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On categories of algebras equivalent to a quasivariety

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Abstract. In a recent paper, B. Banaschewski proved that any *SP*-class of algebras which is category equivalent to a variety (over a possibly different finitary similarity type) is itself a variety. Here we prove the analogous statement obtained by replacing "variety" with "quasivariety". We also present examples which detail some of the difficulties arising when one tries to strengthen the theorem in various ways.

0. Introduction

Let τ be a similarity type of finitary relations and operations and consider the class $\mathcal{M}_\tau$ of all relational structures in type τ to be a category in the usual sense by declaring all homomorphisms (i.e., atomic relation preserving functions) to be morphisms in the category. A class $\mathcal{K} \subseteq \mathcal{M}_\tau$ is then viewed as a full subcategory of $\mathcal{M}_\tau$. In [2] Banaschewski provides examples of classes $\mathcal{K}$ which are equivalent as categories to varieties (= equational classes) of algebras (i.e. where the similarity type consists only of operations), but which are not themselves varieties (e.g. the class of all torsionfree divisible abelian groups is equivalent to the equational class of vector spaces over the rational number field). It may be of interest to note that all of his examples are elementary *P*-classes (universal-existential Horn, in fact). The class of rings of continuous real valued functions with 0-dimensional compact Hausdorff domains is not closed under any of the usual operators on classes (in particular it is neither elementary nor *P*-closed); yet it is equivalent, by the duality theorems of M. H. Stone and Gel'fand–Kolmogorov, to the variety of Boolean algebras.

The main result in [2], somewhat surprising in light of the examples, is the following:

0.1 THEOREM (Banaschewski). *Any SP-class of algebras which is category equivalent to a variety is itself a variety.*

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As an immediate corollary, one can show (see [4]) that the quasivariety of torsionfree abelian groups is not equivalent to any variety.

In the present note we prove the following analogue of (0.1).

0.2 THEOREM. *Any SP-class of algebras which is category equivalent to a quasivariety is itself a quasivariety.*

Both theorems relate "internal" notions (the structure of the *SP*-class as a category) and "external" ones (how that class sits as a subcategory of $\mathcal{M}_\tau$). Although knowledge of (0.1) was instrumental in leading us to think of possible analogues such as (0.2), our methods of proof turned out (not entirely unexpectedly) to be quite different: whereas we use the techniques of Chang, Galvin, Keisler, Shelah, *et al.* (see [5, 7]) in the manipulation of reduced products, Banaschewski's method equates the notions of "equivalence relation" on an object in a category and "congruence" on an algebra.

A word about terminology. A class $\mathcal{K}$ is: (i) a *P-class* if it is closed under the taking of usual direct products; (ii) an *S-class* if it is closed under substructures in $\mathcal{M}_\tau$; (iii) an *elementary* (resp. *universal elementary, universal-existential elementary*, etc., *Horn*) *class* if it is the class of models of a set of sentences (resp. universal sentences, universal-existential sentences, etc., Horn sentences) in type τ (see [5]); and (iv) a *variety* if it is a class of algebras which is the class of models of a set of equations (i.e. universally closed atomic formulas). Any terminology which is not in standard textbooks will be defined as we proceed.

1. Proof of Theorem 0.2

Of central importance in our approach is the reduced product construction in a category (see [1, 3, 4, 6, 9]). Given a category $\mathcal{A}$ with products, a family $\langle A_i : i \in I\rangle$ of $\mathcal{A}$-objects, and a filter D of subsets of I, we construct (when possible) the limit $\prod_D^{\mathcal{A}} A_i = \varinjlim \langle \prod_{i\in J}^{\mathcal{A}} A_i : J \in D\rangle$; where D is directed by reverse inclusion and the connecting morphisms are the "restrictions" $\rho_{JK}^{\mathcal{A}} : \prod_{i\in J}^{\mathcal{A}} A_i \to \prod_{i\in K}^{\mathcal{A}} A_i$, $J \supseteq K \in D$. $\prod_D^{\mathcal{A}} A_i$ is called a *reduced product* (in $\mathcal{A}$) and an *ultraproduct* when D is an ultrafilter. When $A_i = A$ for all $i \in I$, there is a natural "diagonal" morphism $\Delta_D^{\mathcal{A}} : A \to \prod_D^{\mathcal{A}}(A)$ making all relevant diagrams commute. (In [6] this morphism is used to define "finiteness" in a category: A is *$\mathcal{A}$-finite* if $\Delta_D^{\mathcal{A}}$ is an isomorphism for all ultrafilters D. We will use this concept in §2.)

