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R. M. Shortt
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On the Classification of Minimally Free Rings of Continuous Functions

Paul Bankston
Marquette University
and
Robert A. McCoy
Virginia Polytechnic Institute and State University

Abstract

A universal algebra A is *minimally free* if there is a subset X of A such that every function from X into A extends uniquely to an endomorphism on A. If κ is a cardinal number, A is *κ-free* if the set X above can be chosen to have cardinality κ. (0-free is the same as (endomorphism-) rigid.) For a topological space $\mathcal{X}$, we let $C(\mathcal{X})$ be the unital ring of continuous real-valued functions on $\mathcal{X}$. We are interested in the problem of classifying the κ-free rings of the form $C(\mathcal{X})$. In particular we prove that $\mathbf{R} = C(\{\text{point}\})$ is the only 0-free such ring; and for $1 \leq \kappa \leq \omega, C(\mathbf{R}^\kappa)$ and $\mathbf{R}^c = C(\{\text{discrete space of cardinality=continuum}\})$ are the only κ-free such rings.

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0. Introduction

This article partially answers a question posed in [**1**], whether one can classify (effectively list) all κ-free commutative unital rings $C(\mathcal{X})$ of continuous real-valued functions (Question 3.14 of [**1**]). A fair amount of progress was made on the question by the first author at the time [**3**] was written, but still a satisfactory classifaction theorem was lacking. Now, at least, we can classify the κ-free rings $C(\mathcal{X})$ for countable κ : $\mathbf{R} = C(\{\text{point}\})$ is the only 0-free such ring; and for $1 \leq \kappa \leq \omega, C(\mathbf{R}^\kappa)$ and $\mathbf{R}^c = C(\{\text{discrete space of cardinality=continuum}\})$ are the only κ-free such rings.

We follow the notation and terminology of [**1**]. Let A be a universal algebra of arbitrary type. A is *minimally free* if there is a subset X of A, called a *pseudobasis*, such that every mapping of X into A extends uniquely to an endomorphism on A. If κ is a cardinal number, A is *κ-free* if a pseudobasis of cardinality κ can be found. (Our terminology "κ-free" inadvertently clashes with that of Eklof [**6**], et al. Unfortunately, we were unaware of their usage when we wrote [**1**], and offer our apologies.) In [**1**], [**2**] and the Kříž-Pultr paper [**10**], there are theorems that describe how badly a pseudobasis can fail to generate the algebra. Indeed, there are varieties of algebras (e.g., the commutative unital rings) that have arbitrarily large κ-free algebras for any fixed κ (see [**2**], [**10**] for details).

Any pseudobasis is "Marczewski independent" (see Głazek [**8**]): every mapping from X into A extends (uniquely) to a homomorphism from the subalgebra generated by X. Furthermore, if the pseudobasis actually generates A, then X is a free basis for A relative to the smallest variety containing A. (Hall [**9**] introduces the notion "relatively free group" in this way. See also Neumann's book [**11**].)

Here we concentrate on minimally free rings. In our usage, "ring" always means "commutative unital ring", and homomorphisms preserve the unity element. We let RCF be the class of rings $C(\mathcal{X})$ of continuous real-valued functions where $\mathcal{X}$ is a topological space. It is well known [**7**] that $RCF = \{C(\mathcal{X}) : \mathcal{X}$ is realcompact and Tichonov $\}$. Moreover, by the duality theorem of Gel'fand-Kolmogorov, $C(\cdot)$ is a contravariant equivalence of categories; so two realcompact Tichonov spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic if and only if the rings $C(\mathcal{X})$ and $C(\mathcal{Y})$ are isomorphic.

