H-ENRICHMENTS AND THEIR HOMEOMORPHISM GROUPS II

by

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Abstract. H-enrichments of topologies are larger (i.e., inclusive) topologies that also have larger homeomorphism groups. In this paper we continue our study of H-enrichments of the euclidean and rational topologies via generalized Baire category arguments. New results concern lower bounds on the number of H-enrichments, connectedness and separation properties of such topologies, and certain of their cardinal invariants.

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0. INTRODUCTION. This paper seeks to shed further light on the use of Baire category arguments in the study of H-enrichments of topologies, and is a continuation of [7] (whose notation and terminology we follow). To recall the basic definition: Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X, $\mathcal{T} \subseteq \mathcal{T}'$ (i.e., $\mathcal{T}'$ is an enrichment of $\mathcal{T}$). $\mathcal{T}'$ is an *H-enrichment* of $\mathcal{T}$ if every homeomorphism on $(X, \mathcal{T})$ is also a homeomorphism on $(X, \mathcal{T}')$. In the language of groups, this condition may be phrased $H(\mathcal{T}) \leq H(\mathcal{T}')$, where $H(\mathcal{T}) := H(X, \mathcal{T})$ is the group of homeomorphisms on the space $(X, \mathcal{T})$.

The study of H-enrichments arose as an off-shoot of the question of which rings of continuous real-valued functions possess "pseudobases." (Gillman-Jerison [10] is still the main source on the subject of such rings.) Recall [3] that a *pseudobasis* in an abstract algebra A is a subset P such that every function from P into A extends uniquely to an endomorphism on A. Pseudobases that generate an algebra are free bases in the usual sense; however in the case of function rings, pseudobases never generate. A principal result of [1] is that a function ring $C(\mathcal{X})$ has a pseudobasis of cardinality κ (finite or infinite) just in case $\mathcal{X}$ may be taken to be of the form $(\mathbf{R}^\kappa, \mathcal{T})$, where $\mathcal{T}$ is a realcompact topology that is a special kind of H-enrichment (called a "coreflective" enrichment) of the usual product topology $\mathcal{U}^\kappa$. This result was later used [3] to give a complete classification of those function rings that possess countable pseudobases. (There is a more streamlined proof of the classification theorem in [4]. See also [2, 5, 6, 12], as well as their references, for more information on pseudobases and how they tie in with other algebraic work, e.g., the theories of E-rings and of endomorphism-rigid abstract algebras.)

Although certain of our results are quite general, their more interesting applications concern familiar spaces; e.g., the euclidean spaces $(\mathbf{R}^n, \mathcal{U}^n)$, $0 < n < \omega$, and the rational line $(\mathbf{Q}, \mathcal{U}')$. We thus phrase our guiding questions in terms of these spaces. In particular, the following three problems remain unsolved.

0.1 QUESTION. Is there a proper H-enrichment of $\mathcal{U}$ that is both connected and regular?

0.2 QUESTION. How many H-enrichments does $\mathcal{U}$ have?

0.3 QUESTION. Given an H-enrichment $\mathcal{T}$ of the rational topology $\mathcal{U}'$, when is $H(\mathcal{U}')$ a proper subgroup of $H(\mathcal{T})$?

The results presented in the sequel address these (as well as other) questions. Section 1 summarizes the main tool of the paper, a Baire category result due to Hung-Negrepontis [9]. Section 2 gives the most general applications of this result to H-enrichment problems, and Sections 3, 4 carry these applications into the very special realm of euclidean spaces. In Section 5 we consider the situation as regards the rational line, and show that the Baire category approach cannot be used to obtain H-enrichments in the same way as before.

1. THE MAIN BAIRE CATEGORY LEMMA.

If X is an infinite set, we identify the power set $\mathcal{P}(X)$ naturally with the cartesian power 2^X. When we equip the latter with the usual product topology (where 2 is given the discrete topology), the result is the space which we denote $\mathcal{P}^\omega(X)$. This space is compact Hausdorff (even compact metrizable when X is countable), and thus satisfies the Baire category theorem: The intersection of countably many dense open sets is dense. This theorem, especially in the context of completely metrizable spaces of sets, functions, or other structures, has been used repeatedly by combinatorialists (and others, see, e.g., [8, 17]) to prove the existence (and abundance) of interesting objects.

When X is of uncountable cardinality , the usual Baire category theorem is of limited use, because a particular intersection we may wish to be nonempty typically involves $|X|$ factors. However, if $|X|$ is regular (so X cannot be expressed as a union of fewer than $|X|$ subsets, each of cardinality less than $|X|$), a higher cardinal version of the theorem is available, thanks to work of Hung-Negrepontis [9]. We first recall notation from [7]. For any set X and cardinal κ, let $\mathcal{P}^\kappa(X)$ be the space $(\mathcal{P}(X), \mathcal{T})$, where $\mathcal{T}$ is basically generated by sets of the form $[F, G] := \{A \subseteq X : F \subseteq A \subseteq X \setminus G\}$, F and G being subsets of X of cardinality $< \kappa$. When X is infinite and κ is a regular cardinal, then $\mathcal{P}^\kappa(X)$ is canonically homeomorphic to $(2^X, \mathcal{T})$, where $\mathcal{T}$ is the "κ- modification" of the product topology; i.e., new open sets are basically generated by $< \kappa$ intersections of old open sets. (The κ-modification is an example of a coreflective enrichment.) The Main Lemma is now the following.

1.1 LEMMA ([9, Theorem 15.8]). Let X be of infinite regular cardinality κ. Then:

(*i*) $\mathcal{P}^\kappa(X)$ is a "κ-Baire" space (i.e., the intersection of κ dense open sets is dense); and

(*ii*) if κ is not weakly compact, then any intersection of κ dense open subsets of $\mathcal{P}^\kappa(X)$ is homeomorphic to $\mathcal{P}^\kappa(X)$. $\square$

In a κ-Baire space, any set that contains an intersection of κ dense open sets is termed *κ-residual*; complements of κ-residual sets are called *κ-meager*. 1.1 (*ii*) says that if κ is not too "large" (see [13]), then the κ-residual sets in $\mathcal{P}^\kappa(X)$ have maximal cardinality. This fact was used twice in [7], in the proofs of Theorems 3.3 and 3.10. In Section 3 we improve on those results, obviating the need for 1.1 (*ii*) in the process.

