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A note on large minimally free algebras

PAUL BANKSTON

In Memory of Evelyn Nelson

Let Ω be a type of finitary function symbols (in the sense of [5]). An Ω-algebra A is called *minimally free* if there is a subset X of A such that every function from X into A extends uniquely to an endomorphism on A. X is then called a *pseudobasis* for A. If κ is a cardinal number, A is said to be *κ-free* if A has a pseudobasis of cardinality κ. (Thus "0-free" means "(endomorphism)rigid.") The concept of minimal freeness was initiated in [1], and has been subsequently studied in [2] and [10]. The related papers [4] and [6] deal with 1-free groups, under the guise of E-rings, and the earlier works [7], [8], [9] and [14] study 0-free algebras of various types.

The object of this note is to provide a short proof of the Kříž–Pultr theorem [10], that every variety containing arbitrarily large rigid algebras must also contain arbitrarily large κ-free algebras for any fixed κ. We prove a more general theorem which firstly has a much simpler and more direct proof than the Kříž–Pultr result and secondly does not seem to be a consequence of their methods.

Our main tool stems from Birkhoff's 1935 construction [3] of the free algebra on κ generators, over the smallest variety containing a given set of algebras. When the aforementioned set of algebras contains just the algebra A, this contruction may be described as follows: first take the direct power $A^{(A^\kappa)}$ (abbreviated $A \uparrow \kappa$); next let $\Pi = \{\pi_\xi : \xi < \kappa\}$ be the subset consisting of all the projection maps from A^κ to A; and finally let B be the subalgebra of $A \uparrow \kappa$ generated by Π. (This construction of B is also carried out in [5]. In the context of groups, B is called "relatively free," after P. Hall [11]. The author is grateful to B. Banaschewski for informing him of Birkhoff's construction.)

Π is, of course, a pseudobasis for B, but Π generates B as well. We are interested here in constructing minimally free algebras whose cardinalities can be arbitrarily large relative to the cardinalities of their pseudobases, however. As

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is shown below, this task may be easily accomplished by taking A to be rigid and modifying Birkhoff's construction so that the generating set is expanded to include all the constant functions from A^κ to A.

Our first awareness of nongenerating pseudobases came about in connection with the unital ring $C(\mathbf{R}^\kappa)$ of continuous real-valued functions on the topological product of κ copies of the real line. This ring turns out to be κ-free and generally of cardinality greater than $\kappa \cdot \aleph_0$. Morever, $C(\mathbf{R}^\kappa)$ is a subring of $\mathbf{R} \uparrow \kappa$ containing Π and the constant functions, but never generated by any pseudobasis (see [1], [2]).

In early 1983, the author discussed with various colleagues the problem of finding varieties containing arbitrarily large κ-free algebras, and ultimately Kříž and Pultr [10] came up with their extremely interesting result (see [1] for more history). Because of extensive earlier work on rigid algebras, their theorem may be applied to several interesting varieties, e.g., commutative groupoids [14], semigroups [9], commutative untial rings [7], and bounded lattices [8].

Let Ω be a fixed type, A a given Ω-algebra, and κ a cardinal number. Clearly all the projection maps $\pi_\xi : A^\kappa \to A$, $\xi < \kappa$, are distinct when A is nontrivial, so we are interested in when Π forms a pseudobasis for various subalgebras B with $\Pi \subseteq B \subseteq A \uparrow \kappa$. In particular, Π is a pseudobasis for its generated subalgebra, but we want also for the cardinality of B to be at least that of A. Theorem 2 below lets us arrange this.

In addition to the projections π_ξ, $\xi < \kappa$, there are the "higher order" projections $\pi_u : A \uparrow \kappa \to A$, $u \in A^\kappa$. Also important for our consideration is the diagonal embedding $\Delta : A \to A \uparrow \kappa$, taking $a \in A$ to $\bar{a} = \Delta(a)$, the constant map with domain A^κ and value a. We denote by End (A) the set of endomorphism on A (so A is rigid if and only if End $(A) = \{id_A\}$).