As mentioned above, there are arbitrarily large κ-free rings for every κ. However, RCF contains very few of them [**1**], and no κ-free ring in RCF is generated by any of its pseudobases (as we prove in 0.3 below). In brief, let $\mathbf{R}$ be the set of real numbers; and for each cardinal κ, let $\mathcal{U}^\kappa$ be the κ-fold Tichonov product of the usual topology $\mathcal{U}$ on $\mathbf{R}$. We are interested in enrichments of this topology on $\mathbf{R}^\kappa$. If $\mathcal{X} = \langle X, \mathcal{T}\rangle$ and $\mathcal{X}' = \langle X', \mathcal{T}'\rangle$ are any topological spaces, with $f : \mathcal{X} \to \mathcal{X}'$ continuous, define f to be a *coreflection map* if for each continuous $g : \mathcal{X} \to \mathcal{X}'$ there is a unique continuous $h : \mathcal{X} \to \mathcal{X}$ such that $g = f \circ h$. Coreflection maps are discussed more fully in [**1**]; easy to verify is the following.

0.1 PROPOSITION. *(Proposition 3.3 of [**1**]) Coreflection maps with nonempty domains are continuous bijections.*

Clearly every map with empty domain is coreflective; however if $f : \langle X, \mathcal{T}\rangle \to \langle X', \mathcal{T}'\rangle$ is a coreflection map with X nonempty, we may view X and X' as the same set with $\mathcal{T}$ an enrichment of $\mathcal{T}'(\mathcal{T} \supseteq \mathcal{T}')$ satisfying the condition that any $\langle \mathcal{T}, \mathcal{T}'\rangle$-continuous map (i.e., a map that pulls $\mathcal{T}'$-open sets back to $\mathcal{T}$-open sets) is also a $\langle \mathcal{T}, \mathcal{T}\rangle$-continuous map. Under these circumstances we call $\mathcal{T}$ a *coreflective enrichment* of $\mathcal{T}'$. Examples of this phenomenon arise most naturally in connection with topological coreflective functors (e.g., discretization, k-modification, G_δ-modification, to name a few).

The only $A \in RCF$ that is simultaneously 0-free (i.e., (endomorphism-) rigid) and κ-free for some $\kappa > 0$ is clearly the degenerate ring $C(\emptyset)$ (in which case $\kappa = 1$). This said, we henceforth consider only the nondegenerate case.

0.2 THEOREM. *(Theorem 3.10 of [**1**]) $A \in RCF$ is κ-free if and only if there is a coreflective enrichment $\mathcal{T}$ of $\mathcal{U}^\kappa$ such that $\langle \mathbf{R}^\kappa, \mathcal{T}\rangle$ is a realcompact Tichonov space and $A \cong C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$. A pseudobasis for $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$ is the set Π of projection maps from $\mathbf{R}^\kappa$ to $\mathbf{R}$.*

For any set X, let $\mathcal{D}(X)$ be the discrete space with point set X.

0.3 COROLLARY.

(i) *The only 0-free ring in RCF is $\mathbf{R} = C(\{point\})$.*
(ii) *If $\kappa > 0$, the cardinality of any κ-free ring in RCF lies between $\mathfrak{c} \cdot \kappa^\omega$ and $\exp^2(\omega \cdot \kappa)$.*
(iii) *If $\kappa > 0, C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa\rangle)$ is κ-free; and if in addition $\kappa <$ the first measurable cardinal, then $C(\mathcal{D}(\mathbf{R}^\kappa))$ is κ-free as well.*
(iv) *No minimally free ring in RCF is generated by any pseudobasis. (Thus no member of RCF is relatively free.)*

PROOF: Clauses (i)-(iii) are treated in [**1**]. As for clause (iv), assume $A \in RCF$ is minimally free (and nondegenerate), and let $X \subseteq A$ be a pseudobasis of cardinality κ. Let $\mathcal{T}$ be a topology on $\mathbf{R}^\kappa$ prescribed by 0.2, with $f : A \to C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$ an isomorphism. The set Π of projection maps is a pseudobasis for $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$, of cardinality κ; and, given a bijection g between $f[X]$ and Π, there is a unique automorphism on $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$ extending g. Thus there is an isomorphism from A to $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$ taking X to Π. Clearly Π does not generate $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$; consequently X does not generate A. ∎

The classification theorem, which we prove in the next section, is the following.

