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FIRST ORDER REPRESENTATIONS OF COMPACT HAUSDORFF SPACES

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ABSTRACT. If two compact Hausdorff spaces have elementarily equivalent lattices of closed sets, then their unital rings of continuous real-valued functions satisfy the same positive-universal sentences. We briefly indicate how this is proved (with details appearing elsewhere), and offer fairly strong evidence that the weakness of the conclusion is in some sense inevitable.

Let τ be a type of finitary relation and function symbols, and denote by $\mathcal{M}(\tau)$ the category of relational structures of type τ (see [**6**]), together with those maps which preserve atomic relations for morphisms. If $\mathcal{X}$ is a category, then a **first order representation (f.o.r.)** of $\mathcal{X}$ in type τ is just a functor $R : \mathcal{X} \to \mathcal{M}(\tau)$. We denote by $R[\mathcal{X}]$ the class $\{R(X) : X \in \mathcal{X}\}$. This notion was first defined explicitly in [**4**], but was implicit much earlier in [**9**]; and a fairly extensive model theoretic study has been made of f.o.r.'s whose domains are categories of topological spaces.

1 EXAMPLES. For a topological space X, let $F(X)$ (resp. $Z(X), B(X)$) denote the bounded lattices of closed (resp. zero-, clopen) subsets of X; and let $D(X)$ denote the bounded lattice of continuous maps from X to the closed unit interval $[0,1]$. By adding constants c_t for $0 \leq t \leq 1$ to the type of bounded lattices we get the type of "[0,1]-lattices", and let $E(X)$ denote the $[0,1]$-lattice of continuous $[0,1]$-valued maps on X. These f.o.r.'s, along with the unital ring $C(X)$ of continuous real-valued functions on X, are contravariant as functors from the category TOP of spaces and continuous maps. Other examples are legion: among those which are covariant on TOP are the first singular homology group and the edge-path groupoid. The semigroup of self-maps is functorial only when one restricts the morphisms of TOP to be the homeomorphisms.

In this note we will consider the domain $\mathcal{X}$ to be a full subcategory of the category KH of compact Hausdorff spaces and continuous maps, and focus on applications of techniques involving "exotic" ultrapowers. The main result of [**4**]

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX is concerned with the preservation of elementary equivalence (in the usual sense of [6], and denoted by $\equiv$) as we pass from one f.o.r. to another. To be specific, a somewhat more general version of the following is proved.

2 **Theorem.** *[4, Theorem (2.13)]. Let X and Y be compact Hausdorff. If $F(X) \equiv F(Y)$ (or $Z(X) \equiv Z(Y)$) then $C(X)$ and $C(Y)$ satisfy the same positive-universal sentences.*

Proof (Sketch): The positive-universal formulas are built up from the atomic formulas using conjunction, disjunction, and universal quantification. The truth of sentences of this type is well known [6] to be preserved under substructures and homomorphic images. Assuming $F(X) \equiv F(Y)$ (the case $Z(X) \equiv Z(Y)$ is handled similarly), the first step is to use the Keisler-Shelah ultrapower theorem [6] to find an ultrafilter $\mathcal{U}$ so that the ultrapowers $\Pi_{\mathcal{U}} F(X)$ and $\Pi_{\mathcal{U}} F(Y)$ are isomorphic as lattices. Next, inspired by the fact that, by Stone duality, ultrapowers of Boolean algebras correspond to "ultracopowers" of Boolean (= totally disconnected compact Hausdorff) spaces, one examines ultracoproducts of arbitrary compact Hausdorff spaces as inverse limits of coproducts in KH (see [3], [4], [5]). Denoting the ultracopower of X via $\mathcal{U}$ by $\Sigma_{\mathcal{U}} X$, one goes on to prove the crucial lemma that an isomorphism between $\Pi_{\mathcal{U}} F(X)$ and $\Pi_{\mathcal{U}} F(Y)$ induces a homeomorphism between $\Sigma_{\mathcal{U}} X$ and $\Sigma_{\mathcal{U}} Y$. The duality theorem of Gel'fand-Kolmogorov then tells us that $C(X)$ and $C(Y)$ have isomorphic "ultrapowers". Unfortunately this must be taken in the category-theoretic sense. Letting $\Pi'_{\mathcal{U}} C(X)$ denote $C(\Sigma_{\mathcal{U}} X)$, one determines that this ring can be obtained from the usual ultrapower $\Pi_{\mathcal{U}} C(X)$ by taking a quotient of a subring in such a way that there is still a natural embedding from $C(X)$ to $\Pi'_{\mathcal{U}} C(X)$ (induced by the diagonal maps from $C(X)$ into powers of $C(X)$, in the category $C[KH]$). The last step is to use this analysis to make the easy inference that $C(X)$ and $C(Y)$ must satisfy the same positive-universal sentences. ∎

