H-ENRICHMENTS AND THEIR HOMEOMORPHISM GROUPS

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0. Introduction

THE study of H-enrichments of topologies arose as a spin-off of the notion of minimal freeness in universal algebra [**1**], especially as it applies to rings of continuous real-valued functions [**2**]: If $\langle X, \mathcal{T}\rangle$ is a real-compact topological space whose ring of continuous functions is minimally free with pseudobasis of cardinality λ, then we may view $\mathcal{T}$ as an H-enrichment of the usual (product) topology $\mathcal{U}^\lambda$ on the cartesian power $\mathbb{R}^\lambda$ of the real line. H-enrichments were recognized as interesting in their own right, and their study was initiated properly in [**3**]. The present paper is a continuation of that study.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X, with $\mathcal{T}'$ an *enrichment* of $\mathcal{T}$, i.e., $\mathcal{T} \subseteq \mathcal{T}'$. $\mathcal{T}'$ is an *H-enrichment* of $\mathcal{T}$ if every homeomorphism on $\langle X, \mathcal{T}\rangle$ is also a homeomorphism on $\langle X, \mathcal{T}'\rangle$. Letting $H(\mathcal{T})$ denote the group of all homeomorphisms on $\langle X, \mathcal{T}\rangle$ (the underlying set X being understood), this condition simply says that the group $H(\mathcal{T})$ is a subgroup of the group $H(\mathcal{T}')$, in symbols, $H(\mathcal{T}) \leq H(\mathcal{T}')$. We are interested here in part in how $H(\mathcal{T}')$ sits between $H(\mathcal{T})$ and the full symmetric group Sym (X).

For a topological space $\langle X, \mathcal{T}\rangle$, let $HE(\mathcal{T})$ be the set of all H-enrichments of $\mathcal{T}$. $HE(\mathcal{T})$ is a complete lattice under intersection and topological join; it is a bounded lattice, with $\mathcal{T}$ at the bottom and the discrete topology (generically denoted $\mathcal{D}$) at the top. We are also interested in properties of the map $H(\cdot)$ from $HE(\mathcal{T})$ into the lattice of subgroups of Sym (X).

Although certain of our results are stated in a general context, the major applications concern spaces related to the real line $\mathbb{R}$ with its usual topology $\mathcal{U}$ (e.g., $\langle \mathbb{R}^\lambda, \mathcal{U}^\lambda\rangle$, also the rational line $\langle \mathbb{Q}, \mathcal{U}'\rangle$). Our notation is pretty standard regarding ordinal and cardinal numbers: $\omega := \{0, 1, \dots\}$ is the first infinite ordinal (cardinal); each ordinal is the set of its predecessors; κ^+ is the cardinal successor of the cardinal κ; the notation κ^λ, when κ and λ are cardinals, indicates the set of functions from λ into κ, as well as the cardinality of that set; $\kappa^{<\lambda} := \sup\{\kappa^\alpha\colon \alpha < \lambda\}$; $|X|$ is the cardinality of the set X; $\mathfrak{c} := |\mathbb{R}| = 2^\omega$. A subset $Y \subseteq X$ is *small* (resp. *co-small*, a *moiety*) if $|Y| < |X|$ (resp. $|X \setminus Y| < |X|$, $|Y| = |X\setminus Y|$). (The term "moiety" is the coinage of P. M. Neumann.)

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We first collect some results about H-enrichments proved in [**3**]. The first one says they are easy to come by.

0.1 THEOREM ([**3**, Theorem 2.1]). *If $\langle X, \mathcal{T}\rangle$ has a nonclosed set that is nowhere dense, then $\mathcal{T}$ has a proper nondiscrete H-enrichment.*

A natural way to form H-enrichments of $\langle X, \mathcal{T}\rangle$ is to let $\mathcal{A}$ be any family of subsets of X (in symbols, $\mathcal{A} \subseteq \mathcal{P}(X)$), and let $\mathcal{T}_{\mathcal{A}}$ be the smallest H-enrichment of $\mathcal{T}$ containing all the sets in $\mathcal{A}$. A typical basic open set for $\mathcal{T}_{\mathcal{A}}$ looks like $U \cap h_1(A_1) \cap \cdots \cap h_n(A_n)$, where $U \in \mathcal{T}$, $\{h_1, \ldots, h_n\} \subseteq H(\mathcal{T})$, and $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$. (Equivalently, $\mathcal{T} \cup \{h(A)\colon h \in H(\mathcal{T}), A \in \mathcal{A}\}$ forms a subbasis for $\mathcal{T}_{\mathcal{A}}$.) When $\mathcal{A} = \{A\}$, we set $\mathcal{T}_A := \mathcal{T}_{\{A\}}$. One problem is controlling the behavior of $\mathcal{T}_{\mathcal{A}}$, given $\mathcal{T}$ and $\mathcal{A}$.

0.2 THEOREM ([**3**, Theorem 2.4]). *Suppose $\langle X, \mathcal{T}\rangle$ is a space, and the complement of every member of $\mathcal{A} \subseteq \mathcal{P}(X)$ is $\mathcal{T}$-nowhere dense. Then every $\mathcal{T}$-dense set in X is also $\mathcal{T}_{\mathcal{A}}$-dense. Thus if $\mathcal{T}$ is nondiscrete, so also is $\mathcal{T}_{\mathcal{A}}$.*

0.3 PROPOSITION ([**3**, Proposition 2.5]). *Let $A \subseteq \mathbb{R}$ be $\mathcal{U}$-closed but not $\mathcal{U}$-open. Then $\mathcal{U}_A = \mathcal{D}$.*

Several of our main results involve Baire category arguments. Define a space $\langle X, \mathcal{T}\rangle$ to be *κ-Baire,* where κ is an infinite cardinal, if the intersection of at most κ dense open subsets of $\langle X, \mathcal{T}\rangle$ is dense. $Y \subseteq X$ is *κ-residual* if Y contains such an intersection.

0.4 THEOREM ([**3**, Theorem 2.9]). *Let $\langle X, \mathcal{T}\rangle$ be an ω-Baire space, with $\mathcal{A}$ a family of T-ω-residual sets. Then:*
(i) *If $\mathcal{T}$ is connected, so is $\mathcal{T}_A$.*
(ii) *If $\mathcal{T}_{\mathcal{A}} \neq \mathcal{T}$, then $\mathcal{T}_{\mathcal{A}}$ is nonregular.*

Preservation of regularity (complete regularity, normality, etc.) as one passes from $\mathcal{T}$ to $\mathcal{T}' \in HE(\mathcal{T})$ is an important issue that was addressed in [**3**].

0.5 THEOREM ([**3**, Theorem 2.12]). *$\langle \mathbb{Q}, \mathcal{U}'\rangle$ has no proper nondiscrete regular H-enrichment.*

On the other hand:

0.6 THEOREM ([**3**, Theorem 2.13]). *$\langle \mathbb{R}, \mathcal{U}\rangle$ has proper nondiscrete completely regular H-enrichment.*

The topology constructed for 0.6 by R. A. McCoy, using a rather involved double induction on $\mathfrak{c}$, is nonconnected. A question left over from that paper is whether there exists a proper connected completely regular H-enrichment of $\mathcal{U}$.

0.7 THEOREM ([**3**, Theorem 2.16]). *Let $\mathcal{T} \in HE(\mathcal{U})$. Then either $\mathcal{T}$ is totally disconnected or the $\mathcal{T}$-connected subsets of $\mathbb{R}$ are precisely the intervals. In the latter case, $H(\mathcal{T}) = H(\mathcal{U}) = \{$the monotonic bijections on $\mathbb{R}\}$.*

0.8 *Remark.* 0.7 suggests the question: For $\mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{U})$, precisely when is it true that $H(\mathcal{T}_1) = H(\mathcal{T}_2)$? We consider this question in Section 4. Part of the answer is: $H(\mathcal{T}) = H(\mathcal{U})$ if and only if $\mathcal{T}$ is connected (see Proposition 4.2); $H(\mathcal{T}) = H(\mathcal{D}) = \mathrm{Sym}(\mathbb{R})$ if and only if $\mathcal{T} = \mathcal{D}$. The situation in the latter case is quite general (and easy). Let $\langle X, \mathcal{T}\rangle$ be an infinite space such that there is some moiety $Y \subseteq X$ that is $\mathcal{T}$-open, and suppose $H(\mathcal{T}) = \mathrm{Sym}(X)$. Let $x \in X$ be arbitrary, with R, S two disjoint moieties on $X \backslash \{x\}$, and let $g, h \in \mathrm{Sym}(X)$ take Y to $R \cup \{x\}$ and $S \cup \{x\}$ respectively. Then $\{x\} = (R \cup \{x\}) \cap (S \cup \{x\}) \in \mathcal{T}$, so $\mathcal{T} = \mathcal{D}$.

The last preliminary facts we mention concern H-enrichments of the euclidean topologies $\mathcal{U}^n$, $1 \leq n < \omega$.

0.9 THEOREM ([**3**, Theorem 2.19]). *Let $1 \leq n < \omega$, with $\mathcal{T}$ a proper H-enrichment of $\mathcal{U}^n$. Then every $\mathcal{T}$-convergent sequence (i.e., ω-sequence) in $\mathbb{R}^n$ is eventually constant. (Consequently: $\mathbb{R}^n$ has no infinite $\mathcal{T}$-compact subsets; no point of $\mathbb{R}^n$ has a countable neighborhood basis, unless $\mathcal{T} = \mathcal{D}$; and every metrizable subset of $\langle \mathbb{R}^n, \mathcal{T}\rangle$ is discrete in the subspace topology.)*

0.10 THEOREM ([**3**, Theorem 2.21]). *Let $1 \leq n < \omega$, with $\mathcal{T}$ a nondiscrete H-enrichment of $\mathcal{U}^n$. Then every nonempty $\mathcal{T}$-open set has cardinality $\mathfrak{c}$.*

A quick summary of the four sections that make up the sequel is as follows.

Section 1 uses a Baire category result from [**4**] to prove that, under cetain conditions on $\langle X, \mathcal{T}\rangle$, we have that "almost all" (read "residually many") $A \subseteq X$ give rise to H-enrichments $\mathcal{T}_A$ (or $\mathcal{T}_{\{A, X\backslash A\}}$) with a specific property (Theorem 1.8). This theorem is applied repeatedly in the next two sections, so can be fairly regarded as the "keystone" result of the paper.

Section 2 consists almost entirely of direct applications of 1.8 in the context of euclidean space. One such is a simple strengthening of 0.6 to read that the usual topology $\mathcal{T}$ on $\mathbb{R}$ has a nondiscrete 0-dimensional H-enrichment. The price to pay for avoiding a messy inductive argument, however, is the assumption that $\mathfrak{c}$ is a regular cardinal.