Now the categories $\mathcal{M}_\tau$ have reduced products and they are easily seen to be the usual ones. In fact, it is a straightforward consequence of the Łoś Theorem that in an elementary *P*-class $\mathcal{K}$, ultraproducts are defined and they are usual (hence "$\mathcal{K}$-finite" = "finite underlying set"). However, as we shall see (in Example 2.4), being an elementary class with ultraproducts as a category is not enough to ensure that those ultraproducts will be usual.

If $\mathcal{K}$ is a Horn class, a classic result of C. C. Chang – that Horn sentences are preserved by all reduced products – shows that $\mathcal{K}$ has reduced products and they are indeed the usual ones. For future reference we state the most important results concerning how the studies of elementary classes and reduced products relate to one another.

1.1 THEOREM. (i) (*Frayne–Morel–Scott, Kochen*) *$\mathcal{K}$ is an elementary class iff $\mathcal{K}$ is closed under ultraproducts and elementary equivalence.*

(ii) (*Chang, Keisler, Galvin*) *$\mathcal{K}$ is a Horn class iff $\mathcal{K}$ is closed under reduced products and elementary equivalence.*

(iii) (*Keisler, Shelah*) *Two relational structures are elementarily equivalent (i.e., satisfy the same first order sentences) iff they have isomorphic ultrapowers for some ultrafilter.*

(iv) (*Folklore*) *$\mathcal{K}$ is a quasivariety iff $\mathcal{K}$ is an SP-class closed under ultraproducts.*

1.2 *Remark.* 1.1(iv) follows from 1.1(ii, iii) plus the old result of McKinsey (see [7]) that a Horn S-class is a quasivariety. To see this suppose $\mathcal{K}$ is an SP-class closed under ultraproducts. To show $\mathcal{K}$ to be a quasivariety it suffices to show that $\mathcal{K}$ is closed under arbitrary reduced products. So let $\langle A_i : i \in I\rangle$ and D be given, $A_i \in \mathcal{K}$ each $i \in I$, and D is a filter on I. Letting $\mathcal{U} = \{D' : D'$ is an ultrafilter on I, $D \subseteq D'\}$, we note that $D = \bigcap \mathcal{U}$ and therefore that $\prod_D A_i$ naturally embeds in the product $\prod_{D' \in \mathcal{U}} B_{D'}$, where $B_{D'} = \prod_{D'} A_i$.

If $\mathcal{K}$ is any P-class, and if $\prod_D^{\mathcal{K}} A_i$ is defined, then there are "external" and "internal" limit morphisms $\rho_J : \prod_{i\in J} A_i \to \prod_D A_i$, $\rho_J^{\mathcal{K}} : \prod_{i\in J} A_i \to \prod_D^{\mathcal{K}} A_i$ and a morphism $\beta_D^{K} : \prod_D A_i \to \prod_D^{K} A_i$ such that $\beta_D^{\mathcal{K}} \circ \rho_J = \rho_J^{\mathcal{K}}$ for all $J \in D$. Our main lemma is the following.

1.3 Lemma. *Let $\mathcal{K}$ and $\mathcal{L}$ be SP-classes of algebras, and let $\Phi : \mathcal{K} \to \mathcal{L}$ be a category equivalence. If $\langle A_i : i \in I\rangle$ is an indexed family in $\mathcal{K}$, D is a filter on I, and $\beta_D^{\mathcal{L}} : \prod_D \Phi(A_i) \to \prod_D^{\mathcal{L}} \Phi(A_i)$ is an isomorphism then so too is $\beta_D^{\mathcal{K}} : \prod_D A_i \to \prod_D^{\mathcal{K}} A_i$.*

Proof. Without loss of generality we may treat $\prod_D \Phi(A_i)$ and $\prod_D^{\mathcal{L}} \Phi(A_i)$ as identical and define $\prod_D^{\mathcal{K}} A_i$ via the correspondence Φ.

Since $\mathcal{K}$ is an S-class of algebras, it will suffice to show that $\beta_D^{\mathcal{K}} : \prod_D A_i \to \prod_D^{\mathcal{K}} A_i$ is one-one; so let $[f]_D \neq [g]_D$ in the usual reduced product $\prod_D A_i$ ($[f]_D = \{f' \in \prod_{i\in I} A_i : \{i : f'(i) = f(i)\} \in D\}$). Then for each $J \in D$, the restrictions $f \restriction J$, $g \restriction J$