3 **Remarks.** (i) One way to try to prove a preservation theorem about elementary equivalence is to effect a syntactic translation which has desirable semantic qualities. Such an approach was used by A. MacIntyre [9, Theorem (5.1)] to show that elementary equivalence is preserved for all compact Hausdorff (even Tichonov) spaces as we pass from C to Z. (The translation is based on the observation that, given $f, g \in C(X)$, their zero-sets $Z(f)$ and $Z(g)$ are disjoint iff $f^2 + g^2$ has an inverse. Thus $Z(f) \subseteq Z(g)$ iff for each $h \in C(X)$, if $Z(g)$ and $Z(h)$ are disjoint then so are $Z(f)$ and $Z(h)$. This tells us how to translate the atomic formulas in the language of lattices. The rest of the translation follows straightforwardly.) This approach can also be used to show the preservation of elementary equivalence for all spaces when we pass from F or Z to B. (Here, the basis of the translation is the observation that a closed (or zero-) set in X is in $B(X)$ iff that set has a complement.) However, any attempt at translation seems doomed to failure in the case of Theorem (2) because of the following example due to J. Isbell [9, Theorem

(5.2)]. Let X (resp. Y) be the one-point compactification of a countably infinite (resp. uncountable) discrete space. Then $F(X) \equiv F(Y)$ [9, Theorem (3.3)]. However Isbell observed that in $Z(Y)$ every atom is complemented, whereas in $Z(X)$ the atom corresponding to the point at infinity of X is not complemented; so $Z(X) \not\equiv Z(Y)$. By MacIntyre's theorem, $C(X) \not\equiv C(Y)$. (At this point we do not have absolute proof that $Z(X) \equiv Z(Y)$ does not imply $C(X) \equiv C(Y)$, but we have strong suspicions.)

(ii) The f.o.r. $C : KH \to C[KH]$ is very special in that it is a category duality (the Gel'fand-Kolmogorov theorem). This is what allows us to infer that $\Pi'_{\mathcal{U}} C(X) \cong \Pi'_{\mathcal{U}} C(Y)$ above. What prevents the desirable conclusion "$C(X) \equiv C(Y)$" in Theorem (2) is the pathological nature of $C[KH]$ as a class of rings: if it were closed under usual cartesian powers as well as usual ultrapowers (it fails on both counts), then everything would be fine, i.e. the usual ultrapowers would be the categorical ones.

A similar situation arises with the f.o.r. E. As recently shown by B. Banaschewski [1], $E : KH \to E[KH]$ is also a duality. Unlike the category $C[KH]$, $E[KH]$ (= the "separated functionally complete" $[0,1]$-lattices) does have usual products; however ultraproducts in $E[KH]$ are generally quotients of the usual ultraproducts. Thus, techniques similar to those used to prove Theorem (2) work in this setting, and we can infer that $E(X)$ and $E(Y)$ satisfy the same positive-universal sentences whenever $F(X) \equiv F(Y)$ or $Z(X) \equiv Z(Y)$. Unfortunately we see no way to improve on this, even though ultraproducts behave better here than in $C[KH]$.

(iii) Once we have a version of Theorem (2) for E, we can immediately obtain one for D since any sentence in the language of bounded lattices holds in $D(X)$ just in case it holds in $E(X)$. (We note in passing that D is not a duality; although, as is proved in [1, Corollary to Proposition (1)], D is "sharp", i.e. $D(X)$ and $D(Y)$ are non-isomorphic whenever X and Y are non-homeomorphic.)

Theorem (2) can be naturally decomposed into two steps as follows. Define two compact Hausdorff spaces X and Y to be **co-elementarily equivalent** (we write $X \equiv Y$, abusing notation slightly) if there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that the ultracopowers $\Sigma_{\mathcal{U}} X$ and $\Sigma_{\mathcal{V}} Y$ are homeomorphic. (Note that, for Boolean spaces X and Y, $X \equiv Y$ iff $B(X) \equiv B(Y)$ (Stone duality plus the ultrapower theorem). It is not trivial, however, to show that $\equiv$ is an equivalence relation on KH [5].) Step one is to show that $R(X) \equiv R(Y)$ implies $X \equiv Y$; and step two is to show that $X \equiv Y$ implies something about $S(X)$ and $S(Y)$. It turns out that the conjunction of steps one and two is much stronger than Theorem (2) (in which ultrapower issues are completely hidden): one can easily obtain spaces $X, Y \in KH$ such that $X \equiv Y$ but $R(X) \not\equiv R(Y)$, where R is any of F, Z or C. (Let X and Y be Boolean without isolated points, let X be extremally disconnected, and let Y fail to be basically disconnected [7]. Then $B(X)$ and $B(Y)$, being atomless Boolean algebras, are elementarily equivalent. Thus $X \equiv Y$. But clearly $F(X) \not\equiv F(Y)$ and $Z(X) \not\equiv Z(Y)$. $C(X) \not\equiv C(Y)$ now follows by MacIntyre's theorem.)