In Section 3, 1.8 is used to study H-enrichments of $\mathcal{T}$ that are maximal (in the complete lattice $HE(\mathcal{T})$) with respect to having a given property. Theorem 3.4 says that if $\mathfrak{c}$ is a regular cardinal, $1 \leq n < \omega$, and $\mathcal{T} \in HE(\mathcal{U}^n)$ is maximal connected, then $\mathcal{T}$ has no open basis of cardinality $\mathfrak{c}$.

Section 4 is concerned mainly with group-theoretic properties of $H(\mathcal{T})$ for $\mathcal{T} \in HE(\mathcal{U})$. For example, if $\mathcal{T}$ is nonconnected, then $H(\mathcal{T})$ is k-transitive for all $k < \omega$ (Corollary 4.11) ($H(\mathcal{U})$ is not 3-transitive). We also consider the relative positioning of $H(\mathcal{U})$ in $H(\mathcal{T})$ and $H(\mathcal{T})$ in $\mathrm{Sym}(\mathbb{R})$.

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1. H-enrichments via Baire category arguments

A very powerful, but highly nonconstructive, way of getting H-enrichments is to use Baire category arguments applied to power sets fitted out with appropriate topologies.

We first introduce an important mechanism (read "coreflection") for enriching topologies. Let $\langle X, \mathcal{T}\rangle$ be a space, κ an infinite cardinal. Define the topology $(\mathcal{T})_\kappa$ to be the smallest enrichment $\mathcal{T}'$ of $\mathcal{T}$ such that intersections of $<\kappa$ $\mathcal{T}'$-open sets are still $\mathcal{T}'$-open. $(\mathcal{T})_\kappa$ is called the *κ-modification* of $\mathcal{T}$, and is a very special H-enrichment of $\mathcal{T}$. If $\mathcal{X} := \langle X, \mathcal{T}\rangle$, we set $(\mathcal{X})_\kappa := \langle X, (\mathcal{T})_\kappa\rangle$.

If X is an infinite set, we identify the power set $\mathcal{P}(X)$ with the cartesian power 2^X in the standard way: $\langle\text{subset}\rangle \mapsto \langle\text{characteristic map}\rangle$. If κ is an infinite cardinal, $\mathcal{P}^\kappa(X)$ is the space $\langle \mathcal{P}(X), \mathcal{T}\rangle$, where $\mathcal{T}$ is the "inclusion-exclusion" topology basically generated by sets $[F, G] := \{A \subseteq X\colon F \subseteq A \subseteq X\backslash G\}$, $F, G \subseteq X$ of cardinality $<\kappa$. When κ is a regular cardinal, i.e., not the supremum of any increasing sequence $\langle \alpha_\xi\colon \xi < \lambda\rangle$ of ordinals where $\lambda < \kappa$ and $\alpha_\xi < \kappa$ for all $\xi < \lambda$, then it is easy to see that $\mathcal{P}^\kappa(X)$ is just $(2^X)_\kappa$ (2^X being given the usual product topology).

The main results of this section are based on work of H. H. Hung and S. Negrepontis [**4**].

1.1 Theorem ([**4**, Theorem 15.8]). *Let κ be an infinite regular cardinal. Then:* (i) $(2^\kappa)_\kappa$ *is a κ-Baire space; and* (ii) *if κ is not weakly compact, then* *any intersection of* $\leq\kappa$ *dense open subsets of* $(2^\kappa)_\kappa$ *is homeomorphic to* $(2^\kappa)_\kappa$.

1.2 *Remark.* Weak compactness is a "large cardinal" property which is larger than strong inaccessibility, and which we need not elaborate upon here; except to say that ω has the property and every other cardinal we will be dealing with (e.g., ω_1, ω_{57}, $\mathfrak{c}$) does not. It is proved in [**4**, Corollary 15.7] that a cardinal κ is weakly compact if and only if $(2^\kappa)_\kappa$ and $(\kappa^\kappa)_\kappa$ are nonhomeomorphic. Thus, when κ is regular and not weakly compact, $(\kappa^\kappa)_\kappa$ is κ-Baire. (This also holds when $\kappa=\omega$, but in this case complete metrizability is the reason.)

Let $\langle X, \mathcal{T}\rangle$ be a topological space. As noted above, one can parametrize all of $HE(\mathcal{T})$ using $\mathcal{P}(\mathcal{P}(X))$, i.e., for $\mathcal{A}\subseteq\mathcal{P}(X)$, we have $\mathcal{T}_{\mathcal{A}}\in HE(\mathcal{T})$. Let $\kappa=2^{|X|}$. Then appropriate basic open sets for $\mathcal{P}^\kappa(\mathcal{P}(X))$ look like $[\mathcal{F}, \mathcal{G}]$, where $\mathcal{F}, \mathcal{G}\subseteq\mathcal{P}(X)$ are small. Recall that a space is *perfect* if it has no isolated points.

1.3 Proposition. *Let* $\langle X, \mathcal{T}\rangle$ *be an infinite topological space. Then* $\{\mathcal{A}\subseteq\mathcal{P}(X);\ \mathcal{T}_{\mathcal{A}}$ *is nonperfect*$\}$ *is dense open in* $\mathcal{P}^\kappa(\mathcal{P}(X))$, *where* $\kappa=2^{|X|}$.

Proof. Let $[\mathcal{F}, \mathcal{G}]\neq\varnothing$ be a basic open set in $\mathcal{P}^\kappa(\mathcal{P}(X))$. For any infinite set S, there are $2^{|S|}$ moieties on S. [Indeed, if $|\{T\subseteq S\colon |T|<|S|\}|<2^{|S|}$, then the conclusion is immediate; if $|\{T\subseteq S\colon |T|<|S|\}|=2^{|S|}$, then every such T can be expanded to a moiety on S in a one-one fashion.] So let $A\subseteq X$ be a moiety that is not in $\mathcal{G}$. Fix $a\in A$. Since $|X\backslash A|=|X|$, there are κ sets $B\cup\{a\}$, where $B\subseteq X\backslash A$. Find $B\subseteq X\backslash A$ with $B\cup\{a\}\notin\mathcal{G}$, and set $\mathcal{A}:=\mathcal{F}\cup\{A, B\cup\{a\}\}$. Then $\mathcal{A}\in[\mathcal{F}, \mathcal{G}]$ and $\mathcal{T}_{\mathcal{A}}$ contains $\{a\}$. This establishes density.

Next suppose $\mathcal{A}\subseteq\mathcal{P}(X)$ is such that $\mathcal{T}_{\mathcal{A}}$ is nonperfect. Then for some $U\in\mathcal{T}$, $h_1,\ldots,h_n\in H(\mathcal{T})$, and $A_1,\ldots,A_n\in\mathcal{A}$, we have $|U\cap h_1(A_1)\cap\cdots\cap h_n(A_n)|=1$. If $\mathcal{B}$ contains $\{A_1,\ldots,A_n\}$, then $\mathcal{T}_{\mathcal{B}}$ has an isolated point. Thus $[\{A_1,\ldots,A_n\},\varnothing]$ is an open neighborhood of $\mathcal{A}$ (in $\mathcal{P}^\omega(\mathcal{P}(X))$, in fact) such that for every member $\mathcal{B}$, $\mathcal{T}_{\mathcal{B}}$ has an isolated point. This establishes openness.

If $\langle X, \mathcal{T}\rangle$ is a homogeneous topological space (i.e., $H(\mathcal{T})$ acts transitively on X), then clearly every H-enrichment of $\mathcal{T}$ is also homogeneous. In the context of homogeneity, e.g., when we are talking about H-enrichments of $\mathcal{U}^\lambda$ or of $\mathcal{U}'$, "perfect" is synonymous with "nondiscrete."

1.4 Corollary. *Let* $\langle X, \mathcal{T}\rangle$ *be a homogeneous topological space. Then* $\{\mathcal{A}\subseteq\mathcal{P}(X)\colon \mathcal{T}_{\mathcal{A}}=\mathcal{D}\}$ *is dense open in* $\mathcal{P}^\kappa(\mathcal{P}(X))$, *where* $\kappa=2^{|X|}$.

1.5 PROPOSITION. *Let $\langle X, \mathscr{T}\rangle$ be a topological space, $\kappa := |X| \geqslant \omega$. Then $\{A \subseteq X : \mathscr{T}_A$ has a small nonempty set$\}$ is dense in $\mathscr{P}^\kappa(X)$.*

Proof. Let $[F, G] \neq \varnothing$ be a basic open set in $\mathscr{P}^\kappa(X)$. If $F \neq \varnothing$, we have $F \in [F, G]$ and $\mathscr{T}_F$ contains the small set F. If $F = \varnothing$, then since G is small, we can find some $x \in X \backslash G$. Then $\{x\} \in [F, G]$ and $\mathscr{T}_{\{x\}}$ contains the small set $\{x\}$.

1.6 COROLLARY. *Let $1 \leqslant n < \omega$. Then $\{A \subseteq \mathbb{R}^n$: $\mathscr{U}^n_A = \mathscr{D}\}$ is dense in $\mathscr{P}^{\mathfrak{c}}(\mathbb{R}^n)$.*

Proof. Immediate from 1.5 and 0.10.

The main result of this section stands in contrast to 1.3–1.6. Define the *weight* of a topological space $\langle X, \mathscr{T}\rangle$ to be the least cardinal $\kappa \geqslant \omega$ such that $\mathscr{T}$ has an open basis of cardinality κ; denote this number by $w(\mathscr{T})$. (So $w(\mathscr{U}^\lambda) = \omega \cdot \lambda$ for $\lambda \geqslant 1$.) Let $\mathscr{H} \leqslant H(\mathscr{T})$, $\mathscr{A} \subseteq \mathscr{P}(X)$. The *partial H-enrichment* $\mathscr{T}^{\mathscr{H}}_{\mathscr{A}}$ is the smallest enrichment of $\mathscr{T}$ in which each $A \in \mathscr{A}$ is open and such that each $h \in \mathscr{H}$ is a homeomorphism. It is easy to see that if $\mathscr{B} \subseteq \mathscr{T}$ is an open basis for $\mathscr{T}$, then $\mathscr{B} \cup \{h(A)$: $h \in \mathscr{H}, A \in \mathscr{A}\}$ forms a subbasis for $\mathscr{T}^{\mathscr{H}}_{\mathscr{A}}$. When $\mathscr{H} = H(\mathscr{T})$, we have $\mathscr{T}^{\mathscr{H}}_{\mathscr{A}} = \mathscr{T}_{\mathscr{A}}$ as above. (We will have occasion to be given $\mathscr{T}$ and $\mathscr{T}_1 \in HE(\mathscr{T})$, and need to find $\mathscr{T}_2 \in HE(\mathscr{T})$ with $\mathscr{T}_1 \subseteq \mathscr{T}_2$. $\mathscr{T}_2$ will be constructed to be of the form $(\mathscr{T}_1)^{\mathscr{H}}_{\mathscr{A}}$, where $\mathscr{H} = H(\mathscr{T})$.)

