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ON PRODUCTIVE CLASSES OF FUNCTION RINGS

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ABSTRACT. No nontrivial P-class ("P" for "productive") of rings of continuous real-valued functions can be category equivalent to any elementary P-class of finitary universal algebras.

0. Introduction. In this paper, "algebra" means "finitary universal algebra" in the sense of Birkhoff, and a class $\mathfrak{K}$ of algebras will be viewed as a category by allowing all algebra homomorphisms as category morphisms. $\mathfrak{K}$ is *productive* or a *P-class* (resp. *S-class*) if $\mathfrak{K}$ is closed under usual direct products (resp. subalgebras); $\mathfrak{K}$ is *elementary* if there is a set of first order axioms such that $\mathfrak{K}$ is precisely the class of models of those axioms (see **[5]**).

We will be interested in category theoretic properties of classes of function rings, to wit: Let RCF denote the class of all rings of continuous real-valued functions $C(X)$ with topological spaces for domains. We ask which subclasses of RCF can be equivalent to "nice" classes of algebras (e.g. elementary P-classes, SP-classes, varieties, etc.).

0.1 EXAMPLES. (i) $\mathfrak{K}_0 = \{C(X)\colon X$ is zero-dimensional compact Hausdorff$\}$ is equivalent to the variety of Boolean algebras (see **[6]**).

(ii) $\mathfrak{K}_1 = \{C(X) \in \mathfrak{K}_0\colon X$ has no isolated points$\}$ is equivalent to the elementary P-class of atomless Boolean algebras.

There has come to be a growing list of negative results in this area. In **[1]** it is shown that $\mathfrak{K} = \{C(X)\colon X$ is compact Hausdorff$\}$ cannot be equivalent to an SP-class; and in **[3]** it is shown also that $\mathfrak{K}$ cannot be equivalent to a class $\mathfrak{L}$ which is "representable" (i.e. free objects over singletons exist in $\mathfrak{L}$) and is either an elementary P-class or an S-class whose basic alphabet of operation symbols has cardinality less than that of the continuum. The major unsolved problem in this area is whether $\mathfrak{K}$ is equivalent to any elementary P-class at all. In this paper we highlight the importance of the fact that products in $\mathfrak{K}_0$, $\mathfrak{K}_1$ and $\mathfrak{K}$ above are not the usual ones by proving results of which the following is an easy corollary.

0.2 THEOREM. *No nontrivial P-subclass of RCF (i.e. one having more than the isomorphism type of the "degenerate" ring $0 = C(\varnothing)$) can be category equivalent to an elementary P-class.*

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The proof employs the notion of "reduced product" in a category and will be presented in the next section.

1. Main results. We assume the reader to be familiar with the usual notion of reduced product in model theory [**5**]. The key observation, one which is well known, is that if $\mathcal{K}$ is any elementary P-class of finitary relational structures, then ultraproducts in $\mathcal{K}$ are simply direct limits of direct products (see [**2, 3, 4**] for more details and references). This of course can be placed in a categorical context. We will show that most categorical reduced products in a P-subclass of RCF must be trivial (or nonexistent). This will immediately entail (0.2) since the diagonal morphism from an algebra into an ultrapower in an elementary P-class is always a monomorphism.

Our notation regarding reduced products and powers comes from [**5**]: If $\langle A_i\colon i \in I\rangle$ is a family of relational structures of the same finitary type and D is a filter on I then $\prod_D A_i = \prod_D \langle A_i\colon i \in I\rangle$ is the reduced product with elements $a_D = \{a' \in \prod_{i\in I} A_i\colon \{i\colon a'_i = a_i\} \in D\}$; if each A_i is equal to A then the reduced power is denoted $\prod_D A$ and the natural diagonal embedding is denoted $d\colon A \to \prod_D A$.