0.4 THEOREM. *Let $1 \leq \kappa \leq \omega$. Then the only κ-free rings in RCF are $C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa\rangle)$ and $C(\mathcal{D}(\mathbf{R}))$.*

1. Proof of the Classification Theorem

In view of 0.2, all we need show is that, for $1 \leq \kappa \leq \omega$, the κ-fold Tichonov power $\mathcal{U}^\kappa$ of the usual topology $\mathcal{U}$ on $\mathbf{R}$ has no proper nondiscrete coreflective enrichments that are realcompact and Tichonov. As it turns out, we can do much better than this.

The notion "coreflective enrichment" has two successive generalizations, "C-enrichment" and "H-enrichment", which are defined as follows: Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X with $\mathcal{T} \supseteq \mathcal{T}'$. $\mathcal{T}$ is a *C-enrichment* (resp. *H-enrichment*) of $\mathcal{T}'$ if whenever $f : X \to X$ is a $\langle \mathcal{T}', \mathcal{T}'\rangle$-continuous map (resp. a $\langle \mathcal{T}', \mathcal{T}'\rangle$-homeomorphism), f is also a $\langle \mathcal{T}, \mathcal{T}\rangle$-continuous map (resp. a $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphism). The study of H-enrichments is interesting in itself, as part of the theory of bitopological spaces, and is taken up in another paper [4]. What we need in order to finish the proof of 0.4 is the following topological result.

1.1 THEOREM. *Let X be a normed linear space over the real field. Then any proper C-enrichment of the norm-topology is discrete.*

PROOF: Let $X = \langle X, +, 0, \|\cdot\| \rangle$ be a normed linear space, and denote the norm-topology by $\mathcal{E}$. Let $\mathcal{T}$ be any proper C-enrichment of $\mathcal{E}$. Because $\langle X, \mathcal{E} \rangle$ is point-homogeneous and $\mathcal{T}$ is an H-enrichment of $\mathcal{E}$, $\langle X, \mathcal{T} \rangle$ is also point-homogeneous. Thus it suffices to prove that 0 is a $\mathcal{T}$-isolated point. Again using homogeneity and the fact that $\mathcal{T} \neq \mathcal{E}$, there exists a sequence $\langle x_n \rangle = \langle x_0, x_1, \cdots \rangle$ of distinct points of X such that $\langle x_n \rangle$ $\mathcal{E}$-converges to 0 and the set $S = \{x_n : n < \omega\}$ is $\mathcal{T}$-closed in X. Let $U_0 = X$; and, for $n > 0$, let $U_n = \{x : \|x\| < 1/n\}$ be the open $\frac{1}{n}$-ball about 0. For $n = 1, 2, \cdots$ let A_n be the "annulus" $U_{n-1} \setminus U_n$. Then $X = (\bigcup_{n=1}^{\infty} A_n) \cup \{0\}$. For each $x \neq 0$, let R_x be the ray $\{tx : t \geq 0\}$. For $n = 1, 2, \cdots$, define $\ell_n : A_n \to (0,1), m_n : A_n \to (0,1]$, and, for $n \geq 2, M_n : A_n \to (1,2]$ to be $\ell_n(x) = 1/((n+1)\|x\|), m_n(x) = 1/(n\|x\|)$, and $M_n(x) = 1/((n-1)\|x\|)$. Then $R_x \bigcap A_n = \{tx : m_n(x) \leq t < M_n(x)\}, n \geq 2$, and for all $n, R_x \cap A_{n+1} = \{tx : \ell_n(x) \leq t < m_n(x)\}$. The functions ℓ_n, m_n and M_n are clearly continuous where A_n has the inherited norm-topology.