2. GENERAL APPLICATIONS. Our main goal in this section is to show how a small generalization of Theorem 1.8 in [7] can greatly extend its applicability. Let $(X, \mathcal{T})$ be a topological space, with $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \leq H(\mathcal{T})$. We denote by $\mathcal{T}_{\mathcal{A}}^{\mathcal{H}}$ the smallest enrichment $\mathcal{T}'$ of $\mathcal{T}$ such that: (i) $\mathcal{A} \subseteq \mathcal{T}'$; and (ii) $\mathcal{H} \leq H(\mathcal{T}')$. This new topology is called a "partial H-enrichment" of $\mathcal{T}$, and we drop the superscript entirely when $\mathcal{H} = H(\mathcal{T})$. Also we set $\mathcal{T}_A^{\mathcal{H}} := \mathcal{T}_{\{A\}}^{\mathcal{H}}$. The following is easy to prove.

2.1 PROPOSITION. Let $(X, \mathcal{T})$, $\mathcal{A}$ and $\mathcal{H}$ be as above, with $\mathcal{B}$ an open basis for $\mathcal{T}$. Then an open basis for $\mathcal{T}_{\mathcal{A}}^{\mathcal{H}}$ consists of sets of the form $U \cap \bigcap_{h \in H} h(A)$, where $U \in \mathcal{B}$, $A \in \mathcal{A}$, and H is a finite subset of $\mathcal{H}$. $\square$

A little more notation and terminology: κ^+ is the cardinal successor of κ; κ^- is λ if $\kappa = \lambda^+$ for some λ, and is κ otherwise. The *weight* $w(\mathcal{T})$ of a topology $\mathcal{T}$ is the smallest cardinality of a possible open basis for $\mathcal{T}$. Define a space $(X, \mathcal{T})$ to be *even* if: (i) $w(\mathcal{T}) \leq |X|$; and (ii) $|U| = |X|$ for every nonempty $U \in \mathcal{T}$. A subset A of X is *small* if $|A| < |X|$, *cosmall* if $X \setminus A$ is small, and a *moiety* otherwise. In this section, "κ-residual" abbreviates "κ-residual in $\mathcal{P}^\kappa(X)$." Our generalization of Theorem 1.8 in [7] is the following (and the proof is quite similar).

2.2 THEOREM. Let κ be an infinite regular cardinal, $(X, \mathcal{T})$ an even space of cardinality κ, and $\mathcal{H}$ a subgroup of $H(\mathcal{T})$ of cardinality $\leq \kappa$. Then:

(*i*) $\mathcal{R} := \{A \subseteq X :$ for all nonempty $U \in \mathcal{T}$ and all finite $H \subseteq \mathcal{H}, |U \cap \bigcap_{h \in H} h(A)| \geq \kappa^-\}$ is κ-residual.

(*ii*) If $\mathcal{T}$ is a Hausdorff topology and $S \subseteq X$ is either $\mathcal{T}$-open or cosmall, then $\mathcal{S} := \{A \subseteq X :$ for all $U \in \mathcal{T}$ and all finite $H_1,\ H_2 \subseteq \mathcal{H},\ U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$ is either empty or infinite $\}$ is κ-residual.

PROOF. Ad (*i*). Fix an open basis $\mathcal{B}$ for $\mathcal{T}$, $|\mathcal{B}| \leq \kappa$, nonempty $U \in \mathcal{B}$, finite $H \subseteq \mathcal{H}$, and ordinal $0 < \alpha < \kappa$. Let $\mathcal{R}_{U,H,\alpha} := \{A \subseteq X :$ there is a one-one map from α into $U \cap \bigcap_{h \in H} h(A)\}$. There are $\leq \kappa$ such sets $\mathcal{R}_{U,H,\alpha}$, and $\mathcal{R}$ clearly contains their intersection. In view of 1.1, it remains to show each such set is dense open in $\mathcal{P}^\kappa(X)$.

Ad "dense." Fix small disjoint sets $F, G \subseteq X$, and set $A := X \setminus G$. Then $A \in [F, G]$. Also A is cosmall; whence so is $\bigcap_{h \in H} h(A)$. Since $|U| = \kappa$, we have that $|U \cap \bigcap_{h \in H} h(A)| = \kappa$, so $A \in \mathcal{R}_{U,H,\alpha}$.

Ad "open." Suppose $A \in \mathcal{R}_{U,H,\alpha}$, and let f map α one-one into $U \cap \bigcap_{h \in H} h(A)$. Let R be the image of f (so $|R| < \kappa$). Set $F := \bigcup_{h \in H} h^{-1}(R)$. Then F is small and a subset of A. Moreover, if B is any subset of X containing F, then $R \subseteq U \cap \bigcap_{h \in H} h(B)$, so we have $A \in [F, \emptyset] \subseteq \mathcal{R}_{U,H,\alpha}$.

Ad (*ii*). Fix an open basis $\mathcal{B}$ for $\mathcal{T}$ as above, and set $\mathcal{F} := \{(U, H_1, H_2) : U \in \mathcal{B}, H_1$ and H_2 are finite subsets of $\mathcal{H}$, and there is some $r \in U$ with $\{h^{-1}(r) : h \in H_2\} \subseteq S \setminus \{h^{-1}(r) : h \in H_1\}\}$. For each $(U, H_1, H_2) \in \mathcal{F}$, let $\mathcal{S}_{U,H_1,H_2} := \{A \subseteq X : U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$ is nonempty $\}$. Now there are $\leq \kappa$ such sets $\mathcal{S}_{U,H_1,H_2}$, so it remains to show that each of them is dense open in $\mathcal{P}_\kappa(X)$, and that $\mathcal{S}$ contains their intersection. So fix $\mathcal{S}_{U,H_1,H_2}$.

Ad "dense." Fix small disjoint sets $F, G \subseteq X$, and let $V := \{r \in U : \{h^{-1}(r) : h \in H_2\} \subseteq S \setminus \{h^{-1}(r) : h \in H_1\}\}$. V is nonempty because $(U, H_1, H_2) \in \mathcal{F}$, so let $r \in V$ be arbitrary. Since $\mathcal{T}$ is Hausdorff, there are disjoint $\mathcal{T}$-open sets $W_i \supseteq \{h^{-1}(r) : h \in H_i\}, i = 1, 2$. Let $W := U \cap \bigcap_{h \in H_1} h(W_1) \cap \bigcap_{h \in H_2} h(W_2 \cap S)$. Then $r \in W \subseteq V$. If S is $\mathcal{T}$-open (empty or nonempty; if empty, then H_2 is empty as well), then so is W; hence V is $\mathcal{T}$-open also. Thus $|V| = \kappa$. If S is cosmall, then W is easily seen to be an intersection of a nonempty $\mathcal{T}$-open set and a cosmall set. Thus $|V| = \kappa$ in this case too. Now fix $r \in V \setminus (\bigcup_{h \in H_1} h(G) \cup \bigcup_{h \in H_2} h(F))$ and set $A := F \cup \{h^{-1}(r) : h \in H_1\}$. Then $A \in [F, G]$. We proceed to show $A \in \mathcal{S}_{U,H_1,H_2}$. Indeed $r \in U$; if $h \in H_1$, then $h^{-1}(r) \in A$, so $r \in h(A)$. Finally, if $h \in H_2$, then $h^{-1}(r) \in S$ since $r \in V$. Thus $r \in h(S)$. Now suppose $r \in h(A)$. Then, since $r \notin h(F)$, it follows that there is some $g \in H_1$ with $h(g^{-1}(r)) = r$, or $h^{-1}(r) = g^{-1}(r)$. This contradicts the fact that $r \in V$; hence we conclude $r \in h(S \setminus A)$.

