sections on exact equations contain the standard test for exactness and often include a discussion of integrating factors, but I have seen none that develops a test for separability.

This paper gives a simply stated test for separability. In what follows D, the domain of f, is an open convex set in the plane, f is real valued, and partial derivatives of f are indicated by subscripts.

PROPOSITION 1. *If there are differentiable functions $\phi(x)$ and $\psi(y)$ such that $f(x, y) = \phi(x)\psi(y)$ for all $(x, y) \in D$, then $f(x, y)f_{xy}(x, y) = f_x(x, y)f_y(x, y)$ for all $(x, y) \in D$.*

The proof follows directly from $f_x(x, y) = \phi'(x)\psi(y)$ and the corresponding formulas for the other partial derivatives.

PROPOSITION 2. *If f, f_x, f_y and f_{xy} are continuous in D, $f(x, y)$ is never 0 in D, and $f(x, y)f_{xy}(x, y) = f_x(x, y)f_y(x, y)$ for all $(x, y) \in D$, then there are continuously differentiable functions $\phi(x)$ and $\psi(y)$ such that $f(x, y) = \phi(x)\psi(y)$ for all $(x, y) \in D$.*

Proof. f has the same sign throughout D and factoring f is equivalent to factoring $-f$, so we may assume that f is positive on D. Now

$$\frac{\partial}{\partial y}\frac{f_x(x, y)}{f(x, y)} = \frac{f(x, y)f_{xy}(x, y) - f_x(x, y)f_y(x, y)}{f^2(x, y)} = 0.$$

Since $\frac{\partial}{\partial x}\ln(f(x, y)) = f_x(x, y)/f(x, y)$, we conclude that $\frac{\partial}{\partial x}\ln(f(x, y))$ is a function of x alone and can write $\frac{\partial}{\partial x}\ln(f(x, y)) = \alpha(x)$. The function α is continuous because f and f_x are.

Let $\beta(x) = \int\alpha(x)\,dx$. Then there is a function γ such that $\ln(f(x, y)) = \beta(x) + \gamma(y)$. γ is continuously differentiable because f and f_y are continuous.

Let $\phi(x) = \exp(\beta(x))$ and $\psi(y) = \exp(\gamma(y))$. Then $f(x, y) = \phi(x)\psi(y)$, completing the proof.

As the following example shows, the condition that f is nonzero on D cannot be dropped. Define $f(x, y)$ by

$$f(x, y) = \begin{cases} x^2\exp(y), & \text{if } x \geqslant 0, \\ x^2\exp(2y), & \text{if } x \leqslant 0. \end{cases}$$

Direct computation shows that on R^2 we have $ff_{xy} = f_xf_y$. However f does not factor since if it did $f(1, y)$, which is $\exp(y)$, and $f(-1, y)$, which is $\exp(2y)$, would be proportional as functions of y.

One can replace the condition that f is nonzero on D by requiring that $f(x, y)$ be an analytic function of x and y. Then, using Proposition 2 and the fact that if two analytic functions agree in a neighborhood of a point in D they agree on the connected set D, one can show that $ff_{xy} = f_xf_y$ implies that f factors.

TOPOLOGICAL PARTITIONS OF EUCLIDEAN SPACE BY SPHERES

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A space X *partitions* a space Y if there is a family of topological embeddings of X into Y whose images form a cover of Y by pairwise disjoint sets. In [**1**] (whose notation and terminology we follow here) it is shown that the n-sphere $S(n)$ cannot partition euclidean $(n + 1)$-space $\mathbb{R}^{n+1}$ (an easy result) and that if X is any nonempty subspace of $S(n)$, then X partitions $\mathbb{R}^{2n+1}$. An obvious question then arises as to whether $S(n)$ partitions $\mathbb{R}^m$ for $n + 1 < m < 2n + 1$ (Question 3.1 (iv) of [**1**]). In this note we give an affirmative answer based on an unpublished construction due to S. Kakutani and communicated to us by P. R. Halmos [**4**] (who learned of the construction forty years ago). This construction, used to give a partition of $\mathbb{R}^3$ by $S(1)$, is extremely simple, essentially different from those given in [**1**], and easily generalizable. Because several people have looked at the problem since the completion of [**1**], we thought the solution should be publicized as a short postscript to that paper.

