_\delta]\!]_2$, $\delta \in \Delta$, and $[a_\delta, b_\delta] \cap U = \emptyset$ for all δ. Clearly we have the net convergence $\{a_\delta, b_\delta\} \to \{a,b\}$. The hyperspace 2^X is compact, and hence the net $\langle [a_\delta, b_\delta] \rangle$ has a subnet that converges to some $A \in 2^X$. Since subnets of convergent nets converge to the same point, we lose no generality in assuming that $[a_\delta, b_\delta] \to A$.

Arguing as in the proof of Theorem 3.10, and noting that each $[a_\delta, b_\delta]$ is connected, we infer that $A \in \mathcal{K}_{a,b}$; hence that $[a, b] \subseteq A$. But now we have $A \cap U \neq \emptyset$, implying–by the definition of net convergence–that $[a_\delta, b_\delta] \cap U \neq \emptyset$ for some $\delta \in \Delta$, a contradiction. $\square$

Remark 5.10. The argument for the proof of Theorem 5.9 does not allow us to conclude lsc at a pair $\{a, b\}$ where only $[a, b]$ is assumed to be connected; we need the intervals in the net to be connected too. Contrast this situation with the one in Theorem 3.10.

Putting Theorems 5.9 and 5.2 together, we have:

Corollary 5.11. *For any locally connected hereditarily unicoherent continuum,* $[\cdot, \cdot]$ *is continuous at all pairs.*

To obtain a companion to Corollary 3.12 for K-betweenness, we cite an immediate corollary of [4, Theorem 3.2].

Lemma 5.12. *For a locally connected continuum,* K*-betweenness is antisymmetric.*

Now we combine Lemmas 2.15 and 5.12, along with Corollary 5.11.

Corollary 5.13. *For a locally connected hereditarily unicoherent continuum* X*,* $[\cdot, \cdot]$ *is a topological embedding of* $\mathcal{F}_2(X)$ *into* $\mathcal{K}(X)$*.*

And combining Theorems 5.9 and 5.6 gives us:

Corollary 5.14. *Let* X *be an indecomposable continuum that is hereditarily unicoherent and has at least two composants. Then* $[\cdot, \cdot]$ *is continuous at* $\{a, b\}$ *if and only if* a *and* b *lie in different composants of* X*.*

When we add the Fort-Kuratowski Lemma 1.2 to Theorem 5.9, we obtain the following.

Corollary 5.15. *For a hereditarily unicoherent metrizable continuum,* $[\cdot, \cdot]$ *is usc–and hence continuous–at almost every pair.*

Remark 5.16. Note that Lemma 1.2 comes in two versions. One is used to prove Corollary 2.9, the other proves Corollary 5.15.

Finally, adding Lemma 1.2 to Theorems 5.6 and 5.9 gives the following result, which makes no mention of betweenness functions.

Corollary 5.17. *For any nondegenerate hereditarily unicoherent indecomposable metrizable continuum* X*, the set* $\{\{a, b\} : \kappa(a) \neq \kappa(b)\}$ *is dense residual in* $\mathcal{F}_2(X)$*.*

Remark 5.18. Under the hypotheses of Corollary 5.17, $[a, b] = X$ precisely when $\kappa(a) \neq \kappa(b)$. Thus the bracket set of the interval X is dense residual in $\mathcal{F}_2(X)$. Contrast this with the locally connected case in which–by Lemmas 2.15 and 5.12–all bracket sets are singletons.

Question 5.19. Can hereditary unicoherence be removed from the hypothesis of Corollary 5.17?

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Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201-1881, U.S.A.
E-mail address: `paul.bankston@marquette.edu`

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway, Ireland
E-mail address: `aisling.mccluskey@nuigalway.ie`

School of Mathematics and Statistics, University College Dublin, Dublin, Ireland
E-mail address: `richard.smith@ucd.ie`