Define $f : X \to X$ and $g : X \to X$ as follows:

$$f(x) = \begin{cases} \frac{1 - m_{2n}(x)}{M_{2n}(x) - m_{2n}(x)} x_n + \frac{M_{2n}(x) - 1}{M_{2n}(x) - m_{2n}(x)} x_{n+1}, & \text{if } x \in A_{2n} \\ x_n, & \text{if } x \in A_{2n-1} \\ 0, & \text{if } x = 0 \end{cases}$$

$$g(x) = \begin{cases} \left(\frac{1 - m_n(x)}{M_n(x) - m_n(x)} m_n(x) + \frac{M_n(x) - 1}{M_n(x) - m_n(x)} \ell_n(x) \right) x, & \text{if } x \in A_n, n \geq 2 \\ \ell_1(x) x, & \text{if } x \in A_1 \\ 0, & \text{if } x = 0 \end{cases}$$

Now f and g are $\mathcal{E}$-continuous, hence $\mathcal{T}$-continuous. Since $f^{-1}[S] = \bigcup_{n=1}^{\infty} A_{2n-1}$, we know $\bigcup_{n=1}^{\infty} A_{2n-1}$ is $\mathcal{T}$-closed. But $g^{-1}[\bigcup_{n=1}^{\infty} A_{2n-1}] = \bigcup_{n=1}^{\infty} g^{-1}[A_{2n-1}] = \bigcup_{n=1}^{\infty} A_{2n}$, so $\bigcup_{n=1}^{\infty} A_{2n}$ is also $\mathcal{T}$-closed. Therefore $\bigcup_{n=1}^{\infty} A_n$ is $\mathcal{T}$-closed, hence $\{0\}$ is $\mathcal{T}$-open. ∎

1.2 COROLLARY. *Let $1 \leq \kappa \leq \omega$. Then the only proper C-enrichment of $\mathcal{U}^{\kappa}$ is discrete.*

PROOF: For finite κ, we can apply 1.1 directly: the usual euclidean norm gives rise to the euclidean topology. In the infinite case, apply 1.1 to the Banach space ℓ^2 of square-summable real sequences, with the obvious norm.

Then use the celebrated result of Anderson [**12**] that ℓ^2 and $\langle \mathbf{R}^\omega, \mathcal{U}^\omega \rangle$ are homeomorphic. (The topological vector space $\langle \mathbf{R}^\omega, \mathcal{U}^\omega \rangle$ is well known not to be normable.) ∎

1.3 REMARK. *1.1 actually holds for any metrizable locally convex topological vector space. The proof is similar to the above, but less concrete (see [4]). Thus Anderson's theorem can be avoided in 1.2.*

2. Rings in RCF that are κ-free for uncountable κ.

When we pass from countable κ to uncountable κ, the problem of classifying the κ-free rings in RCF seems to become impossibly difficult. By 0.3(iii) there always exists a κ-free ring in RCF, namely $C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa \rangle)$. Moreover, this ring has the attractive property of being ***uniquely*** κ-free; i.e., it is not λ-free for any $\lambda \neq \kappa$ (Theorem 3.18 of [**1**]). (Note that $C(\mathcal{D}(\mathbf{R}))$ is κ-free for all $1 \leq \kappa \leq \omega$.) Recall that κ is *Ulam-measurable* if there exists a countably complete nonprincipal ultrafilter on a set of cardinality κ. Let μ be the smallest Ulam-measurable cardinal (should one exist). Then κ is Ulam-measurable if and only if $\kappa \geq \mu$. When $\kappa < \mu$ we also know from 0.3 (iii) that $C(\mathcal{D}(\mathbf{R}^\kappa))$ is κ-free. (Whether it is uniquely κ-free depends on obvious cardinality issues.) However, for $\kappa \geq \mu, C(\mathcal{D}(\mathbf{R}^\kappa))$ is not minimally free (Theorem 3.12 in [**1**]).

By work of Comfort-Retta [**5**] and Williams [**13**], one can always find at least three κ-free rings in RCF for uncountable κ. In order to state the result of theirs that we need, we adopt the following notation. For any topology $\mathcal{T}$ on a set X, and any $\lambda \geq \omega_1$, let $(\mathcal{T})_\lambda$ be the topology basically generated by intersections of fewer than λ open sets from $\mathcal{T}$. Clearly $(\mathcal{T})_\lambda$ is a coreflective enrichment of $\mathcal{T}$.