Define the space $\langle X, \mathscr{T}\rangle$ to be *even* if: (i) $|U| = |X|$ for every nonempty $U \in \mathscr{T}$; and (ii) $w(\mathscr{T}) \leqslant |X|$.

1.7 EXAMPLES. Let $\lambda \geqslant 1$. Then the generalized euclidean space $\langle \mathbb{R}^\lambda, \mathscr{U}^\lambda\rangle$ is even. The space $\langle \mathbb{Q}, \mathscr{U}'\rangle$ of rational numbers is also even. The argument in [3] to prove 0.9 also shows that if $\mathscr{T} \in HE(\mathscr{U}')$ is proper, then every $\mathscr{T}$-convergent sequence in $\mathbb{Q}$ is eventually constant. Thus if $\mathscr{T}$ is also nondiscrete (a possibility guaranteed by 0.1), then $w(\mathscr{T}) \geqslant \omega_1$; hence $\langle \mathbb{Q}, \mathscr{T}\rangle$ is not even.

1.8 THEOREM. *Assume κ is an infinite regular cardinal, $\langle X, \mathscr{T}\rangle$ is an even space of cardinality κ, and $\mathscr{H} \leqslant H(\mathscr{T})$ has cardinality $\leqslant \kappa$.*

(i) *If $\mathscr{T}$ is a T_1 topology, then $\{A \subseteq X$: $\mathscr{T}^{\mathscr{H}}_A$ is perfect$\}$ is κ-residual in $\mathscr{P}^\kappa(X)$.*

(ii) *If $\mathscr{T}$ is a connected topology, then $\{A \subseteq X$: $\mathscr{T}^{\mathscr{H}}_A$ is connected$\}$ is κ-residual in $\mathscr{P}^\kappa(X)$.*

(iii) *If $\mathscr{T}$ is a Hausdorff topology, then $\{A \subseteq X$: $\mathscr{T}^{\mathscr{H}}_{\{A, X\backslash A\}}$ is perfect$\}$ is κ-residual in $\mathscr{P}^\kappa(X)$.*

Proof. Ad (i). Let $\mathscr{B} \subseteq \mathscr{T}$ be an open basis of cardinality $\leqslant \kappa$. A typical basic open set for $\mathscr{T}^{\mathscr{H}}_A$ then looks like $U \cap \bigcap_{h \in H} h(A)$, where $U \in \mathscr{B}$ and $H \subseteq \mathscr{H}$ is finite. Let $\mathscr{F} = \{\langle U, H\rangle$: $U \in \mathscr{B} \backslash \{\varnothing\}$ and $H \subseteq \mathscr{H}$ is finite$\}$.

Define $\mathcal{S}_{U,H} := \left\{A \subseteq X\colon\ U \cap \bigcap_{h\in H} h(A) \neq \varnothing\right\}$ for $\langle U, H\rangle \in \mathcal{F}$. By hypothesis, $|\mathcal{F}| \leq \kappa$. By 1.1(i) it remains to show that each $\mathcal{S}_{U,H}$ is dense open in $\mathcal{P}^\kappa(X)$ and that $\{A \subseteq X\colon\ \mathcal{T}_A^{\mathcal{H}}$ is perfect$\} \supseteq \bigcap\{\mathcal{S}_{U,H}\colon\ \langle U, H\rangle \in \mathcal{F}\}$.

First let $[F, G] \neq \varnothing$ be basic open in $\mathcal{P}^\kappa(X)$, so $F, G \subseteq X$ are disjoint and of cardinality $<\kappa$. Set $A := X\backslash G$. Then $A \in [F, G]$. Because $|U| = \kappa$, we have $U \cap \bigcap_{h\in H} h(A) = U\backslash\left(\bigcup_{h\in H} h(G)\right) \neq \varnothing$. Thus $A \in [F, G] \cap \mathcal{S}_{U,H}$, and $\mathcal{S}_{U,H}$ is dense in $\mathcal{P}^\kappa(X)$.

Next let $A \in \mathcal{S}_{U,H}$, with $r \in U \cap \bigcap_{h\in H} h(A)$. Let $F := \{h^{-1}(r)\colon\ h \in H\}$. Then $F \subseteq A$ is finite, so $A \in [F, \varnothing]$. Also if $B \in [F, \varnothing]$ then clearly $r \in U \cap \bigcap_{h\in H} h(B)$; whence $B \in \mathcal{S}_{U,H}$. This says that $\mathcal{S}_{U,H}$ is open in $\mathcal{P}^\kappa(X)$ (even in $\mathcal{P}^\omega(X)$).

Finally suppose $A \in \bigcap\{\mathcal{S}_{U,H}\colon\ \langle U, H\rangle \in \mathcal{F}\}$. We must show that for $U \in \mathcal{B}$ and $H \subseteq \mathcal{H}$ finite, $\left|U \cap \bigcap_{h\in H} h(A)\right| \neq 1$. If $U = \varnothing$, the cardinality is 0; if $U \neq \varnothing$, then $\langle U, H\rangle \in \mathcal{F}$. Let $r \in U \cap \bigcap_{h\in H} h(A)$. Then there is some $V \in \mathcal{B}\backslash\{\varnothing\}$, $V \subseteq U\backslash\{r\}$, since $\mathcal{T}$ is a T_1 topology. Then $\langle V, H\rangle \in \mathcal{F}$, so $A \in \mathcal{S}_{V,H}$. This tells us that $U \cap \bigcap_{h\in H} h(A)$ contains elements other than r, and is hence not a singleton.

Ad(ii). Use the proof above to show that $\mathcal{S}_{U,H}$ is dense open in $\mathcal{P}^\kappa(X)$ for $\langle U, H\rangle \in \mathcal{F}$. Assuming $\mathcal{T}$ is connected and $A \in \cap\{\mathcal{S}_{U,H}\colon\ \langle U, H\rangle \in \mathcal{F}\}$, let $\mathcal{V}$ be a cover of X by $\mathcal{T}_A^{\mathcal{H}}$-basic open sets, with $a, b \in X$ arbitrary and $V_a, V_b \in \mathcal{V}$ containing a and b respectively. It suffices to find $V_1, \ldots, V_m \in \mathcal{V}$ with $V_1 = V_a$, $V_m = V_b$, and $V_i \cap V_{i+1} \neq \varnothing$ for $1 \leq i \leq m-1$. For each $\mathcal{T}_A^{\mathcal{H}}$-basic open set V, let $U_V \in \mathcal{B}$, $H_V \subseteq \mathcal{H}$ finite be such that $V = U_V \cap \bigcap_{h\in H_V} h(A)$. Let $\mathcal{V}' = \{U_V\colon V \in \mathcal{V}\}$. Then $\mathcal{V}'$ is a $\mathcal{T}$-open cover of X. Since $\mathcal{T}$ is a connected topology, there exist $V_1, \ldots, V_m \in \mathcal{V}$ such that $V_1 = V_a$, $V_m = V_b$, and setting $U_i = U_{V_i}$, $H_i = H_{V_i}$, $1 \leq i \leq m-1$, we have $U_i \cap U_{i+1} \neq \varnothing$. Now for each i, $V_i \cap V_{i+1} = (U_i \cap U_{i+1}) \cap \bigcap_{h\in H_i\cup H_{i+1}} h(A)$. Since $U_i \cap U_{i+1} \neq \varnothing$, we know that $A \in \mathcal{S}_{U_i\cap U_{i+1}, H_i\cup H_{i+1}}$ for $1 \leq i \leq m-1$. Thus $V_i \cap V_{i+1} \neq \varnothing$, and $\mathcal{T}_A^{\mathcal{H}}$ is therefore connected.

Ad(iii). Let $\mathcal{B}$ be as above. A typical basic open set for $\mathcal{T}^{\mathcal{H}}_{\{A, X\backslash A\}}$ looks like $U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$, where $U \in \mathcal{B}$ and $H_1, H_2 \subseteq \mathcal{H}$ are finite. $\mathcal{T}^{\mathcal{H}}_{\{A, X\backslash A\}}$ is perfect if and only if no such set is a singleton.

Let $\mathcal{F} = \{\langle U, H_1, H_2\rangle\colon\ U \in \mathcal{B}, H_1, H_2 \subseteq \mathcal{H}$ are finite, and there is some $r \in U$ with $\{h^{-1}(r)\colon\ h \in H_1\} \cap \{h^{-1}(r)\colon\ h \in H_2\} = \varnothing\}$. For each

$\langle U, H_1, H_2\rangle \in \mathcal{F}$, let $\mathcal{S}_{U,H_1,H_2} := \left\{A \subseteq X\colon\ U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A) \neq \varnothing\right\}$. We know $|\mathcal{F}| \leq \kappa$, so it suffices to show that each such $\mathcal{S}_{U,H_1,H_2}$ is dense open in $\mathcal{P}^\kappa(X)$ and that $\{A \subseteq X\colon\ \mathcal{T}^{\mathcal{H}}_{\{A, X\backslash A\}}$ is perfect$\} \supseteq \bigcap\{\mathcal{S}_{U,H_1,H_2}\colon\ \langle U, H_1, H_2\rangle \in \mathcal{F}\}$.

First fix $\mathcal{S}_{U,H_1,H_2}$ for $\langle U, H_1, H_2\rangle \in \mathcal{F}$, and let $[F, G] \neq \varnothing$ be basic open in $\mathcal{P}^\kappa(X)$. Let $V := \{r \in U\colon\ \{h^{-1}(r)\colon\ h \in H_1\} \cap \{h^{-1}(r)\colon\ h \in H_2\} = \varnothing\}$. Because $\mathcal{T}$ is a Hausdorff topology and $H_1 \cup H_2$ is finite, there are disjoint $\mathcal{T}$-open sets $W_1 \supseteq \{h^{-1}(r)\colon\ h \in H_1\}$ and $W_2 \supseteq \{h^{-1}(r)\colon\ h \in H_2\}$, where $r \in V$ is arbitrary. Let $W = U \cap \bigcap_{h\in H_1} h(W_1) \cap \bigcap_{h\in H_2} h(W_2)$. Then $W \in \mathcal{T}$ and $r \in W \subseteq V$; whence $V \in \mathcal{T}$. By definition of $\mathcal{F}$, $V \neq \varnothing$; hence $|V| = \kappa$. Let $r \in V\backslash\left(\bigcup_{h\in H_1} h(G) \cup \bigcup_{h\in H_2} h(F)\right)$. Then we have

$$\begin{aligned}\{h^{-1}(r)\colon h \in H_1\} \cap \{h^{-1}(r)\colon h \in H_2\} &= \{h^{-1}(r)\colon\ h \in H_1\} \cap G\\ &= \{h^{-1}(r)\colon\ h \in H_2\} \cap F = \varnothing.\end{aligned}$$

Set $A := F \cup \{h^{-1}(r)\colon\ h \in H_1\}$. Then $A \in [F, G]$. Also $\{h^{-1}(r)\colon\ h \in H_2\} \subseteq X\backslash A$, so we have $r \in U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$. Thus $A \in \mathcal{S}_{U,H_1,H_2}$, establishing density of $\mathcal{S}_{U,H_1,H_2}$ in $\mathcal{P}^\kappa(X)$.