Ad "open." Suppose $A \in \mathcal{S}_{U,H_1,H_2}$. Let $r \in U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$, and set $F := \{h^{-1}(r) : h \in H_1\}, G := \{h^{-1}(r) : h \in H_2\}$. Then $[F, G]$ is a basic open set in $\mathcal{P}^\omega(X)$, and $A \in [F, G]$. If $B \in [F, G]$, then we have $r \in U \cap \bigcap_{h \in H_1} h(B) \cap \bigcap_{h \in H_2} h(S \setminus B)$, so $B \in \mathcal{S}_{U,H_1,H_2}$.

Ad "containment." Suppose A is in $\mathcal{S}_{U,H_1,H_2}$ for all $(U, H_1, H_2) \in \mathcal{F}$. Fix $U \in \mathcal{B}$ and finite $H_1, H_2 \subseteq \mathcal{H}$. We need to show that $U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$ is either empty or infinite. Suppose first that $(U, H_1, H_2) \notin \mathcal{F}$. Then for any $r \in \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$, we have $\{h^{-1}(r) : h \in H_1\} \subseteq A$ and $\{h^{-1}(r) : h \in H_2\} \subseteq S \setminus A \subseteq S \setminus \{h^{-1}(r) : h \in H_1\}$. Thus $r \notin U$, so $U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$ is empty. Next suppose that $(U, H_1, H_2) \in \mathcal{F}$, and $r_0 \in U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$. The set V defined above is infinite, so there is some $U_1 \subseteq U \setminus \{r_0\}, U_1 \in \mathcal{B}$, and $U_1 \cap V \neq \emptyset$. Thus $(U_1, H_1, H_2) \in \mathcal{F}$, so we may pick $r_1 \in U_1 \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$. By induction on ω, we can therefore find distinct $r_0, r_1, r_2, \ldots$ in $U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(S \setminus A)$, and the proof is complete. □

2.3 REMARK. In 2.2 (ii) we tried without success to prove a version in which "infinite" is replaced with "of cardinality $\geq \kappa^-$," à la 2.2(i).

2.4 COROLLARY. Let $(X, \mathcal{T})$ be an even space of infinite regular cardinality κ. Then $\{A \subseteq X : \text{both } A \text{ and } X \setminus A \text{ are } \mathcal{T}\text{-dense}\}$ is κ-residual. Consequently $\mathcal{T}$ is κ-meager.

PROOF. In 2.2 (i), let $\mathcal{H}$ be the trivial subgroup. Then $\{A \subseteq X : A \text{ is } \mathcal{T}\text{-dense}\}$ is κ-residual since it contains $\mathcal{R}$. Now the operation of set complementation is a homeomorphism on $\mathcal{P}^\kappa(X)$; hence $\{A \subseteq X : X \setminus A \text{ is } \mathcal{T}\text{-dense}\}$ is κ-residual. Consequently, the intersection of the two sets, a set that is disjoint from $\mathcal{T}$, is κ-residual as well. □

The following helps to bridge the gap between the combinatorial conditions in 2.2 and the issues of connectedness and separation.

2.5 PROPOSITION. Suppose $(X, \mathcal{T})$ is any topological space and $\mathcal{C} \subseteq \mathcal{P}(X)$. Let $\mathcal{T}'$ be the topology generated by $\mathcal{T} \cup \mathcal{C}$. Then:

(i) If every finite intersection of members of $\mathcal{C}$ is $\mathcal{T}$-dense and $\mathcal{T}$ is a connected topology, then $\mathcal{T}'$ is connected also.

(ii) If every finite intersection of members of $\mathcal{C}$ is $\mathcal{T}$-dense and $\mathcal{T}' \neq \mathcal{T}$, then $\mathcal{T}'$ is nonregular.

(iii) If $\mathcal{C}$ is closed under complementation and $\mathcal{T}$ is a regular (resp. completely regular, 0-dimensional) topology, then $\mathcal{T}'$ is regular (resp. completely regular, 0-dimensional) also.

PROOF. Let $(X, \mathcal{T})$, $\mathcal{C}$, and $\mathcal{T}'$ be as in the hypothesis.

Ad (i). Use the "chain-connectedness" criterion with respect to any basic open cover (see Theorem 25.15 in [19] or the proof of Theorem 1.8 (ii) in [7]).

Ad (ii). Let B be a $\mathcal{T}'$-closed set that is not $\mathcal{T}$-closed. Let $a \in cl_{\mathcal{T}}(B) \setminus B$, and pick $\mathcal{T}'$-open sets V, W such that $B \subseteq V$ and $a \in W$. It suffices to show $V \cap W$ is nonempty. We may assume W is basic open, so there is some $U_1 \in \mathcal{T}$ and finite $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $W = U_1 \cap \bigcap \mathcal{C}_1$. Let $b \in U_1 \cap B$, and find $U_2 \in \mathcal{T}$ and finite $\mathcal{C}_2 \subseteq \mathcal{C}$ such that $b \in U_2 \cap \bigcap \mathcal{C}_2 \subseteq V$. Then $V \cap W \supseteq U_1 \cap U_2 \cap \bigcap(\mathcal{C}_1 \cup \mathcal{C}_2) \neq \emptyset$ since $b \in U_1 \cap U_2$ and $\bigcap(\mathcal{C}_1 \cup \mathcal{C}_2)$ is $\mathcal{T}$-dense.

Ad (iii). This is essentially Proposition 2.5 in [7], and is straightforward to prove. The thing to note is that members of $\mathcal{C}$ are $\mathcal{T}'$-clopen. □

A strengthened version of Theorem 1.8 in [7] that is closer in spirit to the original can now be stated; its proof is an easy application of 2.1–2.5.

2.6 THEOREM. Let $(X, \mathcal{T})$ be an even space of infinite regular cardinality κ, $\mathcal{H}$ a subgroup of $H(\mathcal{T})$ of cardinality $\leq \kappa$. Then:

(i) $\{A \subseteq X : \mathcal{T}_A^{\mathcal{H}}$ is nonregular and every nonempty $\mathcal{T}_A^{\mathcal{H}}$-open set has cardinality $\geq \kappa^-\}$ is κ-residual.