2.1 THEOREM. *(i) (Comfort-Retta) Let $\mathcal{T}$ be a realcompact Tichonov topology on a set X, with $\mathcal{T}'$ any Tichonov topology satisfying $\mathcal{T} \subseteq \mathcal{T}' \subseteq (\mathcal{T})_{\omega_1}$. Then $\mathcal{T}'$ is also realcompact.*

(ii) (Comfort-Retta) Let $\mathcal{T}$ be a realcompact Tichonov topology. Then $(\mathcal{T})_\mu$ is also realcompact Tichonov.

(iii) (Williams) Let $\mathcal{T}$ be a realcompact Tichonov topology, $\alpha < \mu$ a cardinal, and $\mathcal{T}'$ any Tichonov topology satisfying $\mathcal{T} \subseteq \mathcal{T}' \subseteq (\mathcal{T})_{\alpha^+}$ and $\mathcal{T}' = (\mathcal{T}')_{cf(\alpha)}$ (where α^+ is the cardinal successor of α and $cf(\alpha)$ is the cofinality of α). Then $\mathcal{T}'$ is also realcompact.

With the aid of 2.1, we immediately have the following.

2.2 THEOREM. *(Theorem 3.15 of [**1**]) Let κ be a cardinal, and α any cardinal $\leq \mu$. Then $C(\langle \mathbf{R}^\kappa, (\mathcal{U}^\kappa)_\alpha \rangle)$ is κ-free.*

2.3 COROLLARY. *Let* κ *be uncountable. Then* $C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa \rangle)$, $C(\langle \mathbf{R}^\kappa, (\mathcal{U}^\kappa)_{\omega_1} \rangle)$, *and* $C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa)_\mu \rangle)$ *are three (isomorphically) distinct* κ-*free rings in* RCF.

Regarding the question of which κ-free rings in RCF are uniquely κ-free, other than $C(\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa \rangle)$, we have the following result.

2.4 THEOREM. *(i) Assume* κ *is a regular cardinal that is not Ulam-measurable, and assume further that either: (a)* $\kappa^+ = \exp(\kappa)$; *or (b)* $\kappa = \sup\{\exp(\alpha) : \alpha < \kappa\}$. *Then* $C(\langle \mathbf{R}^\kappa, (\mathcal{U}^\kappa)_\kappa \rangle)$ *is uniquely* κ-*free.*

(ii) $C(\langle \mathbf{R}^\mu, (\mathcal{U}^\mu)_\mu \rangle)$ *is uniquely* μ-*free.*