Next suppose $A \in \mathcal{S}_{U,H_1,H_2}$, where $U \in \mathcal{B}$ and $H_1, H_2 \subseteq \mathcal{H}$ are finite. Let $r \in U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$, with $F := \{h^{-1}(r)\colon\ h \in H_1\}$, $G := \{h^{-1}(r)\colon\ h \in H_2\}$. Then $[F, G]$ is a basic open set in $\mathcal{P}^\omega(X)$, and $A \in [F, G]$. If $B \in [F, G]$, then we have $r \in U \cap \bigcap_{h\in H_1} h(B) \cap \bigcap_{h\in H_2} h(X\backslash B)$, so $B \in \mathcal{S}_{U,H_1,H_2}$. This establishes openness of $\mathcal{S}_{U,H_1,H_2}$.

To finish, suppose $A \in \mathcal{S}_{U,H_1,H_2}$ for all $\langle U, H_1, H_2\rangle \in \mathcal{F}$. We need to show that for all $U \in \mathcal{B}$ and finite

$$H_1, H_2 \subseteq \mathcal{H},\ \left|U \cap \bigcap_{h\in H_1} h(A) \cup \bigcap_{h\in H_2} h(X\backslash A)\right| \neq 1.$$

So pick $\langle U, H_1, H_2\rangle$ arbitrary. If $\langle U, H_1, H_2\rangle \in \mathcal{F}$, then let $r_0 \in U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$. We know that

$$V := \{r \in U\colon\ \{h^{-1}(r)\colon\ h \in H_1\} \cap \{h^{-1}(r)\colon\ h \in H_2\} = \varnothing\}$$

is nonempty and $\mathcal{T}$-open. So let $U' \in \mathcal{B}$ be nonempty and contained in $V\backslash\{r_0\}$. Then $\langle U', H_1, H_2\rangle \in \mathcal{F}$, so $A \in \mathcal{S}_{U',H_1,H_2}$. Thus $U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$ contains elements other than r_0, and is hence not a singleton. If

$\langle U, H_1, H_2\rangle \notin \mathcal{F}$, then for all $r \in U$, $\{h^{-1}(r)\colon h \in H_1\} \cap \{h^{-1}(r)\colon h \in H_2\} \neq \varnothing$. Thus for each $r \in U$, $r \notin \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A)$; consequently $U \cap \bigcap_{h\in H_1} h(A) \cap \bigcap_{h\in H_2} h(X\backslash A) = \varnothing$, and is not a singleton in this case either.

2. Applications of 1.8

2.1 THEOREM. *Assume κ is an infinite regular cardinal, $\langle X, \mathcal{T}\rangle$ an even space of cardinality κ, and $|H(\mathcal{T})| \leq \kappa$.*

(i) *If $\mathcal{T}$ is a connected topology, then there is a proper nonempty $A \subseteq X$ such that both $\mathcal{T}_A$ and $\mathcal{T}_{X\backslash A}$ are connected. If $\mathcal{T}$ is also a Hausdorff topology, then A can be found as above so that $\mathcal{T}_{\{A,X\backslash A\}}$ is perfect.*

(ii) *Under the hypothesis that $\kappa = 2^{<\kappa}$, the set A above may be taken to be a moiety of X.*

Proof. Ad(i). The map $B \mapsto X\backslash B$, complementation in $\mathcal{P}(X)$, is a homeomorphism on $\mathcal{P}^\kappa(X)$ (taking $[F, G]$ to $[G, F]$). Thus the sets $\mathcal{S}_1 := \{A \subseteq X\colon \mathcal{T}_A$ is connected$\}$ and $\mathcal{S}_2 := \{A \subseteq X\colon \mathcal{T}_{X\backslash A}$ is connected$\}$ are κ-residual in $\mathcal{P}^\kappa(X)$, by 1.8(ii). Thus $\mathcal{S}_1 \cap \mathcal{S}_2 = \{A \subseteq X\colon$ both $\mathcal{T}_A$ and $\mathcal{T}_{X\backslash A}$ are connected$\}$ is κ-residual in $\mathcal{P}^\kappa(X)$, hence infinite, and A can be found as claimed so that $\varnothing \neq A \neq X$. If $\mathcal{T}$ is also Hausdorff, we set $\mathcal{S}_3 := \{A \subseteq X\colon \mathcal{T}_{\{A,X\backslash A\}}$ is perfect$\}$. $\mathcal{S}_3$ is κ-residual in $\mathcal{P}^\kappa(X)$ by 1.8(iii), so $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$ is κ-residual. We then find A proper nonempty in this intersection.

Ad(ii). Assume $\kappa = 2^{<\kappa}$, and set $\mathcal{S}_4 := \{A \subseteq X\colon A$ is a moiety$\}$. Then $|\mathcal{P}(X)\backslash \mathcal{S}_4| = \kappa$. For each pair F, G of small subsets of X, let $\mathcal{S}_{F,G} = \{A \subseteq X\colon F \neq A \neq X\backslash G\}$. It is easy to show each $\mathcal{S}_{F,G}$ is dense open in $\mathcal{P}^\kappa(X)$, and $\mathcal{S}_4 = \bigcap\{\mathcal{S}_{F,G}\colon F, G$ small in $X\}$. Because $\kappa = 2^{<\kappa}$, $\mathcal{S}_4$ is κ-residual by 1.1(i). Thus the set A we want comes from $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_4$ (or from $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \cap \mathcal{S}_4$, depending).

2.2 THEOREM. *Let $\lambda \geq 1$, and assume $\mathfrak{c}^\lambda$ is a regular cardinal. Then there is a proper nonempty subset $A \subseteq \mathbb{R}^\lambda$, which can be taken to be a moiety if either $\lambda < \omega$ or $\mathfrak{c}^\lambda$ satisfies $2^{<\kappa} = \kappa$, such that $\mathcal{U}^\lambda_A$ and $\mathcal{U}^\lambda_{\mathbb{R}^\lambda/A}$ are both connected, and such that $\mathcal{U}^\lambda_{\{A,\mathbb{R}^\lambda\backslash A\}}$ is perfect.*

Proof. $\langle \mathbb{R}^\lambda, \mathcal{U}^\lambda\rangle$ is an even connected Hausdorff space, and $|H(\mathcal{U}^\lambda)| \leq (\mathfrak{c}^\lambda)^\lambda = \mathfrak{c}^\lambda$ (because $w(\mathcal{U}^\lambda) = \omega \cdot \lambda$, so there is a dense subset of cardinality $\omega \cdot \lambda$). Thus 2.1 applies. This takes care of the theorem, except for the case $\lambda < \omega$ and we want A to be a moiety. But then 0.10 is applicable: If A is not a moiety, then one of $\mathcal{U}^\lambda_A$, $\mathcal{U}^\lambda_{\mathbb{R}^\lambda\backslash A}$ contains a small nonempty open set, forcing discreteness.

The following application of 1.8 shows that the map $H(\cdot)$ from $HE(\mathcal{U})$ to the subgroup lattice of Sym $(\mathbb{R})$ is not join-preserving.

2.3 THEOREM. *Assume* $\mathfrak{c}$ *is a regular cardinal. There are H-enrichments* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *of* $\mathcal{U}$ *such that* $H(\mathcal{T}_1) \vee H(\mathcal{T}_2) = H(\mathcal{U})$, *but* $H(\mathcal{T}_1 \vee \mathcal{T}_2) \neq H(\mathcal{U})$. *Thus* $H(\cdot)$ *does not preserve lattice joins.*

Proof. By 2.2 there is a moiety $A \subseteq \mathbb{R}$ such that both $\mathcal{U}_A$ and $\mathcal{U}_{\mathbb{R}\setminus A}$ are connected. Let $\mathcal{T}_1 = \mathcal{U}_A$, $\mathcal{T}_2 = \mathcal{U}_{\mathbb{R}\setminus A}$. Then $H(\mathcal{T}_1) = H(\mathcal{T}_2) = H(\mathcal{U})$, by 0.7. Thus $H(\mathcal{T}_1) \vee H(\mathcal{T}_2) = H(\mathcal{U})$. However, $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{U}_{\{A, \mathbb{R}\setminus A\}}$ is nonconnected. Hence, by 4.2(i), $H(\mathcal{T}_1 \vee \mathcal{T}_2) \neq H(\mathcal{U})$. (Section 4 deals with how much larger $H(\mathcal{T})$ is than $H(\mathcal{U})$ when $\mathcal{T} \in H(\mathcal{U})$ is nonconnected.)

2.4 *Remark.* The hard work in proving 0.6 went in constructing a set $A \subseteq \mathbb{R}$ such that $\mathcal{U}_{\{A, \mathbb{R}\setminus A\}}$ is nondiscrete (which one can do without assuming $\mathfrak{c}$ is a regular cardinal). Showing complete regularity is easy.

2.5 PROPOSITION. *Let* $\mathcal{T}$ *be a topology on a set* X, *let* $\mathcal{H} \leqslant H(\mathcal{T})$, *and suppose* $\mathcal{A} \subseteq \mathcal{P}(X)$ *is closed under complementation. If* $\mathcal{T}$ *is regular (resp. completely regular, 0-dimensional), then so is* $\mathcal{T}^{\mathcal{H}}_{\mathcal{A}}$.

Proof. Set $\mathcal{T}' := \mathcal{T}^{\mathcal{H}}_{\mathcal{A}}$, and let $x \in T \in \mathcal{T}'$ be given. Assume first that $\mathcal{T}$ is regular. Then we may assume that T is basic open, so that $T = U \cap B$, where $U \in \mathcal{T}$ and $B := h_1(A_1) \cap \cdots \cap h_n(A_n)$ for some $\{h_1, \ldots, h_n\} \subseteq \mathcal{H}$ and $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$. Since each $A \in \mathcal{A}$ is $\mathcal{T}'$-clopen, so is B. Since $\mathcal{T}$ is regular, there is a $\mathcal{T}$-open neighborhood V of x with $cl_{\mathcal{T}}(V)$, the $\mathcal{T}$-closure of V, contained in U. Then we have $x \in V \cap B \subseteq cl_{\mathcal{T}'}(V \cap B) \subseteq cl_{\mathcal{T}'}(V) \cap cl_{\mathcal{T}'}(B) = cl_{\mathcal{T}'}(V) \cap B \subseteq cl_{\mathcal{T}}(V) \cap B \subseteq U \cap B = T$. Thus $\mathcal{T}'$ is regular.