(ii) If $\mathcal{T}$ is a connected topology, then $\{A \subseteq X : \mathcal{T}_A^{\mathcal{H}}$ is nonregular and connected $\}$ is κ-residual.

(iii) If $\mathcal{T}$ is a Hausdorff topology and $S \subseteq X$ is either $\mathcal{T}$-open or cosmall, then $\{A \subseteq X :$ every nonempty $\mathcal{T}_{\{A,S\setminus A\}}^{\mathcal{H}}$-open set is infinite$\}$ is κ-residual.

(iv) In (iii) above, in the case $S := X$, preservation of the separation properties regularity, complete regularity, and 0-dimensionality is also assured. □

The general results above can help to answer Guiding Question 0.2 by giving us lower bounds on the number of H-enrichments a given topology may have. For a topology $\mathcal{T}$ on a set X, let $HE(\mathcal{T})$ be the complete lattice of H-enrichments of $\mathcal{T}$. Recall that a cardinal κ is called *weakly inaccessible* if: (i) κ is regular; and (ii) κ is a limit cardinal (i.e., $\kappa = \kappa^-$). Weak inaccessibility is a very "small" large cardinal axiom (smaller than, say, weak compactness, see [13]).

2.7 THEOREM. Let $(X, \mathcal{T})$ be an even space of weakly inaccessible cardinality κ, and suppose $|H(\mathcal{T})| \leq \kappa$. Then there is a chain in $HE(\mathcal{T})$ of order type κ^+ consisting of even nonregular topologies. If $\mathcal{T}$ is connected, we can obtain a chain in $HE(\mathcal{T})$ of order type ω consisting of even nonregular connected topologies; we can also obtain a family in $HE(\mathcal{T})$ consisting of κ^+ even nonregular connected topologies.

PROOF. We build the chain $\mathcal{C}$ by induction on κ^+. Since κ is weakly inaccessible, we may replace κ^- with κ in 2.6; we apply 2.6 with $\mathcal{H} := H(\mathcal{T})$.

Using 2.6 (i), find $A \subseteq X$ so that $\mathcal{T}_0 := \mathcal{T}_A^{\mathcal{H}}$ is nonregular and every nonempty $\mathcal{T}_0$-open set has cardinality κ. By 2.4, we can ensure that $\mathcal{T}_0 \neq \mathcal{T}$. Since both $w(\mathcal{T})$ and $|\mathcal{H}|$ are $\leq \kappa$, we have $w(\mathcal{T}_0) \leq \kappa$ too. Thus $\mathcal{T}_0$ is an even topology. Now fix ordinal $\alpha < \kappa^+$ and assume, by way of induction, that $\mathcal{C}_\alpha$ is an α-indexed chain in $HE(\mathcal{T})$ such that for each $\beta < \alpha$, $\mathcal{T}_\beta \in \mathcal{C}_\alpha$ is nonregular and even. Let $\mathcal{T}'$ be the join of the topologies $\mathcal{T}_\beta$. Since $|\alpha| \leq \kappa$, $\mathcal{T}'$ is an even H-enrichment of $\mathcal{T}$, so we may invoke 2.6 (i) and 2.4 to find $\mathcal{T}_\alpha$ using $\mathcal{T}'$ in the same way we found $\mathcal{T}_0$ using $\mathcal{T}$. If $\mathcal{T}$ is connected, we use 2.6 (ii) in conjunction with 2.6 (i) and 2.4. Since infinite chain joins of connected topologies are not necessarily connected, we are able to perform only a finite induction in this case.

To obtain κ^+ even nonregular connected topologies in $HE(\mathcal{T})$, let $\mathcal{S} := \{A \subseteq X : \mathcal{T}_A$ is nonregular connected and every nonempty $\mathcal{T}_A$-open set has cardinality $\kappa\}$. By 2.6 (i, ii), $\mathcal{S}$ is κ-residual. For each $A \in \mathcal{S}$, $\mathcal{T}_A$ is also even, hence κ-meager by 2.4. For $A \in \mathcal{S}$, let $[A] := \{B \in \mathcal{S} : \mathcal{T}_B = \mathcal{T}_A\}$. Then $[A] \subseteq \mathcal{T}_A$, so $[A]$ too is κ-meager. The equivalence classes $[A]$ cover the κ-residual set $\mathcal{S}$, so there must be more than κ such classes. Our desired family is then obtained by picking sets from $\mathcal{S}$ from different equivalence classes. □

Antichains in $HE(\mathcal{T})$ seem harder to find than chains. However, using 2.6, we can obtain pairs of perfect (i.e., having no isolated points) H-enrichments that have nonperfect topological joins.

2.8 THEOREM. Let $(X, \mathcal{T})$ be an even space of infinite regular cardinality κ, and suppose $|H(\mathcal{T})| \leq \kappa$. Then there exists a pair of nonregular perfect H-enrichments of $\mathcal{T}$ whose join is nonperfect; if $\mathcal{T}$ is connected, these H-enrichments may be taken to be connected too. If $\mathcal{T}$ is regular (resp. completely regular, 0-dimensional), the H-enrichments may also be taken to be regular (resp. completely regular, 0-dimensional).

PROOF. For each $x \in X$, let $h_x : \mathcal{P}(X) \to \mathcal{P}(X)$ be given by $h_x(A) := (A \setminus (X \setminus \{x\})) \cup ((X \setminus \{x\}) \setminus A)$ (i.e., the symmetric difference of A with $X \setminus \{x\}$).

CLAIM. Each h_x is an involutive homeomorphism on $\mathcal{P}^\kappa(X)$.