A Theorem (2)-style proof could also be easily cooked up to show $F(X) \equiv F(Y)$ implies $B(X) \equiv B(Y)$ (as can be shown much more directly by translation techniques (see Remark (3(i)) above)). Of more interest, though, is step two: $X \equiv Y$ implies $B(X) \equiv B(Y)$ for all $X, Y \in KH$. (This follows directly from Lemma (4.6) of [3] which asserts that $B(\Sigma_{\mathcal{U}} X) \cong \Pi_{\mathcal{U}} B(X)$ for any $X \in KH$ and ultrafilter $\mathcal{U}$.) The fact that B is not sharp when one goes beyond the category BS of Boolean spaces is somewhat irksome, however; and the search is on to find a sharp f.o.r. $R : \mathcal{X} \to \mathcal{M}(\tau)$; where $\mathcal{X}$ is a full subcategory of KH properly containing BS, and $R(X) \equiv R(Y)$ whenever $X \equiv Y$. Because of ultrapower techniques already available, there is a hope that such an R could be found so that $R : \mathcal{X} \to R[\mathcal{X}]$ is a duality and $R[\mathcal{X}]$ is an "elementary P-class" (i.e. the class of models of a first order theory, closed under usual products). Unfortunately, this hope cannot be fulfilled: any R with the above properties must bear a very strong resemblance to Stone duality.

4 THEOREM. *Let $\mathcal{X}$ be a full subcategory of KH, containing BS, and suppose that $R : \mathcal{X} \to \mathcal{M}(\tau)$ is a f.o.r. which is a duality onto $R[\mathcal{X}]$. Assume further that $R[\mathcal{X}]$ is an elementary P-class.*

Then:

(i) [3] $X \in \mathcal{X}$ is finite iff $R(X)$ is finite.

(ii) (Banaschewski [2]) $\mathcal{X} = BS$.

(iii) For all $X \in \mathcal{X}$, $|B(X)| \leq |R(X)|$ ($|\cdot|$ denotes cardinality). Moreover, if X is infinite then $w(X) \leq |R(X)|$ ($w(\cdot)$ denotes "weight", i.e. the smallest cardinality of an open basis).

(iv) If, in $R[\mathcal{X}]$, equalizers are embeddings and coequalizers are surjections then $w(X) = |R(X)|$ for all infinite $X \in \mathcal{X}$.

PROOF: (i) Because $\mathcal{X}$ contains the Stone-Čech compactification $\beta(X)$ for each discrete X, we know that all the KH-copowers of the singleton space $\mathbf{1}$ lie in $\mathcal{X}$. If $\mathcal{U}$ is any ultrafilter then the ultracopower $\Sigma_{\mathcal{U}}\mathbf{1} = \varprojlim\{\beta(X) : X \in \mathcal{U}, X\text{discrete}\}$ is simply $\mathbf{1}$ again [3, Lemma (4.2)]. Thus the same is true in the category $\mathcal{X}$. Let $P = R(\mathbf{1})$. Since $\mathcal{K} = R[\mathcal{X}]$ is closed under usual powers and ultrapowers, the categorical ultrapowers in $\mathcal{K}$ are the usual ones. Consequently P is isomorphic to each of its ultrapowers, and is hence finite. If $X \in \mathcal{X}$ has n elements, n finite, then X is the n-fold copower of $\mathbf{1}$; whence $R(X)$ is the n-fold power P^n, a finite structure. Conversely, if X is infinite then X has infinitely many endomorphisms in $\mathcal{X}$ (because $\mathcal{X} \subseteq KH$ is full and constant maps are continuous). Thus $R(X)$ has infinitely many endomorphisms in $\mathcal{K}$, hence $R(X)$ is infinite.

(ii) We will only sketch the proof. The basic idea behind the clever argument in [2] is to find an object in $\mathcal{K}$ which plays a role analogous (to enough of a degree) to that played by the free Boolean algebra on a singleton. The natural candidate is P^n, where n is the cardinal of P. (If R were Stone duality, P would be the two-point algebra, and P^n would be the four-point algebra.) Although P^n will not in general be free on a singleton, it is enough to show that it is a generator (read "separator") in the category $\mathcal{K}$. This is not trivial to prove (the finiteness of P is crucial); but once it is established, the rest is easy. For let X be the space in $\mathcal{X}$ corresponding to P^n. Then X, a copower of $\mathbf{1}$, is Boolean. But X is also a cogenerator for $\mathcal{X}$. This implies that, whenever $Y \in \mathcal{X}$ and $x, y \in Y$ are distinct elements, there exists a continuous $f : Y \to X$ with $f(x) \neq f(y)$. This is, of course, impossible if Y has a nontrivial connected subset.

