PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 3, November 1983

ON PARTITIONS OF PLANE SETS INTO SIMPLE CLOSED CURVES. II

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ABSTRACT. We answer some questions raised in [**1**]. In particular, we prove: (i) Let F be a compact subset of the euclidean plane E^2 such that no component of F separates E^2. Then $E^2 \setminus F$ can be partitioned into simple closed curves iff F is nonempty and connected. (ii) Let $F \subseteq E^2$ be any subset which is not dense in E^2, and let $\mathbb{S}$ be a partition of $E^2 \setminus F$ into simple closed curves. Then $\mathbb{S}$ has the cardinality of the continuum. We also discuss an application of (i) above to the existence of flows in the plane.

Statement of results. This note is a sequel to [**1**], whose notation and terminology we follow faithfully. Throughout the paper, F is a subset of the euclidean plane E^2, and $\mathbb{S}$ is an alleged partition of $E^2 \setminus F$ into simple closed curves (scc's) (i.e. $\mathbb{S}$ is a cover of $E^2 \setminus F$ by pairwise disjoint topological replicas of the unit circle). We are interested in two kinds of question: (i) (existential) what conditions on F ensure or prohibit the existence of a partition $\mathbb{S}$; and (ii) (spectral) what are the relationships between F and the set of cardinalities of possible partitions $\mathbb{S}$?

Existence questions are considered in [**1, 2**]. We summarize what we know: If the cardinality $|F|$ of F is less than the continuum c, and if either the number of isolated points of F or the number of cluster points of F (in E^2) is finite, then $\mathbb{S}$ exists iff $|F| = 1$. We conjecture that the conclusion is still valid under the weaker hypothesis "$|F| < c$"; however, the conclusion fails when the hypothesis is weakened further to "F is totally disconnected", as is witnessed by a nice construction due to R. Fox [**1**, Theorem 12].

In [**1**] we also raise the question of when $\mathbb{S}$ exists for F compact. This brings us to our first result.

1. THEOREM. *Let F be a compact subset of E^2 such that no component of F separates E^2. Then $E^2 \setminus F$ can be partitioned into scc's iff F is nonempty and connected.*

Questions of spectrum are considered in [**1, 2, 4**]; in particular, in [**1**] we ask: for which F is it necessarily the case that $|\mathbb{S}| = c$ (if it exists at all)?

2. THEOREM. *Let F be any subset of E^2 which is not dense in E^2, and let $\mathbb{S}$ be a partition of $E^2 \setminus F$ into scc's. Then $|\mathbb{S}| = c$.*

Received by the editors December 3, 1982 and, in revised form, February 24, 1983.
1980 *Mathematics Subject Classification*. Primary 54B15, 57N05; Secondary 54H20.
Key words and phrases. Topological partitions, euclidean plane, simple closed curves, continuous flows.

The proof of Theorem 1 uses techniques from [**1**]. The proof of Theorem 2 is inspired by H. Cook's proof [**4**] that every partition of E^2 into closed arcs must have cardinality c.

Proof of Theorem 1. Our first observation (due to the referee of [**1**]) is that the components of F, together with singleton points of $E^2 \setminus F$, form an uppersemicontinuous decomposition of E^2. By a theorem of R. L. Moore [**6**, p. 533] the corresponding quotient space is $\simeq E^2$. In view of this it is easy to get the existence of $\mathcal{S}$ whenever F is a nonempty continuum which fails to separate E^2, so it will suffice to prove

3. THEOREM. *Let $F \subseteq E^2$ be a compact totally disconnected subset of cardinality different from* 1. *Then $E^2 \setminus F$ cannot be partitioned into scc's.*

We can eliminate the case $F = \varnothing$ immediately [**1**, Theorem 1], so assume $|F| > 1$; and, for the sake of contradiction, let $\mathcal{S}$ be a partition of $E^2 \setminus F$ into scc's. As in [**1**] we let $B(S)$ be the bounded component of $E^2 \setminus S$ for any scc S and rely heavily on Schönflies's theorem (i.e. $B(S) \simeq E^2$). Also we will use the partial order $<$ on $\mathcal{S}$, given by $S_1 < S_2$ if $\overline{B(S_1)} = B(S_1) \cup S_1 \subseteq B(S_2)$.