For $J \supseteq K \in D$, let $r_{JK}\colon \prod_{i\in J} A_i \to \prod_{i\in K} A_i$ be the natural restriction morphism. Then the associated direct limit $\varinjlim \langle \prod_{i\in J} A_i\colon J \in D\rangle$ is precisely the reduced product $\prod_D A_i$ in the category of all relational structures. Moreover, if $\mathcal{K}$ is any elementary P-class then categorical ultraproducts in $\mathcal{K}$ are the usual ones. (N.B. It is possible to have an elementary class $\mathcal{K}$ which has unusual ultraproducts as a category (see [**4**]).)

Before we state our main results, we introduce the notion of "commuting system" of homomorphisms. Let $\langle X_i\colon i \in I\rangle$ be topological spaces, let D be a filter on I, and let X be any space. A "commuting system", in this context, is a family $\langle h_J\colon J \in D\rangle$ where, for $J \in D$, $h_J\colon \prod_{i\in J} C(X_i) \to C(X)$ is a homomorphism such that for all $J \supseteq K \in D$, $h_K \circ r_{JK} = h_J$. Our main concern is in the existence of certain commuting systems.

1.1 THEOREM. *Assume $\langle X_i\colon i \in I\rangle$ is a family of topological spaces, D is a free filter on I (i.e. $\bigcap D = \varnothing$), X is a space, there are no uncountable measurable cardinals at most $|I|$ ($=$ the cardinality of I), and $\langle h_J\colon J \in D\rangle$ is a commuting system. Then X is empty.*

PROOF. It is well known (see [**6**]) that we lose no generality by assuming all of the above spaces to be realcompact Tichonov; and we can then invoke Theorem (10.6) of [**6**] to the effect that if Y is realcompact and if $h\colon C(Y) \to C(X)$ is a ring homomorphism (N.B. $h(1) = 1$. Hence $\mathrm{Hom}(0, C(X)) = \varnothing$, unless $X = \varnothing$.) then there is a unique continuous $h'\colon X \to Y$ such that, for all $f \in C(Y)$, $h(f) = f \circ h'$.

Let $\dot\bigcup_{i\in I} X_i$ denote the disjoint union of the spaces X_i; and let p be a z-ultrafilter on $\dot\bigcup_{i\in I} X_i$ with the countable intersection property (c.i.p. $=$ intersections of countable subfamilies of p are nonempty). We show that $\dot\bigcup_{i\in I} X_i$ is realcompact by proving that p must be fixed. Indeed let $g\colon \dot\bigcup_{i\in I} X_i \to I$ take x to i exactly when $x \in X_i$, and let $F = \{J \subseteq I\colon g^{\leftarrow}[J] \in p\}$. One can check easily enough that F is a countably complete ultrafilter on I (e.g. F is closed under superset since $g^{\leftarrow}[J] = \dot\bigcup_{i\in J} X_i$ is always clopen, hence a zero set). Now there are no uncountable measurable cardinals at most $|I|$, hence F must be fixed (= principal). Suppose $\{i\} \in F$. Then $X_i \in p$, and p restricted to X_i is a z-ultrafilter on X_i with the c.i.p. Thus, since X_i is realcompact, p converges.

Now since $\prod_{i \in I} C(X_i)$ and $C(\dot{\bigcup}_{i \in I} X_i)$ are naturally isomorphic, we can consider each h_J as a homomorphism from $C(\dot{\bigcup}_{i \in J} X_i)$ to $C(X)$. Thus look at the "dual system" h'_J: $X \to \dot{\bigcup}_{i \in J} X_i$. Letting e_{JK}: $\dot{\bigcup}_{i \in K} X_i \to \dot{\bigcup}_{i \in J} X_i$ be the natural embedding, $J \supseteq K \in D$ (an inclusion in this case), we note that the uniqueness of each h'_J ensures that all the appropriate diagrams commute (i.e. $e_{JK} \circ h'_K = h'_J$ for each $J \supseteq K \in D$). Since $\bigcap D = \varnothing$, this forces X to be empty. $\square$

1.2 Corollary. *If $\mathcal{K}$ is a P-subclass of RCF then reduced products in $\mathcal{K}$ are "trivial", in the sense that $\prod_D^{\mathcal{K}} A_i$, the reduced product in $\mathcal{K}$, is zero whenever D is a free filter on an index set whose cardinality is less than all uncountable measurable cardinals.* $\square$

1.3 Remark. The measurable cardinal hypothesis is necessary for 1.1 to work. For let D be a free countably complete ultrafilter on a set I, and let each X_i be a singleton. Then $\prod_D C(X_i) \cong \mathbf{R}$ (= the ring of real numbers), by Corollary (4.2.8) in [5].