PROOF: (i) Note that $\kappa = \omega$ is the only cardinal satisfying the hypotheses absolutely; that case being covered already. (The case $\kappa = \omega_1$ relies on either $\omega_2 = \exp(\omega_1)$ or $\omega_1 = c = \exp(\omega)$ holding.) So assume κ is uncountable and let $\mathcal{X} = \langle \mathbf{R}^\kappa, (\mathcal{U}^\kappa)_\kappa \rangle$. In Williams' theorem 2.1(iii), set $\mathcal{T} = \mathcal{U}^\kappa$ and $\mathcal{T}' = (\mathcal{U}^\kappa)_\kappa$. Then we may conclude that $(\mathcal{U}^\kappa)_\kappa$ is a realcompact Tichonov coreflective enrichment of $\mathcal{U}^\kappa$; whence $C(\mathcal{X})$ is κ-free. Suppose $C(\mathcal{X})$ is λ-free, and let $f : \mathcal{X} \to \langle \mathbf{R}^\lambda, \mathcal{U}^\lambda \rangle$ be a coreflective map. f is a continuous bijection by 0.1, and the weight $w(\langle \mathbf{R}^\lambda, \mathcal{U}^\lambda \rangle)$ of $\langle \mathbf{R}^\lambda, \mathcal{U}^\lambda \rangle$ is $\lambda \cdot \omega$. Since κ is uncountable regular, $(\mathcal{U}^\kappa)_\kappa$ is nondiscrete; and the intersection of $\lambda \cdot \omega$ open sets in $\mathcal{X}$ cannot be a singleton if $\lambda < \kappa$. Thus $\lambda \geq \kappa$. Suppose (a) holds. Then $\kappa^+ = \exp(\kappa) = \exp(\lambda) > \lambda$, so $\lambda = \kappa$. Now suppose (b) holds. It is easy to construct a discrete subset D of $\langle \mathbf{R}^\lambda, \mathcal{U}^\lambda \rangle$ of cardinality λ; thus $f^{-1}[D]$ is a discrete subset of $\mathcal{X}$, also of cardinality λ. This forces $w(\mathcal{X}) \geq \lambda$. On the other hand, we have $w(\mathcal{X}) \leq$ the number of intersections of $< \kappa$ open subsets of $\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa \rangle \leq$ the number of intersections of $< \kappa$ open sets from $\mathcal{B}$, where $\mathcal{B}$ is an open basis of $\langle \mathbf{R}^\kappa, \mathcal{U}^\kappa \rangle$ of cardinality κ. This number is in turn $\leq \sup\{\exp(\alpha) : \alpha < \kappa\}$, since κ is a regular cardinal. In short $\lambda \leq w(\mathcal{X}) \leq \kappa$; whence $\kappa = \lambda$.

To prove (ii), we use the Comfort-Retta theorem 2.1(ii) to infer that $(\mathcal{U}^\mu)_\mu$ is realcompact. Thus $C(\langle \mathbf{R}^\mu, (\mathcal{U}^\mu)_\mu \rangle)$ is μ-free. Since measurable cardinals are strongly inaccessible, condition (b) holds, and we can argue as above. ∎

Theorem 1.1 can also be put to use in helping us get a better idea of what kinds of topologies $\mathcal{T} \supseteq \mathcal{U}^\kappa$ make it possible for $C(\langle \mathbf{R}^\kappa, \mathcal{T} \rangle)$ to be κ-free, even when κ is uncountable. For each $J \subseteq \kappa$, let $A_J \subseteq \mathbf{R}^\kappa$ be the "flat" $\{x \in \mathbf{R}^\kappa : x_\xi = 0 \text{ for } \xi \notin J\}$. Let $\pi_J : \mathbf{R}^\kappa \to \mathbf{R}^\kappa$ be the projection map onto A_J. For any topology $\mathcal{T}$ on $\mathbf{R}^\kappa$, let $\mathcal{T}|J$ be the restriction of $\mathcal{T}$ to A_J. The following small collection of facts is easy to prove.

2.5 PROPOSITION. *(i) If* $\mathcal{T}$ *is an* H-*enrichment (resp.* C-*enrichment, coreflective enrichment) of* $\mathcal{U}^\kappa$, *then* $\mathcal{T}|J$ *is an* H-*enrichment (resp.* C-*enrichment, coreflective enrichment) of* $\mathcal{U}^\kappa|J$.

(ii) If $\mathcal{T}$ *is an* H-*enrichment of* $\mathcal{U}^\kappa$, *then all straight lines in* $\mathbf{R}^\kappa$ *(viewed as a vector space) are equivalent via* $\langle \mathcal{T}, \mathcal{T} \rangle$-*homeomorphisms.*

(iii) If $\mathcal{T}$ is a C-enrichment of $\mathcal{U}^\kappa$, then each map π_J is $\langle\mathcal{T},\mathcal{T}\rangle$-continuous. Moreover, if $U \in \mathcal{T}|\xi(=\mathcal{T}|\{\xi\})$, then $\pi_\xi^{-1}[U] \in \mathcal{T}$; whence $\mathcal{T}$ is an enrichment of the product topology $\Pi_{\xi<\kappa}\mathcal{T}|\xi$.