THEOREM. *$S(n)$ partitions $\mathbb{R}^{n+2}$ in such a way that each sphere is tamely embedded, unknotted, and not linked with any other member of the partition.*

Proof. We partition $\mathbb{R}^{n+2}$ into n-spheres as follows: First let C_1 be an open unbounded cylinder in $\mathbb{R}^{n+2}$. To be specific, we let X_1 be a coordinate axis in $\mathbb{R}^{n+1}$ and let C_1 be the product of X_1 with the open unit ball in the $(n+1)$-dimensional vector subspace perpendicular to X_1. Now $\mathbb{R}^{n+2} \setminus C_1$, being homeomorphic with $\mathbb{R}^1 \times [1, \infty) \times S(n)$, is easily partitioned into spheres; and C_1 is homeomorphic with $\mathbb{R}^{n+2}$.

Let C_2 be an open unbounded cylinder in C_1 formed in a U-shape—i.e., let R_1, R_2 be parallel translates of a closed ray in X_1, let L be the straight line segment joining the endpoints of R_1, R_2, and let C_2 be the interior of a regular thickening of $R_1 \cup L \cup R_2$. We then choose a branch of C_2 and form another open unbounded U-shape C_3 in exactly the same fashion, making sure that the base of C_3 is at least a unit distance from the base of C_2. Proceed inductively. For each $n \geqslant 1$, C_n is homeomorphic with $\mathbb{R}^{n+2}$, and $C_n \setminus C_{n+1}$ is homeomorphic with $\mathbb{R}^1 \times [1, \infty) \times S(n)$. Finally, because the U-shapes are pushed unboundedly far from the origin, it is easy to see that $\bigcap_{n=1}^{\infty} C_n = \varnothing$. This completes the construction. (It is obvious that there is no linking or knotting, and each sphere is tamely embedded.) □

The following question now becomes inevitable.

Revised Question 3.1 (iv) [**1**]: Is there a nonempty subspace of $S(2)$ which does not partition $\mathbb{R}^4$? Of course by the above we now need only look in $\mathbb{R}^2 \subseteq S(2)$ for candidates, but this brings us to the question of whether in general there are nonempty subsets of $\mathbb{R}^n$ which do not partition $\mathbb{R}^{n+1}$ This is basically Question 3.1 (ii) in [**1**], and we are little closer to a solution now than we were when the issue first arose.

REMARK. In [**5**], A. Szulkin offers a new solution to the problem of whether $S(1)$ partitions $\mathbb{R}^3$, in response to a re-posing of the question by H. S. Shapiro. (Problems like this have a way of surfacing time and again. See also [**2**] and [**3**] for similar problems.) So far, we know of four distinct solutions, each with its own particular merits. To comment:

(i) Of the two solutions offered in [**1**], one of them uses the Axiom of Choice. While the constructive solution is more visual, the Jordan curves in the partition are quite general. However, in the nonconstructive solution, all of the Jordan curves can be taken to be standard circles (of equal radius). Neither solution generalizes to higher dimensions.

(ii) Szulkin's solution combines the attractive features of both of the solutions in [**1**]; namely it constructively partitions $\mathbb{R}^3$ into circles (whose radii include all positive real numbers). However, it too fails to generalize to higher dimensions.

(iii) While the Kakutani solution (as well as the above generalization) is constructive, it involves highly noncircular Jordan curves. This, of course, prompts the question of whether $\mathbb{R}^{n+2}$ can always be partitioned into standard n-spheres.

References

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