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# Obstacles to duality between classes of relational structures

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*Abstract.* We prove an algebraic result concerning inverse limits of copowers in an $S$-class of relational structures of the same (finitary or infinitary) type. Among several applications to the nonexistence of category dualities is the following Theorem: If there exist arbitrarily large measurable cardinals then any class $\mathscr{K}$ of relational structures containing a nontrivial object $A$ and all of its cartesian powers via nonempty index sets will fail to be dual to any $S$-class. (No large cardinal assumption is necessary if either there is a finite such $A$ or if $\mathscr{K}$ consists only of finitary relational structures and contains the elementary class generated by $A$ and its cartesian powers.)

## 0. Introduction

We are interested in isolating obstacles to category duality in the sense of [8]; in particular we wish to investigate conditions under which no dualities can occur between reasonable categories of relational structures.

Let $\tau$ be a (similarity) type of relation and function symbols of various arities, finite or infinite (in the sense of model theory/universal algebra [5, 7]). If $\alpha$ is an infinite cardinal exceeding all the arities in $\tau$ we say $\tau$ is $<\alpha$-*ary* ("$<\omega$-ary" = "finitary"). We consider the class $\mathcal{M}_\tau$ of all relational structures (empty or nonempty) in type $\tau$ to be a category by declaring morphisms to be the homomorphisms (i.e. those maps which preserve atomic relations). If $\mathscr{K} \subseteq \mathcal{M}_\tau$ is closed under isomorphic images we call $\mathscr{K}$ a *class* and consider it, unless otherwise specified, as a full subcategory of $\mathcal{M}_\tau$. An object $\mathscr{A} \in \mathscr{K}$ is *nontrivial* if $|A| \geq 2$ (i.e. $A$ has at least two elements); $\mathscr{K}$ is *nontrivial* if $\mathscr{K}$ has a nontrivial object. Finally $\mathscr{K} \subseteq \mathcal{M}_\tau$ is an *S-class* if it is closed under substructures in $\mathcal{M}_\tau$; a *P-class* if it is closed under usual (= cartesian) products; and an *elementary class* if $\tau$ is finitary and $\mathscr{K}$ is the class of models of a set of sentences in the first order language appropriate to $\tau$.

We first note that reasonable classes of relational structures can be dual. For example the category of complete atomic Boolean algebras and complete homomorphisms is dual to the class $\mathscr{S}$ of sets ($= \mathcal{M}_\tau$ where $\tau$ is the empty type, see e.g. [8]); and the classical self-duality theorems for the $S$-classes of finite

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abelian groups and finite-dimensional vector spaces over a field provide further examples.

One obvious question is whether two nontrivial $SP$-classes can ever be dual. The answer, implicit in [10], is negative; and is an easy corollary to the result (section (2.3) of [10]) which says that two "algebraic categories with rank" can be category dual only if they are both equivalent to either the trivial monoid or the two-element chain. An $SP$-class is clearly an "algebraic category with rank" in the sense of [10].

Bizarre set-theoretic axioms also enter the picture as follows: We noted above that the class $\mathcal{S}$ of sets is dual to the category of complete Boolean algebras and complete homomorphisms. Another obvious question is whether $\mathcal{S}$ can be dual to any class of relational structures. A large cardinal hypothesis (see [5] for terminology) gives a partial answer, namely the result [11 (Thm. 4.1, p. 348)] that if there are arbitrarily large measurable cardinals then $\mathcal{S}$ cannot be dual to any class of algebras.

In the next section we will prove an algebraic proposition concerning inverse limits of copowers (= "reduced copowers") in an $S$-class. In the last section we will give some applications of this result to the nonexistence of category dualities. I would like to acknowledge here my indebtedness both to Bernhard Banaschewski and to Evelyn Nelson for valuable suggestions regarding the form and content of this paper.

## 1. Main result

We will need some terminology about reduced copowers in a category (see also [1, 2, 3, 4, 5, 6, 9]). Let $\mathcal{A}$ be a category, let $I$ be a nonempty index set, and let $D$ be a filter of subsets of $I$. We denote by $A \cdot I$ the $I$-copower in $\mathcal{A}$ and by $\sigma_i : A \to A \cdot I$ the $i^{\text{th}}$ "injection" morphism. We use $\nabla_I : A \cdot I \to A$ to denote the natural "codiagonal" morphism (a left inverse for each $\sigma_i$) and let $\sigma_{JK} : A \cdot K \to A \cdot J$ denote the natural injection for $K \subseteq J \subseteq I$. We use the "connecting" morphisms $\sigma_{JK}$ $(J \supseteq K \in D)$ to define the *reduced copower* $A(D) = \varprojlim \{A \cdot J : J \in D\}$. The "$D$-codiagonal" morphism is denoted $\nabla_{(D)} : A(D) \to A$ and is equal to $\nabla_J \circ \sigma_J$ for each $J \in D$, where $\sigma_J : A(D) \to A \cdot J$ is the $J^{\text{th}}$ limit morphism.