2.6 THEOREM. *Let $\mathcal{T}$ be a C-enrichment of $\mathcal{U}^\kappa$. Then either: (a) $\mathcal{T}$ is a pathwise connected topology; or (b) $\mathcal{T} \supseteq (\mathcal{U}^\kappa)_{\omega_1}$.*

PROOF: **First Proof.** Let $\xi < \kappa$. By 2.5 (i), $\mathcal{T}|\xi$ is a C-enrichment of $\mathcal{U}^\kappa|\xi$ on the ξ-axis A_ξ. ($\langle A_\xi,\mathcal{U}^\kappa|\xi\rangle$ is naturally homeomorphic to $\langle\mathbf{R},\mathcal{U}\rangle$.) By 2.5 (ii), all the subspaces $\langle A_\xi,\mathcal{T}|\xi\rangle$ are homeomorphic; and by 1.1 they are either all euclidean lines or all discrete. Using 2.5 (ii) again, we know that if all the axes are euclidean, then all straight lines in $\langle\mathbf{R}^\kappa,\mathcal{T}\rangle$ are euclidean; hence $\mathcal{T}$ is a pathwise connected topology.

Suppose all the axes are discrete. For each countable $J \subseteq \kappa$, then, $\mathcal{T}|J$ is a proper C-enrichment of $\mathcal{U}^\kappa|J$ on A_J; so, by 1.2 (which relies on Anderson's theorem: $\langle\mathbf{R}^\omega,\mathcal{U}^\omega\rangle \simeq \ell^2$), $\mathcal{T}|J$ is discrete. ($\langle A_J,\mathcal{U}^\kappa|J\rangle$ is naturally homeomorphic to $\langle\mathbf{R}^\omega,\mathcal{U}^\omega\rangle$.) Because π_J is $\langle\mathcal{T},\mathcal{T}\rangle$-continuous by 2.5 (iii), every set of the form $\Pi_{\xi<\kappa}U_\xi$, where U_ξ is a singleton if $\xi \in J$ and $U_\xi = \mathbf{R}$ if $\xi \notin J$, is $\mathcal{T}$-open. This implies $(\mathcal{U}^\kappa)_{\omega_1} \subseteq \mathcal{T}$. ∎

Second Proof. Repeat the argument in the first paragraph of the proof above. We alter the second paragraph so as not to rely on Anderson's theorem (in 1.2).

Suppose all the axes are discrete, and let $J = \{\xi_n : n < \omega\}$, where $\xi_0 < \xi_1 < \cdots$. Define $f : \mathbf{R}^\kappa \to \mathbf{R}^\kappa$ to be the function taking $\langle x_\xi\rangle$ to $\langle \sum\limits_{n<\omega} \frac{1}{2^n}\frac{|x_{\xi_n}|}{1+|x_{\xi_n}|},0,0,\cdots\rangle \in A_0$. Then f is $\langle\mathcal{U}^\kappa,\mathcal{U}^\kappa\rangle$-continuous, hence $\langle\mathcal{T},\mathcal{T}\rangle$-continuous. Moreover, if $U \in \mathcal{T}$ is such that $U \cap A_0 = \{\langle 0,0,\cdots\rangle\}$, then $f^{-1}[U] = \Pi_{\xi<\kappa}U_\xi$, where $U_\xi = \{0\}$ if $\xi \in J$ and $U_\xi = \mathbf{R}$ if $\xi \notin J$. By homogeneity, this implies $(\mathcal{U}^\kappa)_{\omega_1} \subseteq \mathcal{T}$. ∎