Next let $\mathcal{T}$ be completely regular, $x \in T$ as above, and $T = U \cap B$. Then there is a continuous f: $\langle X, \mathcal{T}\rangle \to \langle \mathbb{R}, \mathcal{U}\rangle$ taking x to 0 and $X \setminus U$ to $\{1\}$. Define g; $X \to \mathbb{R}$ to agree with f on the $\mathcal{T}'$-clopen set B and to be constantly 1 on $X\setminus B$. Then g: $\langle X, \mathcal{T}'\rangle \to \langle \mathbb{R}, \mathcal{U}\rangle$ is clearly continuous, and takes x to 0 and $X\setminus T$ to $\{1\}$.

Finally let $\mathcal{T}$ be 0-dimensional, $x \in T = U \cap B$. Let C be a $\mathcal{T}$-clopen set with $x \in C \subseteq U$. Then $C \cap B$ is a $\mathcal{T}'$-clopen set with $x \in C \cap B \subseteq T$.

2.6 THEOREM. *Assume* κ *is an infinite regular cardinal,* $\langle X, \mathcal{T}\rangle$ *is an even space of cardinality* κ, *and* $|H(\mathcal{T})| \leqslant \kappa$. *If* $\mathcal{T}$ *is regular (resp. completely regular, 0-dimensional) Hausdorff, then* $\{A \subseteq X\colon \mathcal{T}_{\{A, X\setminus A\}}$ *is perfect regular (resp. perfect completely regular, perfect 0-dimensional)*$\}$ *is* κ*-residual in* $\mathcal{P}^{\kappa}(X)$.

Proof. Immediate from 1.8(iii) and 2.5.

2.7 COROLLARY. *Let* $\lambda \geqslant 1$, *and assume* $\mathfrak{c}^{\lambda}$ *is a regular cardinal. Then* $\{A \subseteq \mathbb{R}^{\lambda}\colon \mathcal{U}^{\lambda}_{\{A, \mathbb{R}^{\lambda}\setminus A\}}$ *is a perfect completely regular topology*$\}$ *is* $\mathfrak{c}^{\lambda}$*-residual in* $\mathcal{P}^{\mathfrak{c}^{\lambda}}(\mathbb{R}^{\lambda})$.

2.8 *Remark.* We have achieved in 2.7 a result with more generality than that of 0.6, but at the cost of having to assume cardinal regularity. (It is known that the regularity of $\mathfrak{c}$ is independent of the usual, i.e., *ZFC*, axioms of set theory.) An honest improvement on 0.6 is given in 2.12 below.

2.9 Proposition. *Let $\mathcal{T} \in HE(\mathcal{U})$ be nonconnected. For each $x \in \mathbb{R}$ and $\mathcal{U}$-neighborhood U of x, there is a $\mathcal{T}$-clopen set T with $x \in T \subseteq U$.*

Proof. Suppose $A \subseteq \mathbb{R}$ is a proper nonempty $\mathcal{T}$-clopen set, $a \in \mathbb{R}\backslash A$, and that $A' := (a, \infty) \cap A$ is nonempty. Then $A' = [a, \infty) \cap A$ is $\mathcal{T}$-clopen and bounded below. Pick $x \in A'$ and let h: $\mathbb{R} \to \mathbb{R}$ take $y \in \mathbb{R}$ to $2x - y$. Then $h \in H(\mathcal{T})$, hence $h(A')$ is $\mathcal{T}$-clopen, contains x, and is bounded above. Therefore $A'' := A' \cap h(A')$ is a $\mathcal{T}$-clopen neighborhood of x that is bounded. We can shrink A'' as small as we like, while keeping x fixed, using affine maps. This gives us the set T that we want.

2.10 Corollary. *Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ be closed under complementation, where $\mathcal{A}\backslash\mathcal{U} \neq \varnothing$. Then $\mathcal{U}_{\mathcal{A}}$ is 0-dimensional.*

Proof. Let $\mathcal{T} := \mathcal{U}_{\mathcal{A}}$, and suppose $x \in T \in \mathcal{T}$ is given. As in the proof of 2.5, $T = U \cap B$, where $U \in \mathcal{U}$ and B is $\mathcal{T}$-clopen. Because $\mathcal{A}\backslash\mathcal{U} \neq \varnothing$, $\mathcal{T}$ is nonconnected. Thus, by 2.9, there is a $\mathcal{T}$-clopen T' with $x \in T' \subseteq U$. Consequently, $T' \cap B$ is a $\mathcal{T}$-clopen neighborhood of x contained in T.

2.11 Corollary. *Assume $\mathfrak{c}$ is a regular cardinal. Then $\{A \subseteq \mathbb{R}$: $\mathcal{U}_{\{A, \mathbb{R}\backslash A\}}$ is a perfect 0-dimensional topology$\}$ is $\mathfrak{c}$-residual in $\mathcal{P}^{\mathfrak{c}}(\mathbb{R})$.*

Proof. Immediate from 1.8(iii) and 2.10.

The following is an immediate consequence of 2.10 and the proof of 0.6 in **[3]**.

2.12 Corollary. *$\mathcal{U}$ has a nondiscrete 0-dimensional H-enrichment.*

An interesting consequence of 0.5 and 2.5, concerning the rational line $\langle \mathbb{Q}, \mathcal{U}' \rangle$, is the following.

2.13 Corollary. *Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$ be closed under complementation, where $\mathcal{A}\backslash\mathcal{U}' \neq \varnothing$. Then $\mathcal{U}'_{\mathcal{A}} = \mathcal{D}$.*

2.14 *Remark.* A question remaining from **[3]** is whether $\mathcal{T} \in HE(\mathcal{U})$ can be both connected and (completely) regular. One easy fact, proved (in more generality) in **[3]**, is that if $A \notin \mathcal{U}$ is either residual or co-small, then $\mathcal{U}_A$ is both connected and nonregular. One might try to prove a version of "almost every $\mathcal{U}_A$ is connected and nonregular" by showing that $\{A \subseteq \mathbb{R}$: A is residual$\}$ is $\mathfrak{c}$-residual in $\mathcal{P}^{\mathfrak{c}}(\mathbb{R})$. But this is false, since otherwise we would also have the same for $\{A \subseteq \mathbb{R}$: $\mathbb{R}\backslash A$ is residual$\}$; so

$\varnothing = \{A \subseteq \mathbb{R}$: both A and $\mathbb{R} \backslash A$ are residual$\}$ would be $\mathfrak{c}$-residual in $\mathcal{P}^{\mathfrak{c}}(\mathbb{R})$, an impossibility. The problem looks quite difficult, and at this point we cannot even guess as to the outcome.

3. Maximal H-enrichments

Another way of obtaining H-enrichments is via Zorn's lemma. Let P be any topological property. Call P *inductive* if whenever each member of a chain of topologies on a set X satisfies P, then the join of those topologies satisfies P as well. Clearly the conjunction of any number of inductive properties is inductive; examples of inductive properties include: perfect; T_1; Hausdorff; (completely) regular; totally disconnected; 0-dimensional.

Let $\langle X, \mathcal{T}\rangle$ be a topological space. Then, because of Zorn's lemma and the fact that the partially ordered set $HE(\mathcal{T})$ is closed under topological joins, every member of $HE(\mathcal{T})$ satisfying the inductive property P can be enriched to a member of $HE(\mathcal{T})$ that is *H-maximal P*; i.e. maximal in $HE(\mathcal{T})$ with respect to satisfying P.

3.1 *Remark.* Obvious examples of noninductive properties include: nondiscrete; compact; non-Hausdorff. Less obvious is the property of connectedness. J. A. Guthrie and H. E. Stone have shown (see [**6**]) that there are connected topologies that cannot be enriched to maximal connected topologies. However, as is shown in [**6**] via two (nonconstructive) methods (one due to Guthrie–Stone, the other to M. Wage), $\mathcal{U}$ can be so enriched. Wage's enrichment is not an H-enrichment since it contains countably infinite open sets (viz. 0.10). It does not appear that an H-enrichment can be obtained using the Guthrie–Stone method either, so it seems to be an open question whether the euclidean topologies have H-maximal connected H-enrichments.

3.2 Proposition. *Let $1 \leq n < \omega$, with $\mathcal{T} \in HE(\mathcal{U}^n)$ either H-maximal connected or H-maximal nondiscrete. (N.B.: In this instance, "nondiscrete" is synonymous with "perfect," since $\mathcal{U}^n$ is homogeneous.) Then every small subset of $\mathbb{R}^n$ is $\mathcal{T}$-closed.*

Proof. Suppose $\mathcal{T}$ is an H-maximal connected H-enrichment of $\mathcal{U}^n$, and let $\mathcal{T}' \in HE(\mathcal{U}^n)$ be obtained from $\mathcal{T}$ by adding in all co-small subsets of $\mathbb{R}^n$. Because nonempty $\mathcal{T}$-open sets are not small, $\mathcal{T}'$ must be connected. (Use an argument similar to that in 1.8(ii).) By maximality of $\mathcal{T}$, we have $\mathcal{T} = \mathcal{T}'$. Similarly we have the assertion for $\mathcal{T}$ H-maximal nondiscrete.

The next results concern $w(\mathcal{T})$ when $\mathcal{T} \in HE(\mathcal{U}^n)$, $1 \leq n < \omega$. By 0.9 we know that $w(\mathcal{T}) \geq \omega_1$ when $\mathcal{T}$ is proper. By putting conditions on $\mathcal{T}$, and perhaps on the set-theoretic universe, we can improve this lower bound. It is well known that the hypothesis $2^{<\mathfrak{c}} < 2^{\mathfrak{c}}$ is a consequence of Martin's Axiom, and is hence strictly weaker than the Continuum Hypothesis.

3.3 THEOREM. *Assume $\mathfrak{c}$ is a regular cardinal and $2^{<\mathfrak{c}} < 2^{\mathfrak{c}}$. If $1 \leqslant n < \omega$ and $\mathcal{T} \in HE(\mathcal{U}^n)$ is H-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete), then $w(\mathcal{T}) \geqslant \mathfrak{c}$.*

Proof. Suppose $\mathcal{T}$ is H-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete), and assume $w(\mathcal{T}) < \mathfrak{c}$. By 0.10, $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is even. By 1.8(iii), then, with $\mathcal{H} = H(\mathcal{U}^n)$, we have that $\mathcal{S} := \{A \subseteq \mathbb{R}^n\colon\ \mathcal{T}_{\{A, \mathbb{R}^n \setminus A\}}$ is nondiscrete$\}$ is $\mathfrak{c}$-residual in $\mathcal{P}^{\mathfrak{c}}(\mathbb{R}^n)$. By 1.1(ii), $|\mathcal{S}| = 2^{\mathfrak{c}}$. Since $w(\mathcal{T}) < \mathfrak{c}$ and $2^{<\mathfrak{c}} < 2^{\mathfrak{c}}$, we know $|\mathcal{T}| < 2^{\mathfrak{c}}$; hence there is some $A \in \mathcal{S} \setminus \mathcal{T}$, and $\mathcal{T}^{\mathcal{H}}_{\{A, \mathbb{R}^n \setminus A\}}$ is a nondiscrete H-enrichment of $\mathcal{U}^n$ properly containing $\mathcal{T}$. By 2.5, $\mathcal{T}_{\{A, \mathbb{R}^n \setminus A\}}$ is regular (resp. completely regular, 0-dimensional). This contradicts the maximality of $\mathcal{T}$.