Let $\alpha$ be an infinite cardinal. A filter $D$ on $I$ is *$\alpha$-uniform* if $J \in D$ for each $J \subseteq I$ with $|I \backslash J| < \alpha$. (N.B. "$\alpha$-uniform" gets stronger with increasing $\alpha$, and is simply called "uniform" when $\alpha = |I|$; $D$ is $\omega$-uniform iff $\bigcap D = \varnothing$, i.e. $D$ is a *free* filter.)

Our main result is a technical proposition concerning $D$-codiagonal morphisms in categories which are $S$-classes of relational structures.

1.1. THEOREM. *Let $\alpha$ be a regular cardinal, let $\mathcal{K}$ be a $<\alpha$-ary $S$-class, and let $A \in \mathcal{K}$ be such that for any nonempty $I$ the copower $A \cdot I$ exists in $\mathcal{K}$ and not all injections $\sigma_i$ are equal. If $D$ is an $\alpha$-uniform filter on $I$ then $\nabla_{(D)}$ is not an epimorphism.*

*Proof.* We will first need the following.

LEMMA. *Let $\mathcal{A}$ be a category and suppose all copowers of $A$ exist in $\mathcal{A}$. If $f: B \to A \cdot I$ factors through $\sigma_{IJ}$ and $\sigma_{IK}$ for some pair $J, K$ of complementary subsets of $I$ then for each $i \in I, f = \sigma_i \circ \nabla_I \circ f$.*

*Proof of Lemma.* Suppose $J, K \subseteq I$ are nonempty and complementary, and suppose $f = \sigma_{IJ} \circ g = \sigma_{IK} \circ h$ are the hypothesized factorizations. Fixing $i \in I$, there is a morphism $u: A \cdot I \to A \cdot I$ such that $u \circ \sigma_{IJ} = \sigma_{IJ}$ and $u \circ \sigma_{IK} = \sigma_i \circ \nabla_K$ (since $A \cdot I$ is a coproduct of $A \cdot J$ and $A \cdot K$). Then $u \circ f = u \circ \sigma_{IJ} \circ g = \sigma_{IJ} \circ g = f$. Also $u \circ f = u \circ \sigma_{IK} \circ h = \sigma_i \circ \nabla_K \circ h$, so $f = \sigma_i \circ \nabla_K \circ h$. Hence $\nabla_I \circ f = \nabla_I \circ \sigma_i \circ \nabla_K \circ h = \nabla_K \circ h$, so $f = \sigma_i \circ \nabla_I \circ f$.

To prove (1.1), let $\alpha, \mathcal{K}, A, I$ and $D$ be as in the hypothesis. Assuming $A(D)$ exists, let $x \in A(D)$. Then, since $\alpha$ is a regular cardinal and $\mathcal{K}$ is $<\alpha$-ary, there is a set $J \subseteq I$ with $|J| < \alpha$ and $\sigma_I(x) \in \sigma_{IJ}[A \cdot J]$. (Indeed, for any structure $B$ in the $S$-class $\mathcal{K}$, $\bigcup \{\sigma_{IJ}[B \cdot J]: J \subseteq I$ of cardinality $<\alpha\}$ is an $\alpha$-directed union of substructures of $B \cdot I$ and is therefore a substructure of $B \cdot I$. This union clearly behaves like the copower and must therefore be the copower.)