PROOF OF CLAIM. Fix $x \in X$ and set $h := h_x$. h is clearly an involution, so it suffices to show h is an open map. For each $A \subseteq X$, we have

$$h(A) = (A \cap \{x\}) \cup (X \setminus (A \cup \{x\})) = \begin{cases} \{x\} \cup (X \setminus A) & \text{if } x \in A \\ X \setminus (A \cup \{x\}) & \text{if } x \notin A. \end{cases}$$

To show h is open, let $F, G \subseteq X$ have cardinality $< \kappa$, and assume $A \in [F, G]$. We require $F', G' \subseteq X$, also of cardinality $< \kappa$, such that $h(A) \in [F', G'] \subseteq h([F, G])$.
CASE 1. $x \in A$. Then $h(A) = \{x\} \cup (X \setminus A)$. Since $A \in [F, G]$, we have $F \subseteq A \subseteq X \setminus G$, so $G \subseteq X \setminus A \subseteq X \setminus F$. Thus $G \cup \{x\} \subseteq h(A) \subseteq (X \setminus F) \cup \{x\} = X \setminus (F \setminus \{x\})$, so $h(A) \in [G \cup \{x\},\ F \setminus \{x\}]$. Set $F' := G \cup \{x\}$, $G' := F \setminus \{x\}$, and assume $F' \subseteq B \subseteq X \setminus G'$. Since h is an involution, it suffices to show $F \subseteq h(B) \subseteq X \setminus G$. Now $x \in B$, so $h(B) = \{x\} \cup (X \setminus B)$. Since $G \cup \{x\} \subseteq B$, we have $X \setminus B \subseteq (X \setminus G) \setminus \{x\}$, so $\{x\} \cup (X \setminus B) \subseteq (X \setminus G) \cup \{x\}$. But $x \in A \subseteq X \setminus G$, so $h(B) \subseteq X \setminus G$. Finally, $B \subseteq X \setminus (F \setminus \{x\})$, so $F \setminus \{x\} \subseteq X \setminus B$, hence $F \cup \{x\} \subseteq h(B)$. Thus $F \subseteq h(B)$.
CASE 2. $x \notin A$. Then $h(A) = X \setminus (A \cup \{x\})$. Again we have $G \subseteq X \setminus A \subseteq X \setminus F$, so $G \setminus \{x\} \subseteq h(A) \subseteq (X \setminus F) \setminus \{x\}$, hence $h(A) \in [G \setminus \{x\},\ F \cup \{x\}]$. Set $F' := G \setminus \{x\}$, $G' := F \cup \{x\}$, and assume $F' \subseteq B \subseteq X \setminus G'$. We need to show $F \subseteq h(B) \subseteq X \setminus G$. Since $x \notin B$, we have $h(B) = X \setminus (B \cup \{x\}) = (X \setminus B) \setminus \{x\}$. Now $X \setminus B \subseteq X \setminus (G \setminus \{x\})$, so $h(B) \subseteq (X \setminus (G \setminus \{x\})) \setminus \{x\} = (X \setminus G) \setminus \{x\} \subseteq X \setminus G$. Finally, $F \cup \{x\} \subseteq X \setminus B$, so $(F \cup \{x\}) \setminus \{x\} \subseteq h(B)$. But $(F \cup \{x\}) \setminus \{x\} = F \setminus \{x\}$, and $x \notin A$. Thus $x \notin F$, hence $F \subseteq h(B)$. This proves the claim.

To complete the proof of the Theorem, fix $x \in X$. By 2.6 (i), $\mathcal{S} := \{A \subseteq X : \mathcal{T}_A$ is nonregular perfect$\}$ is κ-residual. Since h_x is an involutive homeomorphism on $\mathcal{P}^\kappa(X)$, $h_x(\mathcal{S}) = \{A \in X : \mathcal{T}_{h_x(A)}$ is nonregular perfect$\}$ is also κ-residual, so $\mathcal{S} \cap h_x(\mathcal{S})$ is dense. Let $A \in [\{x\}, \emptyset] \cap \mathcal{S} \cap h_x(\mathcal{S})$. Then $\mathcal{T}_A$ and $\mathcal{T}_{h_x(A)}$ are both nonregular perfect H-enrichments of $\mathcal{T}$ whose join is nonperfect, since it contains $A \cap h_x(A) = A \cap (\{x\} \cup (X \setminus A)) = \{x\}$.

To handle the other assertions, use the rest of 2.6, where appropriate. □

3. APPLICATIONS TO EUCLIDEAN SPACE. In this section we confine our attention to the euclidean spaces $(\mathbf{R}^n, \mathcal{U}^n)$, $1 \leq n < \omega$. These spaces are even, and $|H(\mathcal{U}^n)| = |\mathbf{R}^n| = \mathbf{c}$. However, in order to apply the results of Section 2, we must assume (as we did in [7]) the reasonably modest axiom that $\mathbf{c}$ is a regular cardinal. We then have a strong analogue to 2.7; fortunately we do not need to assume that $\mathbf{c}$ is weakly inaccessible (a consistent assumption, by the way, see [13]). We first need some preliminary facts.

3.1 LEMMA ([4, Theorem 2.21]). Let $\mathcal{T}$ be a nondiscrete H-enrichment of $\mathcal{U}^n$. Then every nonempty $\mathcal{T}$-open set has cardinality $\mathbf{c}$. □

Recall from the theory of symmetry groups that if G is a subgroup of $\mathrm{Sym}(X)$, the symmetric group of all permutations on the set X, and if $n < \omega$, G is *n-transitive* if any bijection between two n-element subsets of X can be extended to an element of G. G is *highly transitive* if G is n-transitive for all $n < \omega$. It is well known that $H(\mathcal{U}^n)$ is highly transitive for all $1 < n < \omega$.

3.2 LEMMA. Suppose $(X, \mathcal{T})$ is a topological space and $\mathcal{T}' \in HE(\mathcal{T})$. If $H(\mathcal{T})$ is 3-transitive, then $\mathcal{T}'$ is either connected or totally disconnected.

PROOF. Fix distinct $x, y \in X$, and suppose T is a $\mathcal{T}'$-clopen set containing both points. Assuming $\mathcal{T}'$ to be nonconnected, we may assume there is some $z \in X \setminus T$. Using 3-transitivity, there is some $h \in H(\mathcal{T}')$ that fixes x and switches y and z. (Actually, we use the weaker fact that the stabilizer in $H(\mathcal{T})$ of any point in X is 2-transitive on the complement.) Then $h(T)$ is a $\mathcal{T}'$-clopen set containing x and missing y. □

Thus H-enrichments of the multidimensional euclidean topologies are either connected or totally disconnected. The same is true in the unidimensional case for different reasons (see [4, Theorem 2.16]).

3.3 THEOREM. Assume $\mathbf{c}$ is a regular cardinal.

(*i*) There is a chain in $HE(\mathcal{U}^n)$ of order type $\mathbf{c}^+$ consisting of even nonregular topologies; there is a chain of order type ω consisting of even nonregular connected topologies. There is also a family in $HE(\mathcal{U}^n)$ consisting of $\mathbf{c}^+$ even nonregular connected topologies.

(*ii*) There is a chain in $HE(\mathcal{U}^n)$ of order type $\mathbf{c}^+$ consisting of even completely regular totally disconnected topologies.

(*iii*) There is a pair of even nonregular connected H-enrichments of $\mathcal{U}^n$ whose join is discrete. There is also a pair of even completely regular totally disconnected H-enrichments of $\mathcal{U}^n$ whose join is discrete.