4. LEMMA. *If F is totally disconnected and $\mathfrak{M} \subseteq \mathcal{S}$ is a maximal chain, then $\bigcap \{B(S): S \in \mathfrak{M}\}$ is a singleton subset of F.*

PROOF. This is proved in [**1**, Lemma 4]. $\square$

Now for any $S \in \mathcal{S}$, $B(S) \cap F$ is a nonempty clopen subset of F, so for each clopen $G \subseteq F$ let $\mathcal{S}_G = \{S \in \mathcal{S}: B(S) \cap F = G\}$. Then $\mathcal{S} = \bigcup \{\mathcal{S}_G: G \subseteq F \text{ is clopen}\}$ is a (countable) union of pairwise disjoint subcollections, each of which is a chain under the $<$-ordering. Let $U_G = \bigcup \mathcal{S}_G$. Then the collection $\{U_G: G \subseteq F \text{ is clopen}\}$ is a cover of $E^2 \setminus F$ by pairwise disjoint sets ("annuli"). We will show that each U_G is open. By a theorem of Kuratowski-Knaster [**7**], to the effect that X separates E^2 only if a connected subset of X separates E^2, we know that $E^2 \setminus F$ is connected. Hence, $U_G = \varnothing$ for all but one clopen $G \subseteq F$. We will show that, in fact, $\mathcal{S}_F \neq \varnothing$, hence $\mathcal{S} = \mathcal{S}_F$. This will mean that $\mathcal{S}$ is a chain all of whose members enclose F, contradicting Lemma 4.

We will be done, therefore, once we prove the following two assertions.

5. LEMMA. $\mathcal{S}_F \neq \varnothing$.

PROOF. Although we could argue as in the proof of [**1**, Lemma 9], the following approach (suggested by the referee) is more elementary.

View E^2 as $S^2 \setminus \{p\}$ (i.e. the two-sphere minus the point at infinity); and for each scc $S \subseteq E^2$ let $U(S)$ be the complement of $B(S) \cup S$ in S^2. Since the collection of $<$-maximal elements of $\mathcal{S}$ is at most countable and each chain in $\mathcal{S}$ without a $<$-maximal element has countable cofinality, we can find a countable collection $S_1, S_2, \ldots$ in $\mathcal{S}$ which includes the $<$-maximal elements and such that $\overline{\bigcup_{n=1}^{\infty} B(S_n)} = \bigcup \{B(S): S \in \mathcal{S}\}$. For $m = 1, 2, \ldots$, let $C_m = \bigcap_{n=1}^{m} \overline{U(S_n)}$. Then $C_1, C_2, \ldots$ is a decreasing chain of continua containing p. Suppose $C_m \cap F \neq \varnothing$ for each m. Since F is compact, we have that $C = \bigcap_{m=1}^{\infty} C_m$ is a continuum which intersects F and contains p. By a theorem of Sierpiński [**6**, p. 173; **1**, Lemma 8(ii)], to the effect that no locally compact connected Hausdorff space can be partitioned into countably many proper compact subsets, C must contain a point x not in $F \cup \{p\} \cup \bigcup_{n=1}^{\infty} S_n$. But $x \in S$ for some $S \in \mathbb{S}$, and S cannot be $<$-maximal. Thus $x \in B(S_n)$ for some n, a contradiction. Thus for some m, $F \cap C_m = \varnothing$; hence $F \subseteq \bigcup_{n=1}^{m} B(S_n)$.

For each $x \in F$ let $\mathbb{S}_x = \{S \in \mathbb{S}\colon x \in \overline{B(S)}\}$. By [**1**, Theorem 1], $\mathbb{S} = \bigcup_{x \in F} \mathbb{S}_x$, and each $\mathbb{S}_x$ is a chain. Let $G_x = \bigcup \{\overline{B(S)}\colon S \in \mathbb{S}_x\}$. By the above argument, $x \in G_x$, G_x is a closed disk if $\mathbb{S}_x$ has a $<$-maximal element, and G_x is a chain union of open disks if $\mathbb{S}_x$ has no $<$-maximal element. Furthermore, the collection $\{G_x\colon x \in F\}$ is a finite partition of E^2. But the complement of a finite nontrivial union of disjoint closed disks is multiply connected. Hence $G_x = E^2$ for each $x \in F$ and $F \subseteq B(S)$ for some $S \in \mathbb{S}$. □