2.7 REMARKS. *(i) In 2.6, conclusion (a) implies that $C(\langle\mathbf{R}^\kappa,\mathcal{T}\rangle)$ is a connected ring; i.e., there are no nontrivial idempotents. When κ is countable, there is the much stronger conclusion that $\mathcal{T} = \mathcal{U}^\kappa$. However, for uncountable κ, there is at least one other topology that could possibly work, namely the k-modification of $\mathcal{U}^\kappa$. Recall that a space $\langle X,\mathcal{T}\rangle$ is a k-space if for any $C \subseteq X$, C is $\mathcal{T}$-closed if and only if $C \cap K$ is closed in K for every compact subspace K of $\langle X,\mathcal{T}\rangle$. The k-modification $k(\mathcal{T})$ of $\mathcal{T}$ is that topology on X, whose closed sets are precisely those sets whose intersections with the compact subspaces of $\langle X,\mathcal{T}\rangle$ are closed in those subspaces (see [**12**]). It is well known that $k(\mathcal{T})$ is always a coreflective enrichment of $\mathcal{T}$, and that $\langle\mathbf{R}^\kappa,\mathcal{U}^\kappa\rangle$ is not a k-space for $\kappa \geq \omega_1$. Also, since the compact subspaces of $\langle X,k(\mathcal{T})\rangle$ are exactly those of $\langle X,\mathcal{T}\rangle$. the modified topology is pathwise connected whenever the old topology is. Thus $k(\mathcal{U}^\kappa)$ is a proper C-enrichment of $\mathcal{U}^\kappa$ for which conclusion (a) obtains. Unfortunately we do not know whether this* *topology is realcompact, and hence do not know whether $C(\langle \mathbf{R}^\kappa, k(\mathcal{U}^\kappa)\rangle)$ is κ-free.*

*(ii) If conclusion (b) holds in 2.6, then $\langle \mathbf{R}^\kappa, \mathcal{T}\rangle$ is a space with the following properties. (1) The space is totally disconnected; (2) the only compact subspaces are the finite ones; (3) the only convergent sequences are eventually constant; (4) the only first countable subspaces are discrete; and (5) all countable subsets are closed. We do not know whether $\mathcal{T}$ is itself closed under countable intersections (i.e., a P-space topology). If it were, and Tichonov as well, we could conclude that $C(\langle \mathbf{R}^\kappa, \mathcal{T}\rangle)$ is a (von Neumann) regular ring (see Exercise 4J in [**7**]).*

References

1. P. Bankston and R. Schutt, *On minimally free algebras*, Can. J. Math. **37** (1985), 963-978.
2. P. Bankston, *A note on large minimally free algebras*, Algebra Universalis (to appear).
3. P. Bankston, *Characterizing Minimally Free Rings of Continuous Functions*, Marquette University Technical Report #278 (February, 1988).
4. P. Bankston and R. A. McCoy, *H-enrichments of topologies*, (in preparation).
5. W. W. Comfort and T. Retta, *Generalized perfect maps and a theorem of I. Juhász*, in "Rings of Continuous Functions" (Proc. A.M.S. Special Session, Cincinnati, Ohio, 1982, C. E. Aull, ed.) Marcel Dekker, New York, 1985, pp. 79-102..
6. P. Eklof, *Infinitary equivalence of abelian groups*, Fund. Math. **81** (1974), 305-314.
7. L. Gillman and M. Jerison, "Rings of Continuous Functions," Van Nostrand, Princeton, 1960.
8. K. Głazek, *Some old and new problems in the independence theory*, Coll. Math. **42** (1979), 127-189.
9. P. Hall, *The splitting properties of relatively free groups*, Proc. London Math. Soc. **3** (1954), 343-356.
10. I. Kříž and A. Pultr, *Large k-free algebras*, Algebra Universalis **21** (1985), 46-53.
11. H. Neumann, "Varieties of Groups," Ergebnisse der Math. und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin-New York, 1967.
12. S. Willard, "General Topology," Addison Wesley, Reading, MA., 1970.
13. S. W. Williams, *More realcompact spaces*, in "Rings of Continuous Functions" (Proc. A.M.S. Special Session, Cincinnati, Ohio, 1982; C. E. Aull, ed.) Marcel Dekker,New York, 1985, pp. 289-300.

Department of Mathematics, Statistics and Computer Science, Milwaukee, WI 53233 (first author)
Department of Mathematics, Blacksburg, VA 24061 (second author)