PROOF. Ad (i). The proof proceeds exactly as in the proof of 2.7, except here we use 3.1 instead of the assumption that $\mathbf{c}^- = \mathbf{c}$.

Ad (ii). Proceed as in the proof of 2.7, but use 2.6 (iii, iv), with $S := \mathbf{R}^n$, instead of 2.6 (i, ii). Use 3.1 here as well. (Note that $\mathcal{T}'$, a chain join of completely regular topologies, is completely regular. Also note that at each stage of the induction we add new clopen sets, so the topologies are nonconnected, hence totally disconnected.)

Ad (iii). Use 2.8 and 3.1. □

3.4 REMARK. Getting back to our Guiding Question 0.1, we know from 2.6 (ii) that, assuming $\mathbf{c}$ is regular, $\mathcal{U}^n_A$ is nonregular and connected for "almost" every $A \subseteq \mathbf{R}^n$. This simplest of Baire category arguments will therefore not accord us proper regular connected H-enrichments of the euclidean topologies. The generalization of [7, Theorem 1.8 (iii)] obtained by incorporating the "wild card" set S in 2.6 (iii) came about from a suggestion by R. A. McCoy [15], who thought it possible that a judicious choice of $S \subseteq \mathbf{R}$ (the set of irrationals, say) may lead to regular connected H-enrichments when 2.6 (iii) is applied to $\mathcal{U}$. So far that question remains open.

We next look at the issue of calculating the weight $w(\mathcal{T})$ when $\mathcal{T} \in HE(\mathcal{U}^n)$. Thanks to the results of Section 2, we can improve on results in [7] (especially Theorems 3.3 and 3.10 therein). We first recall an important fact.

3.5 LEMMA ([4, Theorem 2.19]). Let $\mathcal{T}$ be a proper H-enrichment of $\mathcal{U}^n$. Then every $\mathcal{T}$-convergent ω-sequence in $\mathbf{R}^n$ is eventually constant. Consequently, $w(\mathcal{T}) \geq \omega_1$. □

Let P be any topological property, $(X, \mathcal{T})$ a space. We say $\mathcal{T}' \in HE(\mathcal{T})$ is *H-maximal-P* if $\mathcal{T}'$ satisfies P, and for any $\mathcal{T}'' \in HE(\mathcal{T})$, if $\mathcal{T}'' \supseteq \mathcal{T}'$ and $\mathcal{T}''$ satisfies P, then $\mathcal{T}'' = \mathcal{T}'$. The existence of H-maximal-P H-enrichments of $\mathcal{T}$ is assured by Zorn's lemma provided: (i) some $\mathcal{T}' \in HE(\mathcal{T})$ satisfies P; and (ii) P is *inductive*, i.e., preserved under chain joins of topologies. Inductive properties include perfectness, (complete) regularity, total disconnectedness and 0-dimensionality. A notable exception is connectedness. Guthrie-Stone and Wage [11] show via two (nonconstructive) methods that $\mathcal{U}^n$ can be enriched to a maximal connected topology, but neither method seems to yield an H-enrichment. (Wage's method yields a topology having countably infinite open sets, and is hence not an H-enrichment by 3.1.) Thus it is an open question whether the euclidean topologies have H-maximal-connected H-enrichments. On the other hand, 3.3 assures us the existence of H-maximal-P H-enrichments of $\mathcal{U}^n$, where P is perfectness, either alone or in conjunction with (complete) regularity and/or total disconnectedness. Our improved result on weight is the following.

3.6 THEOREM. Assume $\mathbf{c}$ is a regular cardinal. If $\mathcal{T} \in HE(\mathcal{U}^n)$ is H-maximal-P, where P is any one of the properties perfectness, connectedness, total disconnectedness, (complete) regularity, 0-dimensionality (all in conjunction with perfectness), then $w(\mathcal{T}) \geq \mathbf{c}^+$.

PROOF. Suppose $\mathcal{T} \in HE(\mathcal{U}^n)$ is perfect and of weight $\leq \mathbf{c}$. Then, by 3.1, $\mathcal{T}$ is even. By 2.4, $\mathcal{T}$ is $\mathbf{c}$-meager in $\mathcal{P}^{\mathbf{c}}(\mathbf{R}^n)$. Let property P be as above. If $\mathcal{T}$ satisfies P, then by appropriate application of 2.6, there exists $\mathcal{T}' \in HE(\mathcal{U}^n)$, $\mathcal{T}'$ a proper enrichment of $\mathcal{T}$, such that $\mathcal{T}'$ also satisfies P. Thus $\mathcal{T}$ cannot be H-maximal-P. □

A simple consequence of 3.5 concerns minimal members of $HE(\mathcal{U}^n)$. Let $A := \mathbf{R}^n \setminus \{1/n : n = 1, 2, \ldots\}$; i.e., the complement of a simple sequence on the first axis. Let $\mathcal{T} := \mathcal{U}^n_A$. The following is easy to show.

3.7 PROPOSITION. $\mathcal{T}$ is the unique minimal proper H-enrichment of $\mathcal{U}^n$; i.e., the "false bottom" of $HE(\mathcal{U}^n)$. $\mathcal{T}$ is connected nonregular, of uncountable weight $\leq \mathbf{c}$. However its (hereditary) density, cellularity, π-weight and (hereditary) Lindelöf degree are all countable. □

3.8 REMARK. On the other hand, by 3.3 (*iii*), there are at least two H-maximal-perfect H-enrichments of $\mathcal{U}^n$. Indeed, perfectness may also be conjoined with total disconnectedness and (complete) regularity; and, in the case $n = 1$, with 0-dimensionality (see 4.2 (*iii*)).

4. APPLICATIONS TO THE REAL LINE.

The usual topology $\mathcal{U}$ on the real line $\mathbf{R}$ is the topology for which we have the most complete information as regards H-enrichments. In this case, we can further sharpen 3.3 because of the following.

4.1 LEMMA ([7, Corollary 2.10]). Let $\mathcal{A} \subseteq \mathcal{P}(\mathbf{R})$ be closed under complementation, where $\mathcal{A} \not\subseteq \mathcal{U}$. Then $\mathcal{U}_{\mathcal{A}}$ is 0-dimensional. □

4.2 THEOREM. Assume $\mathbf{c}$ is a regular cardinal.

(*i*) There is a chain in $HE(\mathcal{U})$ of order type $\mathbf{c}^+$ consisting of even nonregular topologies; there is a chain of order type ω consisting of even nonregular connected topologies. There is also a family in $HE(\mathcal{U})$ consisting of $\mathbf{c}^+$ even nonregular connected topologies.