6. LEMMA. *Each U_G is an open set.*

PROOF. Let $G \subseteq F$ be clopen and assume $\mathbb{S}_G \neq \varnothing$. Since $\mathbb{S}_G$ is a chain it will suffice to show that $\mathbb{S}_G$ has no $<$-minimal or $<$-maximal element. Let $S \in \mathbb{S}_G$. Since $B(S) \simeq E^2$ and G is compact, we can apply Lemma 5 relativized to $B(S)$. Thus, there is a scc $S' \in \mathbb{S}$ with $G \subseteq B(S') \subseteq \overline{B(S')} \subseteq B(S)$. Clearly $S' \in \mathbb{S}_G$, so $\mathbb{S}_G$ has no $<$-minimal element.

To see that $\mathbb{S}_G$ has no $<$-maximal element, we "exchange" the point p at infinity for any element of G. (G is nonempty.) The ordering $<$ is reversed and we apply the above argument to the compact set $(F \backslash G) \cup \{p\}$. This finishes the proof of the lemma, and hence of Theorem 1. □

Proof of Theorem 2. Suppose F is a subset of E^2 which is not dense in E^2, and let $\mathbb{S}$ be a partition of $E^2 \backslash F$ into scc's. Let D be a standard open disk with boundary circle C such that $\overline{D} \cap F = \varnothing$. Then no $S \in \mathbb{S}$ lies in D [**1**, Theorem 1]; so for each $S \in \mathbb{S}$, $S \cap D$ is a countable disjoint union of open arcs with distinct endpoints on C. Since these arcs form a partition of D and each such arc is a subarc of a member of $\mathbb{S}$, it will suffice to show that it takes c arcs to do the job. Let A be one of the arcs and let D_A be a component ($\cong E^2$) of $D \backslash A$. We show that c arcs are necessary to fill D_A by proving the following.

7. LEMMA. *Let* $[0,1]^2$ *denote the closed unit square and let* $\mathcal{Q}$ *be a partition of the open unit square* $(0,1)^2$ *by open arcs* (*i.e. homeomorphs of* $(0,1)$), *each with distinct endpoints on* $[0.1] \times \{0\}$. *Then* $|\mathcal{Q}| = c$.

PROOF. The following argument is similar to that given by H. Cook in [**4**] to show that E^2 cannot be partitioned into $< c$ closed arcs.

By the Baire Category Theorem applied to $(0,1)^2$ (each $A \in \mathcal{Q}$ is closed as well as nowhere dense in $(0,1)^2$), we know that $\mathcal{Q}$ is uncountable; hence there is a real $\delta > 0$ and an uncountable $\mathcal{Q}_0 \subseteq \mathcal{Q}$ such that the endpoints of each $A \in \mathcal{Q}_0$ have a distance apart of at least δ. For $A \in \mathcal{Q}$, let $l(A)$ (resp. $r(A)$) denote the left (resp. right) endpoint of A, and let $B(A)$ denote the region bounded by A and $[l(A), r(A)] \times \{0\}$. We order $\mathcal{Q}$ by writing $A_1 < A_2$ if $B(A_1) \subseteq B(A_2)$ and $A_1 \neq A_2$. Now suppose n is any whole number such that $n\delta \geqslant 1$. Then $\mathcal{Q}_0$ has at most n maximal chains under

$<$. (This follows from the fact that $<$ is a tree ordering; hence, if there were $> n$ maximal chains in $\mathcal{Q}_0$, then there would be $> n$ arcs $A \in \mathcal{Q}_0$ such that the regions $B(A)$ are pairwise disjoint. Since their endpoints have a distance apart of $\geqslant \delta$, this is impossible.) Thus there is an uncountable $\mathcal{Q}_1 \subseteq \mathcal{Q}_0$ which is a chain under the $<$-order. Assume $|\mathcal{Q}_1| < c$, and let $\mathcal{C}$ denote the space of subcontinua of $[0,1]^2$. Under the well-known Hausdorff metric, $\mathcal{C}$ is a compact metric space. Let $\overline{\mathcal{Q}}_1$ denote the closure of $\mathcal{Q}_1$ in $\mathcal{C}$. Then $|\overline{\mathcal{Q}}_1| = c$. (To see this we use the facts that $\mathcal{C}$ is hereditarily Lindelöf, and in such spaces scattered subsets are countable. Since $\overline{\mathcal{Q}}_1$ is uncountable it is not scattered; hence, it has a nonempty closed subset without isolated points. This subset, being also compact metric, contains Cantor sets.)