We are now ready to prove our result.

1 LEMMA. *Let A be any rigid algebra, and assume B is a subalgebra of $A \uparrow \kappa$ containing the set $\Delta[A]$ of constant maps. If $\phi \in$ End (B) then $\phi(\bar{a}) = \bar{a}$ for all $a \in A$.*

Proof. For each $u \in A^\kappa$, the composition $\pi_u \circ \phi \circ \Delta : A \to A$ is id_A, by rigidity. Thus, for each $a \in A$, $u \in A^\kappa$, $a = \pi_u(\phi(\bar{a})) = (\phi(\bar{a}))(u)$; whence $\phi(\bar{a}) = \bar{a}$.

2 THEOREM. *Let A be any nontrivial rigid algebra, and let $B \subseteq A \uparrow \kappa$ be the subalgebra generated by $\Delta[A] \cup \Pi$. Then B is κ-free with pseudobasis Π.*

Proof. Let $\langle f_\xi : \xi < \kappa \rangle$ be any κ-sequence of elements of B. We wish to find a $\phi \in$ End (B) such that $\phi(\pi_\xi) = f_\xi$ for each $\xi < \kappa$, and prove this ϕ is unique. B is generated by $\Delta[A] \cup \Pi$; thus if we can show the existence of such a ϕ, uniqueness will be guaranteed by Lemma 1. Definite $\phi: B \to A \uparrow \kappa$ by the assignment $(\phi(g)(u) = g(\langle f_\xi(u) : \xi < \kappa \rangle)$, $u \in A^\kappa$. Clearly, for all $\zeta < \kappa$, $(\phi(\pi_\zeta))(u) = \pi_\zeta(\langle f_\xi(u) : \xi < \kappa \rangle) = f_\zeta(u)$; so $\phi(\pi_\zeta) = f_\zeta$. ϕ is easily seen to be an Ω-homomorphism, hence it remains to show $\phi[B] \subseteq B$. But this is immediate since ϕ is a homomorphism, B is a subalgebra, $\phi[\Delta[A] \cup \Pi] \subseteq B$, and $\Delta[A] \cup \Pi$ generates B.

Noting further that any π_u serves as a left inverse for the embedding $\Delta: A \to B$, we have the following.

3 COROLLARY. *Any nontrivial rigid algebra A can be embedded as a retract in a κ-free algebra which lies in the smallest class containing A and closed under direct powers and subalgebras.*

4 COROLLARY. *Let $\mathcal{K}$ be any class of Ω-algebras closed under direct powers and subalgebras (e.g., a variety). If $\mathcal{K}$ contains arbitrarily large rigid algebras, then $\mathcal{K}$ also contains arbitrarily large κ-free algebras for any fixed κ.*

Remarks. (i) Corollary 4, where $\mathcal{K}$ is assumed to be a variety, follows from a general result (expressed in category-theoretic language) in [10]. The Kříž–Pultr approach is entirely different from (and more elaborate than) ours, and involves the use of quotients and coproducts.

(ii) R. Schutt independently obtained a special case of Theorem 2 in the context of commutative unital rings, with $\kappa = 1$. He also has many other interesting results concerning minimally free rings [13].

(iii) The variety of groups is a very important case in which Corollary 4 is useless: only the trivial group is rigid. There are, nevertheless, arbitrarily large (abelian) κ-free groups for any $\kappa > 0$. This result is due essentially to a theorem of Dugas–Mader–Vinsonhaler [6] to the effect that arbitrarily large E-rings exist. A commutative ring R is an *E-ring* [4] if R is isomorphic to the endomorphism ring of the additive group of R. In [1] we showed the connection between E-rings and minimal freeness: the 1-free groups, automatically abelian, are precisely the groups that are the additive groups of E-rings; the endomorphism ring of a 1-free group is an E-ring. Furthermore an E-ring of cardinality $\exp(\exp(\exp(\aleph_0)))$ was constructed in [1]. (There seemed to be no way to carry that construction to higher cardinalities.) The main result of [6] is a construction, using S. Shelah's "black box", of E-rings of arbitrarily large cardinality. In order to produce a large κ-free (abelian) group, then, simply take an appropriately large 1-free group and form the direct sum of κ copies of that group.