(*ii*) There is a chain in $HE(\mathcal{U})$ of order type $\mathbf{c}^+$ consisting of even 0-dimensional topologies.

(*iii*) There is a pair of even nonregular connected H-enrichments of $\mathcal{U}$ whose join is discrete. There is also a pair of even 0-dimensional H-enrichments of $\mathcal{U}$ whose join is discrete.

PROOF. Ad (*i*). Same as 3.3 (*i*).
Ad (*ii*). Use 3.3 (*ii*) and 4.1.
Ad (*iii*). Use 3.3 (*iii*) and 4.1. □

As mentioned above, 3.6 is an improvement on Theorem 3.10 in [7], a result which takes the form: "If $\mathcal{T} \in HE(\mathcal{U})$ is H-maximal-P and $w(\mathcal{T}) \leq \mathbf{c}$, then $H(\mathcal{T}) \geq \kappa$." Of course we now know the hypothesis is false, but the conclusion is still interesting. Fortunately we are able to salvage much of the argument that gives the conclusion, replacing an impossible hypothesis with a reasonable one. (Also we do not need to assume $\mathbf{c}$ is regular.) We first recall notation from [7]. Let $(X, \mathcal{T})$ be a topological space, with $\mathcal{T}' \in HE(\mathcal{T})$. Define $h \in \mathrm{Sym}(X)$ to be a *$\mathcal{T}'/\mathcal{T}$-homeomorphism* if there exists a "witnessing" family $\{(T_i, h_i) : i \in I\}$ where $\{T_i : i \in I\}$ is a $\mathcal{T}'$-open cover of X and $\{h_i : i \in I\}$ is a family of $\mathcal{T}$-homeomorphisms such that for each $i \in I, h|T_i = h_i|T_i$. The set of $\mathcal{T}'/\mathcal{T}$-homeomorphisms is denoted $H(\mathcal{T}'/\mathcal{T})$, and is a subgroup of $\mathrm{Sym}(X)$ lying between $H(\mathcal{T})$ and $H(\mathcal{T}')$ (see Proposition 3.6 in [7]). It is worthy of note that all the known examples of members of $H(\mathcal{T}) \setminus H(\mathcal{U})$ for $\mathcal{T}$ a nonconnected H-enrichment of $\mathcal{U}$ are actually in $H(\mathcal{T}/\mathcal{U})$. Moreover, in the witnessing family, the $\mathcal{T}$-open cover is a clopen partition.

4.3 THEOREM. Let κ be a regular cardinal, $\mathbf{c}^+ \leq \kappa \leq 2^{\mathbf{c}}$. If $\mathcal{T} \in HE(\mathcal{U})$ and there are κ $\mathcal{T}$-clopen sets, then $H(\mathcal{T}/\mathcal{U}) \geq \kappa$.

PROOF. Assume κ is as above, and $\{T_\alpha : \alpha < \kappa\}$ is a family of distinct $\mathcal{T}$-clopen sets. Now $|H(\mathcal{U})| = \mathbf{c}$, so we may assume that for $\alpha < \beta < \kappa$, there is no $h \in H(\mathcal{U})$ taking T_α to T_β; i.e., the sets T_α are in distinct $H(\mathcal{U})$-orbits. Also, by 3.1, we may assume that each T_α is a moiety in $\mathbf{R}$. (The status of moiety is not really necessary here; infinite-coinfinite will do.) For each $\alpha < \kappa$, there is a linear shift s_α taking T_α to a $\mathcal{T}$-clopen set not containing 0. The sets $s_\alpha(T_\alpha)$ are all distinct, since the sets T_α all lie in different $H(\mathcal{U})$-orbits. So without loss of generality, we may assume that the sets T_α are all distinct $\mathcal{T}$-clopen moieties not containing 0.

Let $T_\alpha^+ := T_\alpha \cap (0, \infty) = T_\alpha \cap [0, \infty)$, and $T_\alpha^- := T_\alpha \cap (-\infty, 0) = T_\alpha \cap (-\infty, 0]$. Then the sets T_α^+, T_α^- are all $\mathcal{T}$-clopen. Suppose $|\{T_\alpha^+ : \alpha < \kappa\}| < \kappa$. Then, since κ is regular, there is some $X \subseteq \kappa, |X| \geq \kappa$, such that for $\alpha, \beta \in X$, $T_\alpha^+ = T_\beta^+$. But then for all $\alpha, \beta \in X, T_\alpha^- \neq T_\beta^-$. The upshot of all this is that we may assume without loss of generality that we have a family $\{T_\alpha : \alpha < \kappa\}$ of distinct $\mathcal{T}$-clopen moieties, each lying in the positive half-line $(0, \infty)$. For each $\alpha < \kappa$, let $T'_\alpha := T_\alpha \cup \{-x : x \in T_\alpha\}$. Then the sets T'_α are all distinct, and each is a $\mathcal{T}$-clopen moiety that is symmetric about (but does not contain) the origin. For each $\alpha < \kappa$, define $h_\alpha \in \mathrm{Sym}(\mathbf{R})$ to fix x when $x \in T'_\alpha$, and to take x to $-x$ otherwise. Then $\{h_\alpha : \alpha < \kappa\}$ is a family of distinct $\mathcal{T}/\mathcal{U}$-homeomorphisms. $\square$

4.4 REMARK. The hypothesis of 4.3 is indeed nonvacuous; for if we assume $\mathbf{c}$ is a regular cardinal and $\mathcal{T} \in HE(\mathcal{U})$ contains an H-maximal-(perfect 0-dimensional) H-enrichment of $\mathcal{U}$, then by 3.6, $\mathcal{T}$ has $\mathbf{c}^+$ clopen sets.

4.5 REMARK. An example of an enrichment of $\mathcal{U}$ that is completely regular and connected, but not quite an H-enrichment, is the well known density topology $\mathcal{U}_d$. Studied extensively in [14, 18], this topology is defined by taking a set $A \subseteq \mathbf{R}$ to be $\mathcal{U}_d$-open just in case:

(*i*) A is Lebesgue measurable; and

(ii) for every $x \in A, \lim_{r \to 0^+} \frac{m(A \cap [x-r, x+r])}{2r} = 1$, where m is Lebesgue measure.

This topology (not obviously a topology from the definition) is an enrichment of $\mathcal{U}$, but it is not an H-enrichment. To see this, use Theorem 13.1 in [17] (a theorem whose proof uses a Baire category argument, by the way). Pick $A \subseteq [0,1]$ of positive measure. Then there is some $h \in H(\mathcal{U})$ (which can be taken to be the identity outside $(0,1)$) such that $h(A)$ has measure 0. Then $h(A)$ is $\mathcal{U}_d$-meager, but A is $\mathcal{U}_d$-nonmeager. Thus $h \notin H(\mathcal{U}_d)$. The reason the density topology is interesting from the point of view of Guiding Question 0.1 is that it is trying hard to masquerade as a regular connected H-enrichment of $\mathcal{U}$. Here is what we mean. (Facts we cite concerning H-enrichments of $\mathcal{U}$ are proved in [4].)