Under other maximality assumptions, we can get better bounds, as well as dispense with the hypothesis $2^{<\mathfrak{c}} < 2^{\mathfrak{c}}$.

3.4 THEOREM. *Assume $\mathfrak{c}$ is a regular cardinal, $1 \leqslant n < \omega$. If $\mathcal{T} \in HE(\mathcal{U}^n)$ is H-maximal connected, then $w(\mathcal{T}) \geqslant \mathfrak{c}^+$.*

Proof. Assume $w(\mathcal{T}) \leqslant \mathfrak{c}$ and $\mathcal{T}$ is H-maximal connected. $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is even by 0.10; so by 1.8(ii), with $\mathcal{H} = H(\mathcal{U}^n)$, there is a set $A \subseteq \mathbb{R}^n$ such that both $\mathcal{T}^{\mathcal{H}}_A$ and $\mathcal{T}^{\mathcal{H}}_{\mathbb{R}^n \setminus A}$ are connected. (A is, of course, a moiety, by 0.10.) Each of these topologies is in $HE(\mathcal{U}^n)$; so since $\mathcal{T}$ is maximal connected in $HE(\mathcal{U}^n)$, we have $\mathcal{T} = \mathcal{T}^{\mathcal{H}}_A$ and $\mathcal{T} = \mathcal{T}^{\mathcal{H}}_{\mathbb{R}^n \setminus A}$. This means that the moiety A is clopen in $\mathcal{T}$, a contradiction. Thus $w(\mathcal{T}) \geqslant \mathfrak{c}^+$.

3.5 COROLLARY. *Assume $\mathfrak{c}$ is a regular cardinal, $1 \leqslant n < \omega$. If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^n)$ has cardinality $\leqslant \mathfrak{c}$, then $\mathcal{U}^n_{\mathcal{A}}$ is not H-maximal connected.*

Aside from 0.7, we know very little about homeomorphism groups of H-enrichments. As we see in the sequel, one way of specifying $\mathcal{T}'$-homeomorphisms when $\mathcal{T}' \in HE(\mathcal{T})$ is via piecewise definition. Let $\langle X, \mathcal{T} \rangle$ be a topological space, with $\mathcal{T}' \in HE(\mathcal{T})$. Define $h \in \operatorname{Sym}(X)$ to be a *$\mathcal{T}'/\mathcal{T}$-homeomorphism* if there exists a family $\{\langle T_i, h_i \rangle\colon\ i \in I\}$ where $\{T_i\colon\ i \in I\}$ is a $\mathcal{T}'$-open cover of X and $\{h_i\colon\ i \in I\}$ is a family of $\mathcal{T}$-homeomorphisms such that for each $i \in I$, $h \mid T_i = h_i \mid T_i$. The family $\{\langle T_i, h_i \rangle\colon\ i \in I\}$ is said to be a *witness* for h; the set of $\mathcal{T}'/\mathcal{T}$-homeomorphisms is denoted $H(\mathcal{T}'/\mathcal{T})$.

In the case $X = \mathbb{R}$ and $\mathcal{T} = \mathcal{U}$, homeomorphisms in $H(\mathcal{T}'/\mathcal{U})$ may be appropriately described as "piecewise monotonic."

3.6 PROPOSITION. *Let $\langle X, \mathcal{T}\rangle$ be a topological space, with $T' \in HE(\mathcal{T})$. Then $H(\mathcal{T}) \leq H(\mathcal{T}'/\mathcal{T}) \leq H(\mathcal{T}')$.*

Proof. If $h \in H(\mathcal{T})$, then $\{\langle X, h\rangle\}$ is a witness for h; so $h \in H(\mathcal{T}'/\mathcal{T})$. If $h \in H(\mathcal{T}'/\mathcal{T})$ has witness $\{\langle T_i, h_i\rangle: i \in I\}$ and $T \in \mathcal{T}'$, then $h(T) = \bigcup_{i \in I} h(T \cap T_i) = \bigcup_{i \in I} h_i(T \cap T_i) \in \mathcal{T}'$. Also h^{-1} is witnessed by $\{\langle h_i(T_i), h_i^{-1}\rangle: i \in I\}$, so $h^{-1}(T) \in \mathcal{T}'$ as well. Thus $h \in H(\mathcal{T}')$.

We must show $H(\mathcal{T}'/\mathcal{T})$ is a group. From the last paragraph, we know $H(\mathcal{T}'/\mathcal{T})$ is closed under inverses; so let $g, h \in H(\mathcal{T}'/\mathcal{T})$, say g (resp. h) is witnessed by $\{\langle S_i, g_i\rangle: i \in I\}$ (resp. $\{\langle T_j, h_j\rangle: j \in J\}$). Then $g \circ h$ is witnessed by $\{\langle h_j^{-1}(S_i) \cap T_j, g_i \circ h_j\rangle: \langle i, j\rangle \in I \times J\}$.

3.7 COROLLARY. *Suppose $\langle X, \mathcal{T}\rangle$ is a space with $\mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{T})$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Then $H(\mathcal{T}_1/\mathcal{T}) \leq H(\mathcal{T}_2/\mathcal{T})$.*

Proof. It suffices to show $H(\mathcal{T}_1/\mathcal{T}) \subseteq H(\mathcal{T}_2)$, in view of 3.6 and the hypothesis that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Let $h \in H(\mathcal{T}_1/\mathcal{T})$ be witnessed by $\{\langle T_i, h_i\rangle: i \in I\}$. Then $h(T) = \bigcup_{i \in I} h_i(T \cap T_i) \in \mathcal{T}_2$ whenever $T \in \mathcal{T}_2$. Similarly $h^{-1}(T) \in \mathcal{T}_2$, so $h \in H(\mathcal{T}_2)$.

3.8 *Remark.* 3.7 says that the operator $H(\cdot/\mathcal{T})$ is order-preserving as a mapping between lattices; 2.3 says that $H(\cdot)$ is not join-preserving. In fact, if one picks $\mathcal{T}_1 = \mathcal{U}_A$ and $\mathcal{T}_2 = \mathcal{U}_{\mathbb{R}\setminus A}$ as in 2.3, then clearly $H(\mathcal{T}_1/\mathcal{U}) \vee H(\mathcal{T}_2/\mathcal{U}) = H(\mathcal{U})$. However, by 4.2(i), $H(\mathcal{T}_1 \vee \mathcal{T}_2/\mathcal{U}) \neq H(\mathcal{U})$. Thus $H(\cdot/\mathcal{T})$ is not generally join-preserving. The question of whether $H(\cdot)$ is generally order-preserving is more difficult and remains open.

Our next result concerns the cardinality of $H(\mathcal{T}/\mathcal{U})$ in 0-dimensional H-enrichments.

Call a set $A \subseteq \mathbb{R}$ *symmetric* if $-x \in A$ whenever $x \in A$. Clearly the set of symmetric subsets of $\mathbb{R}$ is a complete Boolean subalgebra of $\mathcal{P}(\mathbb{R})$. If $A \subseteq \mathbb{R}$, then we set $-A := \{-x: x \in A\}$, and observe that $A \cup (-A)$ is symmetric. The operation $A \mapsto A \cup (-A)$ is not one-one in general, but is so once we restrict attention to sets lying in $[0, \infty)$ (or in $(-\infty, 0]$).

Let $A \subseteq \mathbb{R}$ be symmetric, and define the function h_A on $\mathbb{R}$ to fix $x \in A$ and to take $x \in \mathbb{R}\setminus A$ to $-x$. Then clearly $h_A \in \mathrm{Sym}(\mathbb{R})$. Also, if $A, B \subseteq \mathbb{R}$ are symmetric sets and $A\setminus\{0\} \neq B\setminus\{0\}$, then $h_A \neq h_B$. Finally, if $\mathcal{T} \in HE(\mathcal{U})$ and $T \subseteq \mathbb{R}$ is a symmetric $\mathcal{T}$-clopen set, then $h_T \in H(\mathcal{T}/\mathcal{U})$.

The *character* of a topology, $\chi(\mathcal{T})$, is defined to be the smallest cardinality of a $\mathcal{T}$-neighborhood basis of a point. If $\langle X, \mathcal{T}\rangle$ is topological space, then $\chi(\mathcal{T}) \leq w(\mathcal{T}) \leq \chi(\mathcal{T}) \cdot |X|$. Because of 2.11, the hypothesis of the next result is reasonably general.

3.9 PROPOSITION. *Assume $\mathcal{T} \in HE(\mathcal{U})$ is 0-dimensional. Then $\chi(\mathcal{T}) \leqslant |H(\mathcal{T}/\mathcal{U})|$.*

Proof. Let $\chi(\mathcal{T}) = \kappa$. Because of homogeneity of $\mathcal{U}$ and the fact that $\mathcal{T}$ is 0-dimensional, there is a family $\{T_\xi\colon \xi < \kappa\}$ of distinct $\mathcal{T}$-clopen sets, each lying in the interval $(0, 1)$. Let $T'_\xi := T_\xi \cup (-T_\xi)$. Then T'_ξ is symmetric $\mathcal{T}$-clopen, and the sets T'_ξ, $\xi < \kappa$ are all distinct. Thus the $\mathcal{T}/\mathcal{U}$-homeomorphisms $h_{T'_\xi}$ are all distinct, so $|H(\mathcal{T}/\mathcal{U})| \geqslant \kappa$.

The following result is a companion to 3.3 and 3.4.