are distinct in $\prod_{i\in J} A_i$. Let $F(x)$ be the free $\mathscr{K}$-algebra over the singleton set $\{x\}$ (the underlying set functors for SP-classes have left adjoints), and let $\phi_J(x) = f \restriction J$, $\gamma_J(x) = g \restriction J$, $J \in D$. Letting $\eta_J : E_J \to F(x)$ be the equalizer of ϕ_J, γ_J, we note that η_J is an inclusion which is not an isomorphism; and that for all $J \supseteq K \in D$ there is an inclusion $\eta_{JK} : E_J \to E_K$. Also note that for each $J \supseteq K \in D$, $\phi_K = \rho_{JK} \circ \phi_J$, $\gamma_K = \rho_{JK} \circ \gamma_J$; and hence the compositions $\phi_D = \rho_J \circ \phi_J$, $\gamma_D = \rho_J \circ \gamma_J$, $\phi_D^{\mathscr{K}} = \rho_J^{\mathscr{K}} \circ \phi_J$, $\gamma_D^{\mathscr{K}} = \rho_J^{\mathscr{K}} \circ \gamma_J$ are independent of $J \in D$. Thus

$$\beta_D^{\mathscr{K}}([f]_D) = \beta_D^{\mathscr{K}}(\phi_D(x)) = \phi_D^{\mathscr{K}}(x), \text{ and}$$

$$\beta_D^{\mathscr{K}}([g]_D) = \beta_D(\gamma_D^{\mathscr{K}}(x)) = \gamma_D^{\mathscr{K}}(x).$$

Since homomorphisms which agree at x must agree everywhere, it will suffice to prove $\phi_D^{\mathscr{K}} \neq \gamma_D^{\mathscr{K}}$.

Since each E_J is a proper subalgebra of $F(x)$, we know that $x \notin E = \bigcup_{J\in D} E_J = \varinjlim \langle E_J : J \in D\rangle$. It thus follows that the inclusion $\eta : E \to F(x)$ is not an isomorphism; hence that $\Phi(\eta) : \Phi(E) \to \Phi(F(x))$ also fails to be an isomorphism.

Now $\mathscr{L}$ is an SP-class and $\Phi(\eta)$ is a monomorphism. Thus $\Phi(\eta)$ is one-one, hence must fail to be onto. So let $y \in \Phi(F(x))$ fail to be in the image of $\Phi(\eta)$. Then, since $\Phi(\eta)$ is the limit morphism for the equalizers $\Phi(\eta_J)$, each pair $\Phi(\phi_J)$, $\Phi(\gamma_J)$ must disagree at y. But $\prod_D^{\mathscr{L}} \Phi(A_i)$ is the usual reduced product. This means that $\Phi(\phi_D^{\mathscr{K}})$ and $\Phi(\gamma_D^{\mathscr{K}})$ also must disagree at y; therefore $\phi_D^{\mathscr{K}} \neq \gamma_D^{\mathscr{K}}$ as desired. □

The proof of (0.2) is now easy, given the classic results (1.1). Suppose $\mathscr{K}$ is an SP-class which is category equivalent to a quasivariety $\mathscr{L}$. Then $\mathscr{L}$ is an SP-class and has usual reduced products. By (1.3), $\mathscr{K}$ is closed under usual reduced products, and is hence a quasivariety. □

2. A question and some examples.

In (1.3) we never used the full strength of the hypothesis that $\mathscr{L}$ is an SP-class, and it would be interesting if we could draw the conclusions of (0.2) from the weaker assumptions that $\mathscr{K}$ is equivalent to an elementary P-class (resp. Horn class). In particular we would like an answer to the following.

2.1 QUESTION. If $\mathscr{K}$ is an SP-class which is category equivalent to an elementary P-class (resp. Horn class), does $\mathscr{K}$ necessarily have usual ultraproducts (resp. reduced products)?

The problem in trying to prove an appropriately strengthened (1.3) is that, although it is easy to see that $\Phi(\phi_J) \neq \Phi(\gamma_J)$ for all $J \in D$, we need the existence of a single $y \in \Phi(F(x))$ at which each pair of morphisms fails to agree. We have explored various hypotheses under which such a y can be found, but the assumption that $\mathscr{L}$ is an SP-class is the only one which seems reasonably elegant.

Here are some examples à propos of this question. First of all, we cannot remove the property of SP-closure from $\mathscr{K}$.

2.2 EXAMPLE. A class $\mathscr{K}$ of algebras which is category equivalent to an equational class and which has unusual ultraproducts.

Construction. As mentioned before, the class $\mathscr{K}$ of rings of continuous real-valued functions with 0-dimensional compact Hausdorff domains is equivalent to the equational class of Boolean algebras. To see that $\mathscr{K}$ does not have usual ultraproducts we note that $A \in \mathscr{K}$ is $\mathscr{K}$-finite iff A is a finite power of the ring of real numbers. □

Although a negative answer to (2.1) would be hard (if not impossible) to find, it is relatively easy to specify an SP-class of algebras with unusual ultraproducts.