Since we are assuming $|\mathcal{Q}_1| < c$, we have $|\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1| = c$. Also, since $\mathcal{Q}_1$ is a chain, each element of $\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1$ is a limit either from above or below of distinct arcs in $\mathcal{Q}_1$, say $B = \lim_{n\to\infty} A_n$ where $A_{n+1} \underline{<} A_n$. Hence, B intersects at most one arc in $\mathcal{Q}_1$ and at most one other continuum in $\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1$. Let

$$r = \inf\{r(A) \colon A \in \mathcal{Q}_1\}, \quad l = \sup\{l(A) \colon A \in \mathcal{Q}_1\},$$

and let $\langle a, 0\rangle$ be the midpoint of the segment $[l, r] \times \{0\}$ $(r - l \geqslant \delta)$. Let L be the vertical segment $\{a\} \times [0,1]$. Then each $B \in \overline{\mathcal{Q}}_1$ intersects both $[0, a) \times \{0\}$ and $(a, 1] \times \{0\}$. Let $f \colon (\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1) \to L$ take a continuum B to a point of $B \cap (L \setminus \{\langle a, 0\rangle\})$. Then the image $f[\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1]$ has cardinality c since the fibers of f have at most two elements.

Finally, it is plain that if x_1, x_2, x_3 are three distinct points of $f[\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1]$ then some arc of $\mathcal{Q}_1$ separates two of them in $[0,1]^2$. Thus no member of the original family $\mathcal{Q}$ can contain more than two points of L. Since every point of $f[\overline{\mathcal{Q}}_1 \setminus \mathcal{Q}_1]$ lies on exactly one arc in $\mathcal{Q}$, this says that $|\mathcal{Q}| = c$. $\square$

8. REMARK. Theorem 2 contrasts nicely with the fact **[5, 9]** that, under hypotheses consistent with the usual axioms of set theory, E^2 can be covered by $< c$ (possibly overlapping) scc's.

An application to the theory of flows. In this section we follow the terminology found in Beck **[3]**. A *flow* in E^2 is a continuous surjection $f \colon E^1 \times E^2 \to E^2$ with the "group property" $f(s + t, x) = f(s, f(t, x))$. We define a flow to be *periodic* if for each $x \in E^2$, either x is a fixed point of f (i.e. $f(t, x) = x$ for all $t \in E^1$) or $p_f(x) = \inf\{t > 0 \colon f(t, x) = x\}$ is finite and positive. It is an easy exercise (see **[3]**) to show that x is a fixed point iff $p_f(x) = 0$; and the *orbit* of x, $\{f(t, x) \colon t \in E^1\}$, is a scc iff $0 < p_f(x) < \infty$.

9. THEOREM (BECK **[3**, COROLLARY 6.20**]**). *Let $F \subseteq E^2$ be a compact set whose complement is homeomorphic with the open annulus $\{x \in E^2 \colon 1 < |x| < 2\}$. Then there exists a periodic flow on E^2 whose set of fixed points is F.*

Let $F \subseteq E^2$ be nonempty and compact, such that both F and $E^2 \setminus F$ are connected. Letting $\mathcal{G}_F$ be the uppersemicontinuous decomposition of E^2 into F together with singletons of $E^2 \setminus F$, we have immediately from Moore's theorem (i.e. $E^2/\mathcal{G}_F \simeq E^2$) that $E^2 \setminus F$ is homeomorphic with an open annulus. Putting Theorem 9 and our Theorem 1 together we have the following existence theorem for flows.

10. THEOREM. *Let $F \subseteq E^2$ be compact, no component of which separates E^2. The following are equivalent*:

(i) *F is nonempty and connected.*

(ii) *There exists a periodic flow on E^2 whose fixed point set is F.*

11. REMARK. The (ii) $\Rightarrow$ (i) direction is a very weak corollary of Theorem 1, which in effect offers a "static" (rather than "dynamic") argument for the nonexistence of flows.

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