3.10 THEOREM. *Assume $\mathfrak{c}$ is a regular cardinal and that κ is any regular cardinal with $\mathfrak{c}^+ \leqslant \kappa \leqslant 2^{\mathfrak{c}}$. If $\mathcal{T} \in HE(\mathcal{U})$ is H-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete) and $w(\mathcal{T}) \leqslant \mathfrak{c}$, then $|H(\mathcal{T}/\mathcal{U})| \geqslant \kappa$.*

Proof. Assume $w(\mathcal{T}) \leqslant \mathfrak{c}$ and $\mathcal{T} \in HE(\mathcal{U})$ is H-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete). By 1.8(iii), with $\mathcal{H} = H(\mathcal{U})$, and 0.10, we have that $\mathcal{S} := \{A \subseteq \mathbb{R}\colon \mathcal{T}^{\mathcal{H}}_{\{A, \mathbb{R}\setminus A\}}$ is nondiscrete$\}$ is $\mathfrak{c}$-residual in $\mathcal{P}^{\mathfrak{c}}(\mathbb{R})$. By 2.5, each $\mathcal{T}^{\mathcal{H}}_{\{A, \mathbb{R}\setminus A\}}$ is regular (resp. completely regular, 0-dimensional); whence by maximality of $\mathcal{T}$, we have $\mathcal{T} = \mathcal{T}^{\mathcal{H}}_{\{A, \mathbb{R}\setminus A\}}$ for all $A \in \mathcal{S}$. Thus $\mathcal{T}$ contains a $\mathfrak{c}$-residual family of clopen sets. By 1.1, $|\mathcal{S}| = 2^{\mathfrak{c}}$, so there is a family $\{T_\xi\colon \xi < 2^{\mathfrak{c}}\}$ of distinct $\mathcal{T}$-clopen sets. Now $|H(\mathcal{U})| = \mathfrak{c}$, hence we may assume that for $\xi < \nu < 2^{\mathfrak{c}}$, there is no $h \in H(\mathcal{U})$ taking T_ξ to T_ν; i.e., the sets T_ξ are all in distinct $H(\mathcal{U})$-orbits. We may also assume, by 0.10, that each T_ξ is a moiety in $\mathbb{R}$. For each $\xi < 2^{\mathfrak{c}}$, there is a linear shift s_ξ (i.e., of the form $x \mapsto x + b$) taking T_ξ to a $\mathcal{T}$-clopen set that does not contain 0. All the sets $s_\xi(T_\xi)$ are distinct since the sets T_ξ lie in distinct $H(\mathcal{U})$-orbits.

Thus, without loss of generality, we may assume each T_ξ is a $\mathcal{T}$-clopen moiety that does not contain 0, all the sets T_ξ being distinct, $\xi < 2^{\mathfrak{c}}$. Let $T^+_\xi := T_\xi \cap (0, \infty)$ and $T^-_\xi := T_\xi \cap (-\infty, 0)$. Then $T^+_\xi = T_\xi \cap [0, \infty)$ and $T^-_\xi = T_\xi \cap (-\infty, 0]$, so the sets T^+_ξ, T^-_ξ are all $\mathcal{T}$-clopen. Now suppose that $|\{T^+_\xi\colon \xi < 2^{\mathfrak{c}}\}| < \kappa$, where κ is any regular cardinal with $\mathfrak{c}^+ \leqslant \kappa \leqslant 2^{\mathfrak{c}}$. Then there is a set $X \subseteq 2^{\mathfrak{c}}$, $|X| \geqslant \kappa$, such that for $\xi, \nu \in X$, $T^+_\xi = T^+_\nu$. Since for all $\xi \neq \nu$ in X we know $T_\xi \neq T_\nu$, it follows that the sets T^-_ξ are all distinct for $\xi \in X$. These observations allow us to assume without loss of generality that there is a family $\{T_\xi\colon \xi < \kappa\}$ of distinct $\mathcal{T}$-clopen moieties, each lying in the interval $(0, \infty)$. From this we conclude that the symmetric $\mathcal{T}$-clopen moieties $T'_\xi := T_\xi \cup (-T_\xi)$ are also distinct. Thus the corresponding $\mathcal{T}/\mathcal{U}$-homeomorphisms $h_{T'_\xi}$ are distinct as well, and we conclude $|H(\mathcal{T}/\mathcal{U})| \geqslant \kappa$.

3.11 COROLLARY. *Assume $\mathfrak{c}$ is a regular cardinal. If $\mathcal{T} \in HE(\mathcal{U})$ contains an H-maximal 0-dimensional nondiscrete H-enrichment of $\mathcal{U}$, then $|H(\mathcal{T}/\mathcal{U})| \geqslant \mathfrak{c}^+$.*

Proof. Assume first that $\mathcal{T}'$ itself is an H-maximal 0-dimensional nondiscrete H-enrichment of $\mathcal{U}$. If $w(\mathcal{T}') \leqslant \mathfrak{c}$, we infer $|H(\mathcal{T}'/\mathcal{U})| \geqslant \mathfrak{c}^+$ by 3.10. If $w(\mathcal{T}') \geqslant \mathfrak{c}^+$, then $\chi(\mathcal{T}') \geqslant \mathfrak{c}^+$; so $|H(\mathcal{T}'/\mathcal{U})| \geqslant \mathfrak{c}^+$ by 3.9. If $\mathcal{T} \in HE(\mathcal{U})$ contains $\mathcal{T}'$ as above, then $|H(\mathcal{T}/\mathcal{U})| \geqslant \mathfrak{c}^+$, by 3.7.

4. Permutations of the Real Line

In this section we concern ourselves with properties of the various homeomorphism groups $H(\mathcal{T})$ and $H(\mathcal{T}/\mathcal{U})$, for $\mathcal{T} \in HE(\mathcal{U})$, and their interrelationships.

For $h \in \mathrm{Sym}\,(X)$, we let $\mathrm{supp}\,(h) := \{x \in X\colon\ h(x) \neq x\}$, the *support* of h. Note that if $\langle X, S\rangle$ and $\langle Y, \mathcal{T}\rangle$ are topological spaces, $\mathcal{T}$ is a Hausdorff topology, and f, g: $\langle X, \mathcal{S}\rangle \to \langle Y, \mathcal{T}\rangle$ are continuous, then $\{x \in X\colon\ f(x) \neq g(x)\}$ is $\mathcal{S}$-open. In particular, we have:

4.1 Proposition. *Let $\langle X, \mathcal{T}\rangle$ be a Hausdorff space, $h \in H(\mathcal{T})$. Then $\mathrm{supp}\,(h)$ is $\mathcal{T}$-open.*

Let $\mathcal{T} \in HE(\mathcal{U})$. By 0.7, $H(\mathcal{T}) = H(\mathcal{U})$ if $\mathcal{T}$ is connected; by 0.8, $H(\mathcal{T}) = \mathrm{Sym}\,(\mathbb{R})$ if and only if $\mathcal{T} = \mathcal{D}$. To answer further the question (raised in 0.8) of exactly when $H(\mathcal{T}_1) = H(\mathcal{T}_2)$ is true for $\mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{U})$, we offer the following.

4.2 Proposition.

(i) *Let $\mathcal{T} \in HE(\mathcal{U})$. Then $H(\mathcal{T}) = H(\mathcal{U})$ if and only if $H(\mathcal{T}/\mathcal{U}) = H(\mathcal{U})$, if and only if $\mathcal{T}$ is connected.*

(ii) *Let $\mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{U})$, where $\mathcal{T}_1$ is 0-dimensional and $\mathcal{T}_1 \nsubseteq \mathcal{T}_2$. Then $H(\mathcal{T}_1/\mathcal{U}) \nsubseteq H(\mathcal{T}_2)$.*

Proof. Ad(i). If $\mathcal{T}$ is connected, then $H(\mathcal{T}) = H(\mathcal{T}/\mathcal{U}) = H(\mathcal{U})$, by 0.7. Suppose $\mathcal{T}$ is nonconnected. It suffices to show $H(\mathcal{T}/\mathcal{U}) \neq H(\mathcal{U})$. By 2.9, there is a $\mathcal{T}$-clopen moiety $T \subseteq (0, 1)$. As in the proof of 3.9, $T' := T \cup (-T)$ is a symmetric $\mathcal{T}$-clopen moiety, and $h_{T'} \in H(\mathcal{T}/\mathcal{U}) \setminus H(\mathcal{U})$.

Ad(ii). Suppose $\mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{U})$, $\mathcal{T}_1$ is 0-dimensional, and $\mathcal{T}_1 \nsubseteq \mathcal{T}_2$. Since the bounded $\mathcal{T}_1$-clopen sets form a basis for $\mathcal{T}_1$, there is a $\mathcal{T}_1$-clopen set $T \subseteq (0, 1)$ with $T \notin \mathcal{T}_2$. Then $T' := T \cup (-T)$ is also $\mathcal{T}_1$-clopen. However, if $T' \in \mathcal{T}_2$, then $T = T' \cap (0, 1) \in \mathcal{T}_2$ also. Consequently $T' \notin \mathcal{T}_2$. However, if $h = -h_{T'}$, i.e.,

$$h(x) = \begin{cases} -x & \text{if } x \in T' \\ x & \text{if } x \in \mathbb{R} \setminus T' \end{cases},$$

then $h \in H(\mathcal{T}_1/\mathcal{U})$. But $\mathrm{supp}\,(h) = T' \notin \mathcal{T}_2$, so $h \notin H(\mathcal{T}_2)$, by 4.1.

One can improve on 4.2(i) to show that when $\mathcal{T} \in HE(\mathcal{U})$ is not connected, $H(\mathcal{T}/\mathcal{U})$ is "significantly larger" than $H(\mathcal{U})$. By way of notation, if H is a subgroup of the group G, then $|G:H|$ is the index of H in G.

4.3 THEOREM. *Let $\mathcal{T} \in HE(\mathcal{U})$. Then there is a subgroup K of $H(\mathcal{T}/\mathcal{U})$ such that:*

(i) *K has exponent 2 (i.e., K consists of involutions), so is abelian.*
(ii) $|K \cap H(\mathcal{U})| = 2$.
(iii) *No $H(\mathcal{U})$-coset in $H(\mathcal{T}/\mathcal{U})$ contains more than 2 elements from K.*
(iv) *If $\mathcal{T}$ is nonconnected, then $|K| \geq \mathfrak{c}$; hence $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq \mathfrak{c}$.*

Proof. For $A \subseteq \mathbb{R}$ symmetric, define $h_A \in \mathrm{Sym}\,(\mathbb{R})$ as before:

$$h_A(x) = \begin{cases} x & \text{if } x \in A \\ -x & \text{if } x \in \mathbb{R}\backslash A \end{cases}.$$

Clearly h_A is an involution on $\mathbb{R}$, so has order 2 in $\mathrm{Sym}\,(\mathbb{R})$. Also note that if $\min\{|A|, |\mathbb{R}\backslash A|\} \geq 2$, then h_A is nonmonotonic, hence not in $H(\mathcal{U})$. Also $h_{\mathbb{R}\backslash A} = -h_A$, and $h_A \circ h_B = h_{(A\cap B)\cup(\mathbb{R}\backslash(A\cup B))}$ whenever $A, B \subseteq \mathbb{R}$ are both symmetric.