(i) $\mathcal{U}_d$ is completely regular (but nonnormal) and connected. In fact, like any connected H-enrichment of $\mathcal{U}$, the only $\mathcal{U}_d$-connected subsets of $\mathbf{R}$ are the intervals. Consequently, $H(\mathcal{U}_d) \leq H(\mathcal{U})$. (Equality holds for a connected H-enrichment.)

(ii) Let $\mathcal{A}$ be the affine homeomorphisms on $\mathbf{R}$; i.e., those maps of the form $x \to ax + b, a \neq 0$. Then it is easy to show $\mathcal{A} \leq H(\mathcal{U}_d)$, so we have that $H(\mathcal{U}_d)$ is 2-transitive. (It is not 3-transitive, by (i) above.)

(iii) Every nonempty $\mathcal{U}_d$-open set has cardinality $\mathbf{c}$, a property shared by any nondiscrete H-enrichment of $\mathcal{U}$. On the other hand, every small subset of $\mathbf{R}$ is $\mathcal{U}_d$- closed. This property is shared by H-maximal-P H-enrichments of $\mathcal{U}$, where P is either perfectness or connectedness [7].

(iv) As is the case with any proper H-enrichment of $\mathcal{U}$, the only $\mathcal{U}_d$-convergent ω-sequences are the eventually constant ones.

5. APPLICATIONS TO THE RATIONAL LINE.

The rational line $(\mathbf{Q}, \mathcal{U}')$ is an even space, but its homeomorphism group (studied in, e.g., [16]) is too large of cardinality to permit use of 2.6 to get H-enrichments; we must content ourselves with partial H-enrichments via countable subgroups of $H(\mathcal{U}')$. Thus for "almost" every $A \subseteq \mathbf{Q}$ it is true that $(\mathcal{U}')^{\mathcal{H}}_A$ is nonregular, and that $(\mathcal{U}')^{\mathcal{H}}_{\{A, \mathbf{Q} \setminus A\}}$ is perfect 0-dimensional. The story for full H-enrichments is quite different however: For "almost" every $A \subseteq \mathbf{Q}$, $\mathcal{U}'_A$ is discrete. This is proved in 5.3 below.

5.1 LEMMA ([4, Theorem 2.13]). No proper nondiscrete H-enrichment of $\mathcal{U}'$ is regular. □

For any space $(X, \mathcal{T})$, let $B(\mathcal{T})$ be the collection of $\mathcal{T}$-clopen sets. Then we have the following easy consequence of 2.5 and 5.1.

5.2 PROPOSITION. Suppose $\mathcal{T} \in HE(\mathcal{U}')$ is nondiscrete. Then $B(\mathcal{T}) = B(\mathcal{U}')$.

PROOF. Let $T \in B(\mathcal{T}) \setminus B(\mathcal{U}')$. Then $\mathcal{T} \supseteq \mathcal{U}'_{\{T, \mathbf{Q} \setminus T\}}$. By 2.5, $\mathcal{U}'_{\{T, \mathbf{Q} \setminus T\}}$ is 0-dimensional; it is also a proper H-enrichment of $\mathcal{U}'$. By 5.1, then, $\mathcal{T}$ is discrete. □

5.3 THEOREM. $\{A \subseteq \mathbf{Q} : \mathcal{U}'_A$ is discrete $\}$ is ω-residual.

PROOF. Let $\mathcal{S} := \{A \subseteq \mathbf{Q}$: both A and $\mathbf{Q} \setminus A$ are $\mathcal{U}'$-dense $\}$. By 2.4, $\mathcal{S}$ is ω-residual. Now if $A \in \mathcal{S}$, let $\{a_0, a_1, \ldots\}$ be a well ordering of $\mathbf{Q}$ so that $A = \{a_n : n$ even $\}$. By a back-and-forth argument, it is easy to construct an order isomorphism h on $\mathbf{Q}$ such that $h(A) = \mathbf{Q} \setminus A$. Because $h \in H(\mathcal{U}')$, we have $\mathcal{U}'_A = \mathcal{U}'_{\{A, \mathbf{Q} \setminus A\}}$. And since $A \notin \mathcal{U}'$, we infer that $\mathcal{U}'_A$ is discrete by 5.2. The set in question thus contains $\mathcal{S}$, and is ω-residual. $\square$

Because $H(\mathcal{U}')$ is uncountable, we cannot hope for analogues of 3.3 and 4.2 using 2.6. We can, however, get chains in $HE(\mathcal{U}')$ of order type ω using constructive means. (These means would also work in the euclidean case, by the way.)

5.4 THEOREM. There is a chain in $HE(\mathcal{U}')$ of order type ω.

PROOF. For each bounded interval I in $\mathbf{Q}$, set $S_I := \{a + \frac{(b-a)}{n} : n = 2, 3, \ldots\}$, where $a < b$ are the endpoints of I. Define the scattered subsets $S_0, S_1, \ldots$ of $(0,1)$ inductively: $S_0 := S_{(0,1)}; S_{n+1} := S_n \cup \bigcup\{S_I : I$ is a maximal interval in $(0,1) \setminus S_n\}$. For each $n < \omega$, let $A_n := \mathbf{Q} \setminus S_n$, and put $\mathcal{T}_n := \mathcal{U}'_{A_n}$. We immediately have $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ because for each $n < \omega$, there is an order isomorphism on $\mathbf{Q}$ taking $(\frac{1}{2}, 1) \cap A_{n+1}$ onto A_n. The topologies $\mathcal{T}_n$ are all distinct; for by a straightforward argument involving Cantor-Bendixson rank, 0 is in the $\mathcal{T}_n$-closure, but not in the $\mathcal{T}_{n+1}$-closure, of S_{n+1}. $\square$

5.5 REMARKS.

(i) The topology $\mathcal{T}_0$ in the proof above is the "false bottom" of $HE(\mathcal{U}')$; i.e., the smallest proper H-enrichment of $\mathcal{U}'$. Although not explicitly stated in [4], there is a version of 3.5 [4, Theorem 2.19] that holds for $\mathcal{U}'$, and the same proof works.

(ii) Re Guiding Question 0.3, it is still open whether $H(\mathcal{U}')$ is a proper subgroup of $H(\mathcal{T})$ when $\mathcal{T}$ is a nondiscrete H-enrichment of $\mathcal{U}'$.

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