2.3 EXAMPLE. An SP-class $\mathscr{K}$ of algebras which has unusual ultraproducts.

Construction. Let $\mathcal{M}_\tau$ be all algebras equipped with a countable sequence of unary operations, and let $\mathscr{K} = \{\langle A, \langle f_n \rangle_{n<\omega} \rangle : \text{if } a \neq b \text{ then } f_n(a) \neq f_n(b) \text{ for some } n < \omega\}$. Now let $A = \omega + 1 = \{0, 1, 2, \ldots, \omega\}$, let

$$f_n(m) = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n, \end{cases} \quad m < \omega, \text{ and } f_n(\omega) = 0 \text{ for all } n < \omega.$$

Then it is not hard to check that $\mathscr{K}$ is an S-class, $\mathscr{K}$-reduced products exist, and the algebra A above is infinite and $\mathscr{K}$-finite. Consequently, $\mathscr{K}$ has the properties claimed. □

We close with an example which answers in the negative a question raised in private conversation with B. Banaschewski, to wit: If $\mathscr{K}$ is an elementary class which has ultraproducts as a category, are those ultraproducts necessarily the usual ones?

2.4 EXAMPLE. A universal elementary class $\mathscr{K}$ of relational structures which has unusual ultraproducts.

Construction. Let $\mathcal{M}_\tau$ be all relational structures equipped with one unary operation and one unary relation, and let $\mathscr{K} = \{\langle A, f, V \rangle : a \in V \text{ iff } f(a) \neq a, \text{ all } a \in A\}$. Clearly, $\mathscr{K}$ is the class of models of a single universal sentence in the appropriate first order language. ($\mathcal{K}$ is thus an S-class.) Given $\langle A, f, V\rangle \in \mathcal{K}$, define $U = \{a \in A : f(a) = a\} = A - V$, $U^{(0)} = U$, $U^{(n+1)} = \{a \in A : f^{n+1}(a) \in U$ and $f^n(a) \in V\}$ (where $n < \omega$ and f^n denotes the n-fold iterate of f), and $U^{(\omega)} = \{a \in A : f^n(a) \in V$, all $n < \omega\}$. Then $\{U^{(\alpha)} : \alpha \leq \omega\}$ is a partition of A; and if h is any homomorphism between members of $\mathcal{K}$ then h preserves all sets $U^{(\alpha)}$ (because h must preserve V). Thus if $A_i = \langle A_i, f_i, V_i\rangle$, $i \in I$, then $\prod^{\mathcal{K}}_{i\in I} A_i = \bigcup_{\alpha \leq \omega} \prod_{i \in I} U_i^{(\alpha)} \subseteq \prod_{i\in I} A_i$. If D is an ultrafilter on I then $\prod^{\mathcal{K}}_D A_i = \bigcup_{\alpha\leq\omega} \prod_D U_i^{(\alpha)} \subseteq \prod_D A_i$, hence it follows that A is $\mathcal{K}$-finite iff $U^{(\alpha)}$ is finite for all $\alpha \leq \omega$. Let $A = \omega$, define $f(0) = 0$, and for each $n \geq 1$ define $f(n) = n - 1$. Of course $U^{(n)} = \{n\}$, $n < \omega$, and $U^{(\omega)} = \emptyset$. Thus A is infinite and $\mathcal{K}$-finite, so $\mathcal{K}$ has unusual ultraproducts. $\square$

2.5 *Remark*. The class $\mathcal{K}$ in (2.4) is not equivalent to an elementary P-class. To see this it clearly suffices to find a $\mathcal{K}$-finite A with an infinite set of endomorphisms. So let $A = \{0, 1, 1', 2, 2', \ldots\}$, and define $f(0) = 0$, $f(k) = f(k') = k - 1$ for $k > 0$. Then $U^{(0)} = \{0\}$, $U^{(n)} = \{n, n'\}$, $n > 0$, and $U^{(\omega)} = \emptyset$. For each $n < \omega$ define $e_n : A \to A$ by

$$e_n(0) = 0, \qquad e_n(k) = k, \qquad e_n(k') = \begin{cases} k & \text{if } k \leq n \\ k' & \text{if } k > n. \end{cases}$$

This gives an infinite family of endomorphisms on A.

It would be interesting to find an elementary class $\mathcal{K}$ which is equivalent to an elementary P-class and which has unusual ultraproducts.

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