(iii) First of all, it is easy to see that the structure P has cardinality ≥ 2. Thus if $X \in \mathcal{X}$ is finite of cardinality n then $|B(X)| = 2^n \leq |P|^n = |R(X)|$. Suppose X is infinite. Then $\alpha = |\ B(X)|$ is infinite and $\alpha = |\hom_{\mathcal{X}}(X, \mathbf{2})|$, where $\mathbf{2}$ is the doubleton space. Thus $\alpha = |\hom_{\mathcal{K}}(P^2, R(X))| \leq |R(X)|$ since P is finite and $R(X)$ is infinite. If X is infinite, we have, from (ii), that X is Boolean. Hence $w(X) = |B(X)|$.

(iv) Given any $X \in \mathcal{X}$, let X^d denote X with the discrete topology. Then the obvious continuous surjection $f : \beta(X^d) \to X$ is a coequalizer in $\mathcal{X}$ since, by (ii), $\mathcal{X} = BS$. Hence $R(X)$ embeds as a substructure of $R(\beta(X^d)) \cong P^\alpha$, where $\alpha = |X|$. This says that every structure in $\mathcal{K}$ is a substructure of a power of P. Now let τ be the type appropriate to $\mathcal{K}$. Then we lose no generality in assuming that τ is countable: for if two distinguished relation or function symbols agree on the finite structure P, they must agree on each structure in $\mathcal{K}$.

Now suppose X is infinite, and let $\alpha = w(X)$. Since, by (ii), X is Boolean, there is an embedding of X into the Cantor discontinuum $\mathbf{2}^\alpha$. Since embeddings in BS are equalizers, we know that $R(X)$ is a homomorphic image of the α-fold $\mathcal{K}$- copower of P^2. The result that $|R(X)| \leq \alpha$ (and hence, by (iii), that $|R(X)| = w(X)$) will be immediate once we prove the

CLAIM. Let $A^{\mathcal{K}}$ be the $\mathcal{K}$-coproduct of the family $\langle A_\xi : \xi < \alpha \rangle$, where $|A_\xi| \leq \alpha$ for each $\xi < \alpha$. Then $|A^{\mathcal{K}}| \leq \alpha$.

To see this, let $\sigma_\xi^{\mathcal{K}} : A_\xi \to A^{\mathcal{K}}$ be the canonical injection, $\xi < \alpha$, and let $A \in \mathcal{M}(\tau)$ be the usual free product of the A_ξ''s (τ is a countable type), with injections $\sigma_\xi : A_\xi \to A$. Since the images of the maps σ_ξ generate A (see [8]), we know that $|A| \leq \alpha$. Now there is a unique homomorphism $\varphi : A \to A^{\mathcal{K}}$ such that $\varphi \circ \sigma_\xi = \sigma_\xi^{\mathcal{K}}$ for all $\xi < \alpha$. Let $B \subseteq A^{\mathcal{K}}$ be the image of φ. By the Löwenheim - Skolem theorem, there is an elementary substructure $A' \subseteq A^{\mathcal{K}}$ with $B \subseteq A'$ and, because $|B| \leq \alpha, |A'| \leq \alpha$. Since $\mathcal{K}$ is an elementary class, $A' \in \mathcal{K}$. And since A' behaves like the $\mathcal{K}$-coproduct, it follows that $A' = A^{\mathcal{K}}$. This completes the proof of the Claim and of the Theorem.∎

5 Remark. Theorem (4(ii)) answers Question (5.1) of [3], namely whether KH can be category dual to an elementary P-class. This question was also answered independently by J. Rosický [10]; however the techniques in [10] are more special than Banaschewski's and do not seem to be directly applicable to yield Theorem (4(ii)).

6 Questions. (i) Under hypotheses similar to those of Theorem (4(iv)), can it be shown that $R(X)$ is actually the structure of continuous functions from X to P (with the discrete topology)?

* (ii) How many co-elementary equivalence classes are there in KH (This question is also raised in [5].) Since $F(X) \equiv F(Y)$ implies $X \equiv Y$, and the language of lattices is countable, we know this number is at most the power of the continuum. On the other hand, by Stone duality and the fact that the theory of Boolean algebras has countably many complete extensions [6], we know the number of co-elementary equivalence classes in BS is countable.

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* The answer is c.