We can get the conclusion of 1.1 with altered hypotheses and more model theoretic techniques.

1.4 Theorem. *Assume $\langle X_i: i \in I \rangle$ is a family of spaces, D is a countably incomplete ultrafilter on I, X is a space, and $\langle h_J: J \in D \rangle$ is a commuting system. Then X is empty.*

Proof. Suppose, to the contrary, that there is a nonempty space X for which a commuting system exists. If $\{i: X_i = \varnothing\} = J \in D$ then h_J: $\prod_{i \in J} C(X_i) \to C(X)$, being a ring homomorphism, forces X to be empty. Since D is an ultrafilter, we lose no generality by assuming that $X_i \neq \varnothing$ for each $i \in I$. For each $J \in D$ let p_J: $\prod_{i \in J} C(X_i) \to \prod_D C(X_i)$ be the natural projection homomorphism. By properties of direct limits there is a unique homomorphism h: $\prod_D C(X_i) \to C(X)$ such that for all $J \in D$, $h \circ p_J = h_J$. Now for each $i \in I$ let d_i: $\mathbf{R} \to C(X_i)$ be the diagonal embedding. Then the ultraproduct mapping $\prod_D d_i$: $\prod_D \mathbf{R} \to \prod_D C(X_i)$ is a homomorphism. Now D is an ultrafilter, hence $\prod_D \mathbf{R}$ is a field by the Łoś Theorem. Therefore $h \circ \prod_D d_i$ is a homomorphism from a field into a nontrivial ring, hence an embedding. Let d: $\mathbf{R} \to \prod_D \mathbf{R}$, e: $\mathbf{R} \to C(X)$ be the diagonal embeddings. It is a straightforward algebraic fact that there can be no other homomorphism e': $\mathbf{R} \to C(X)$, since $C(X)$ is a "diagonal" subring of a power of $\mathbf{R}$ (use the fact that the identity map is the only ring endomorphism on $\mathbf{R}$). Therefore $e = h \circ \prod_D d_i \circ d$, and $\prod_D \mathbf{R}$ embeds as a diagonal subring of $C(X)$. Since D is countably incomplete, this ultrapower is ω_1-saturated. We will obtain a contradiction once we prove the

Lemma. *No diagonal subring of a power of $\mathbf{R}$ is ω_1-saturated.*

Proof of Lemma. Let $A \subseteq \mathbf{R}^I$ be a diagonal subring which is ω_1-saturated. For each $n \in \omega$, let $\phi_n(x)$ be the first order formula which says of x that $x - n$ (= the result when the constantly n function is subtracted from the function x) has a square root. $\phi_n(x)$ can be expressed in the first order language of rings with countably many additional constants. Clearly $\Phi(x) = \{\phi_n(x)\colon n \in \omega\}$ is finitely satisfiable in A: if $\Phi_0(x) = \{\phi_{n_1}(x),\ldots,\phi_{n_k}(x)\}$ then $A \models \phi_{n_i}[\max\{n_1,\ldots,n_k\}]$ for $i = 1,\ldots,k$ since $\mathbf{R} \subseteq A$. So by ω_1-saturicity, there is an $a \in A$ such that $a - n$ has a square root for each $n \in \omega$. That is, for each $i \in I$, the ith coördinate a_i of a exceeds n for all $n \in \omega$, a contradiction. $\square$

1.5 COROLLARY. *If $\mathcal{K}$ is a P-subclass of RCF then ultraproducts in $\mathcal{K}$ are "trivial", in the sense that $\prod_D^{\mathcal{K}} A_i$ is zero whenever D is a countably incomplete ultrafilter.* $\square$

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