Since $D$ is $\alpha$-uniform, $K = I \backslash J \in D$; hence $\sigma_I(x) \in \sigma_{IK}[A \cdot K]$. Now apply the Lemma as follows: Let $B =$ the substructure of $A(D)$ generated by $x$, and let $f = \sigma_I \restriction B$. Since $\sigma_I(x) \in \sigma_{IJ}[A \cdot J] \cap \sigma_{IK}[A \cdot K]$, we have that $f$ factors through $\sigma_{IJ}$ and $\sigma_{IK}$. To see this note that $\sigma_{IJ}$ is an isomorphism onto $\sigma_{IJ}[A \cdot J]$. (Indeed, letting $r: I \to J$ be a set retraction, we let $\rho: A \cdot I \to A \cdot J$ be the copower morphism arising from the family $\{\phi_i : i \in I\}$ from $A$ to $A \cdot J$, where $\phi_i = \sigma_{r(i)}$. Then $\rho$ is a left inverse for $\sigma_{IJ}$.) We thus let $g = \sigma_{IJ}^{-1} \circ f$ and $h = \sigma_K \restriction B$. Now let $i, j \in I$ be two indices such that $\sigma_i, \sigma_j : A \to A \cdot I$ are distinct. Then $\sigma_i \circ \nabla_I \circ (\sigma_I \restriction B) = \sigma_j \circ \nabla_I \circ (\sigma_I \restriction B)$. But the $B$"s cover $A(D)$, so $\sigma_i \circ \nabla_{(D)} = \sigma_i \circ \nabla_I \circ \sigma_I = \sigma_j \circ \nabla_I \circ \sigma_I = \sigma_j \circ \nabla_{(D)}$. Hence $\nabla_{(D)}$ is not an epimorphism. (N.B. In the event $A(D)$ is empty, the above argument works even without the Lemma.) □

## 2. Applications

Under certain hypotheses concerning reduced powers in a category $\mathcal{A}$, we can state that $\mathcal{A}$ cannot be dual to an $S$-class.

2.1 COROLLARY. *Let $\mathcal{K}$ be a class of relational structures containing a* *nontrivial object $A$ and all of its cartesian powers via nonempty index sets. Then $\mathcal{K}$ is not dual to any $S$-class, provided one of the following holds:*

(i) *There exist arbitrarily large measurable cardinals;*

(ii) *$A$ is finite; or*

(iii) *$\mathcal{K}$ is finitary and contains the elementary class generated by $A$ and its cartesian powers.*

*Proof.* Let $A^I$ denote the cartesian power. We note that if the powers of $A$ exist in $\mathcal{K}$ and if the usual model theoretic reduced power $A^{(D)}$ is also in $\mathcal{K}$ then $A^{(D)} \cong \varinjlim \{A^J : J \in D\}$ where the "connecting" morphisms are the restrictions $A^J \to A^K, J \supseteq K \in D$. Thus the usual reduced power can be dual to the reduced copower defined above. Now the "$D$-diagonal" homomorphism $\Delta_{(D)}: A \to A^{(D)}$ is always a monomorphism, and this is our main obstacle to duality.

So let $\mathcal{K}$ and $A$ be as in the hypothesis, and assume (i) holds. Given a regular cardinal $\alpha$ we show $\mathcal{K}$ cannot be dual to a $<\alpha$-ary $S$-class by finding an $\alpha$-uniform filter $D$ such that the reduced power $A^{(D)}$ exists in $\mathcal{K}$ and the $D$-diagonal is a monomorphism. We can then apply (1.1) since $A$ is nontrivial and the projections $A^I \to A$ are all distinct.

Let $\mu$ be a measurable cardinal such that $\alpha + |A| < \mu$, and let $D$ be a free $\mu$-complete ultrafilter on a set of cardinality $\mu$. Then $D$ is clearly $\mu$-uniform, hence $\alpha$-uniform; and the $D$-diagonal is an isomorphism, so the result is immediate.

If (ii) holds and $D$ is an ultrafilter then $\Delta_{(D)}: A \to A^{(D)}$ is an isomorphism. So let $D$ be any $\alpha$-uniform ultrafilter. Again the result is immediate.

If (iii) holds then all usual ultrapowers of $A$ lie in $\mathcal{K}$ (and are the appropriate limits). Then $\Delta_D$ is always a monomorphism. □

(1.1) can also be applied to duality questions involving classes $\mathcal{K}$ such that every nontrivial $A \in \mathcal{K}$ has a cartesian power lying outside of $\mathcal{K}$.

2.2. COROLLARY. *The class of torsion abelian groups is not dual to any $S$-class.*

*Proof.* The class $\mathcal{K}$ of torsion abelian groups has all reduced powers as a category (see [3, 6]); namely form the usual reduced power, itself an abelian group, and take the torsion part $T(\cdot)$. Thus if $A \in \mathcal{K}$ is nontrivial and $I$ is any nonempty index set then the projection maps from $T(A^I)$ to $A$ are all distinct. Also it is easy to check that the $D$-diagonal $\Delta_{(D)}: A \to T(A^{(D)})$ is always a monomorphism. □

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