(iv) It is tempting to conjecture that if $\mathcal{V}$ is a variety and $A \in \mathcal{V}$ is 1-free, then the copower of κ copies of A exists and is κ-free. While this is true in the variety of abelian groups and, more generally, in the variety of R-modules for R a commutative ring, the simple proof in these cases seems to rely on the fact that coproducts are direct sums and embed nicely in direct products. Nonetheless, we are attracted to the conjecture that there is a reasonable version of Corollary 4 that says a suitable class with large λ-free algebras will also contain large κ-algebras for $\kappa \geq \lambda$.

(v) If $\Delta[A] \cup \Pi \subseteq B \subseteq A \uparrow \kappa$ and B is not generated by $\Delta[A] \cup \Pi$, it is still fairly easy to extend maps from Π into B to endomorphisms on B (e.g., when $B = A \uparrow \kappa$). However, uniqueness is a problem. Many examples are available in special cases. For instance, let A be the unital ring $\mathbf{R}$ of real numbers. Then we can take B to be $C(\mathbf{R}^\kappa)$; and even $\mathbf{R} \uparrow \kappa$ itself, provided κ is less than all measurable cardinals. The second assertion is a consequence of the fact that $\mathbf{R}$ is a rigid field. It also points to the difficult problem of providing general (universal) algebraic conditions under which $A \uparrow \kappa$ is κ-free. (Schutt [13] has been working on problems of this nature in the context of ring theory.)

(vi) R. Schutt [13] has informed the author that if A is a unital ring and $A^A = A \uparrow 1$ has pseudobasis $\{id_A\}$, then A must be rigid. We can actually say a bit more; the rigidity assumption in Theorem 2 is nearly always necessary: if Ω contains a constant symbol, $\Delta[A] \cup \Pi \subseteq B \subseteq A \uparrow \kappa$, and Π is a pseudobasis for B, then A is rigid. To see this, suppose $c \in A$ is named by a constant symbol in Ω and that A is not rigid. Assume ϕ, $\psi \in \mathrm{End}(A)$ disagree on $a \in A$. Let $u \in A^\kappa$ be the "constantly c" map. (What is important is that $u \in \bigcap_{\xi<\kappa} \pi_\xi^{-1}[c]$.) Set $\phi' = \phi \circ (\pi_u \mid B)$, $\psi' = \psi \circ (\pi_u \mid B)$. Then ϕ' and ψ' are homomorphisms from B to A. Moreover, for each $\xi < \kappa$, $\phi'(\pi_\xi) = \phi(\pi_u(\pi_\xi)) = \phi(\pi_\xi(u)) = \phi(c) = c$. Likewise $\psi'(\pi_\xi) = c$, and ϕ' and ψ' therefore agree on Π. However, $\phi'(\bar{a}) = \phi(a) \neq \psi(a) = \psi'(\bar{a})$. Thus $\Delta \circ \phi'$ and $\Delta \circ \psi'$ are distinct endomorphisms on B which argee on Π. This says Π is not a pseudobasis for B.

(vii) The class of fields, far from being a variety, enjoys a position quite the reverse of the variety of groups as regards minimal freeness. P. Pröhle [12] has shown that there are arbitrarily large 0-free fields. But of course there are no κ-free fields for $\kappa > 0$, since members of a would-be pseudobasis could never be sent to 0.

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Department of Mathematics
Marquette University
Milwaukee, Wisconsin