Suppose $\mathcal{T} \in HE(\mathcal{U})$ is nondiscrete. Then every nonempty proper $\mathcal{T}$-clopen set is a moiety by 0.10. Define $K := \{h_T$: T is a symmetric $\mathcal{T}$-clopen set$\}$.Then clearly $K \cap H(\mathcal{U}) = \{id_{\mathbb{R}}, -id_{\mathbb{R}}\}$. K is a subgroup since if S, T are symmetric and $\mathcal{T}$-clopen, then so is $(S \cap T) \cup (\mathbb{R}\backslash(S \cup T))$. Also K clearly has exponent 2. This establishes (i) and (ii) above. To establish (iii), suppose h_S, $h_T \in K$ are distinct. Then $S \neq T$. h_S and h_T lie in the same $H(\mathcal{U})$-coset if and only if $h_S \circ h_T^{-1} = h_S \circ h_T = h_{(S\cap T)\cup(\mathbb{R}\backslash(S\cup T))} \in H(\mathcal{U})$, if and only if $(S \cap T) \cup (\mathbb{R}\backslash(S \cup T)) \in \{\mathbb{R}, \varnothing\}$, if and only if $S = T$ or $S = \mathbb{R}\backslash T$. Thus no $H(\mathcal{U})$-coset in $H(\mathcal{T}/\mathcal{U})$ contains more than two elements of K.

Now suppose $\mathcal{T}$ is nonconnected, and let $n \in \{1, 2, \ldots\}$. By 2.9 there is a $\mathcal{T}$-clopen neighborhood S_n of n diameter $<\frac{1}{3}$. Let $T_n := S_n \cup (-S_n)$. For each $X \subseteq \{1, 2, \ldots\}$, let $T_X = \bigcup_{n\in X} T_n$. Then T_X is a symmetric $\mathcal{T}$-clopen moiety. Of course if X and Y are distinct subsets of $\{1, 2, \ldots\}$, then $T_X \neq T_Y$, hence $h_{T_X} \neq h_{T_Y}$. Thus $|K| \geq \mathfrak{c}$. Since no $H(\mathcal{U})$-coset in $H(\mathcal{T}/\mathcal{U})$ can contain more than two elements of K, we infer that $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq \mathfrak{c}$.

In the event $\mathcal{T}$ is discrete, we replace $\mathcal{T}$ with a nondiscrete H-enrichment of $\mathcal{U}$. Of course, in this case $|H(\mathcal{T}/\mathcal{U})| = |H(\mathcal{T})| = 2^{\mathfrak{c}}$; consequently $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| = 2^{\mathfrak{c}}$.

4.4 *Remarks.*

(i) Of course, under the hypothesis of 3.11, we have $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq \mathfrak{c}^+$. Thus the problem of when one can improve the lower bound in 4.3(iv) is inevitable.

(ii) Let $\mathrm{Sym}^{\kappa}(X)$ denote $\mathrm{Sym}(X)$ viewed as a subspace of $(X^X)_{\kappa}$. It is easy to show that $H(\mathcal{U})$ is closed in $\mathrm{Sym}^{\omega}(\mathbb{R})$, as well as nowhere dense in $\mathrm{Sym}^{c}(\mathbb{R})$. For a subgroup G of $\mathrm{Sym}(X)$, to be closed in $\mathrm{Sym}^{\omega}(X)$ means that there is a first order relational structure $\mathcal{R}$ on X such that $G = \mathrm{Aut}(X, \mathcal{R})$, the group of $\mathcal{R}$-automorphisms.
In the particular instance of $\mathcal{U}$, the structure $\mathcal{R}$ is the ternary relation of betweenness.

(iii) The subgroup G of $\mathrm{Sym}(X)$ is *α-transitive,* for α a cardinal, if whenever $f \in \mathrm{Sym}(X)$ and $F \subseteq X$ has cardinality α, there is some $g \in G$ such that $g \mid F = f \mid F$. So to say that G is dense in $\mathrm{Sym}^{\kappa}(X)$, for κ an infinite cardinal, is to say G is α-transitive for all $\alpha < \kappa$. $H(\mathcal{U})$ is 2-transitive but not 3-transitive. By contrast, $H(\mathcal{U}^n)$, $n \geqslant 2$, is k-transitive for all $k < \omega$; i.e., $H(\mathcal{U}^n)$ is "highly transitive." Equivalently, $H(\mathcal{U}^n)$ is dense in $\mathrm{Sym}^{\omega}(\mathbb{R}^n)$.

4.5 PROPOSITION. *Suppose $\mathcal{T} \in HE(\mathcal{U})$ is nonconnected. Then $H(\mathcal{U})$ is closed nowhere dense in $H(\mathcal{T}/\mathcal{U}) \subseteq \mathrm{Sym}^{\omega}(\mathbb{R})$.*

Proof. $H(\mathcal{U})$ is closed in $\mathrm{Sym}^{\omega}(\mathbb{R})$, so it is closed in the subspace $H(\mathcal{T}/\mathcal{U})$. To show $H(\mathcal{U})$ is nowhere dense, it suffices to show $H(\mathcal{T}/\mathcal{U})\backslash H(\mathcal{U})$ is dense in $H(\mathcal{T}/\mathcal{U})$. Let $h \in H(\mathcal{U})$, with $F \subseteq \mathbb{R}$ finite. Let $N := \{g \in H(\mathcal{T}/\mathcal{U}): g \mid F = h \mid F\}$. We show $N \cap (H(\mathcal{T}/\mathcal{U})\backslash H(\mathcal{U})) \neq \varnothing$. By the argument in 4.3 (iv) there is a $\mathcal{T}$-clopen symmetric moiety T with $F \subseteq T$. Let $g = h \circ h_T$. Since $H(\mathcal{T}/\mathcal{U})$ is a group (by 3.6), $g \in H(\mathcal{T}/\mathcal{U})$. If $x \in F$, then we have $g(x) = h(x)$, so $g \in N$. T is a moiety, and g is monotonic in opposite directions on T and $\mathbb{R}\backslash T$. Thus $g \notin H(\mathcal{U})$.

Another way of showing $H(T/\mathcal{U})$ is "significantly larger" than $H(\mathcal{U})$ for $\mathcal{T} \in HE(\mathcal{U})$ nonconnected is to show $H(\mathcal{U})$ to be nonmaximal in $H(\mathcal{T}/\mathcal{U})$, i.e., there is some $h \in H(\mathcal{T}/\mathcal{U})\backslash H(\mathcal{U})$ such that the subgroup of $H(\mathcal{T}/\mathcal{U})$ generated by $H(\mathcal{U}) \cup \{h\}$ is a proper subgroup.

4.6 THEOREM. *Suppose $\mathcal{T} \in HE(\mathcal{U})$ is nonconnected. Then $H(\mathcal{U})$ is not maximal in $H(\mathcal{T}/\mathcal{U})$.*

Proof. Define $g \in H(\mathcal{T}/\mathcal{U})$ to be *pleasant* if: (i) for all $A \subseteq \mathbb{R}$, A is bounded if and only if $g(A)$ is bounded; and (ii) there is a bounded set R_1 and a cobounded set R_2 such that $R_1 \cap R_2 = \varnothing$ and g is monotonic on both R_1 and R_2, but with opposite parity (e.g., g is increasing on R_1 and decreasing on R_2). Let $P := \{g \in H(\mathcal{T}/\mathcal{U});\ g \text{ is pleasant}\}$.

LEMMA. *P is a subgroup of $H(\mathcal{T}/\mathcal{U})$ that contains $H(\mathcal{U})$.*

Proof (*Lemma*). Clearly $H(\mathcal{U}) \subseteq P$. If $g \in P$ then obviously g^{-1} satisfies (i) above. Suppose g is increasing on R_1 (bounded) and decreasing on R_2 (cobounded). Then g^{-1} is increasing on $g(R_1)$ (bounded) and decreasing on $g(R_2)$ (cobounded). Also $g(R_1) \cap g(R_2) = g(R_1 \cap R_2) = g(\emptyset) = \emptyset$. The other cases are treated similarly, and we conclude P is closed under inverses.

Now suppose $g, h \in P$. Then clearly $g \circ h$ satisfies (i) above. Suppose g is increasing on R_1 (bounded), decreasing on R_2 (cobounded), $R_1 \cap R_2 = \emptyset$; and suppose h is increasing on S_1 (cobounded), decreasing on S_2 (bounded), $S_1 \cap S_2 = \emptyset$. Then $g \circ h$ is increasing on $S_1 \cap h^{-1}(R_1)$ as well as on $S_2 \cap h^{-1}(R_2)$; $g \circ h$ is decreasing on $S_1 \cap h^{-1}(R_2)$, as well as on $S_2 \cap h^{-1}(R_1)$. Thus $g \circ h$ is increasing on $S_1 \cap h^{-1}(R_1)$ (bounded) and decreasing on $S_1 \cap h^{-1}(R_2)$ (cobounded). These two sets have empty intersection, so we infer that $g \circ h \in P$. The remaining cases are treated similarly, so we conclude that P is a subgroup of $H(\mathcal{T}/\mathcal{U})$. (Lemma)

Suppose T is a symmetric $\mathcal{T}$-clopen set. Then for any $A \subseteq \mathbb{R}$, $h_T(A) = h_T((A \cap T) \cup (A \backslash T)) = h_T(A \cap T) \cup h_T(A \backslash T) = (A \cap T) \cup (-(A \backslash T))$ (where h_T is defined as in 4.3). Thus h_T satisfies clause (i) in the definition of "pleasant." Suppose now T is a bounded symmetric $\mathcal{T}$-clopen set that is also a moiety. Then $h_T \in P \backslash H(\mathcal{U})$. By 2.9 we know such sets T are in abundance; in particular $P \neq H(\mathcal{U})$. We can also find symmetric $\mathcal{T}$-clopen sets T such that both T and $\mathbb{R} \backslash T$ are unbounded. For such a T, $h_T \notin P$. Thus $H(\mathcal{U})$ fails (quite dramatically) to be maximal in $H(\mathcal{T}/\mathcal{U})$.

The results 4.3, 4.5 and 4.6 discuss ways in which we can say $H(\mathcal{U})$ is "small" (or at least "not very large") in $H(\mathcal{T}/\mathcal{U})$ whenever $\mathcal{T} \in HE(\mathcal{U})$ is nonconnected. Now we shift perspective and look at how $H(\mathcal{T})$ sits in $\mathrm{Sym}(\mathbb{R})$. In one sense, that of index, we show that $H(\mathcal{T})$ is "small" whenever $\mathcal{T} \neq \mathcal{D}$. We first cite an obvious consequence of the Lemma on p. 581 in [**5**].

4.7 THEOREM ([**5**, Lemma for Theorem 2]). *Let X be an infinite set of cardinality κ, with $G \leqslant \mathrm{Sym}(X)$ a subgroup of index $<2^\kappa$ (i.e., of "small index"). Then there is a set $M \subseteq X$, with $|X \backslash M| = \kappa$, such that if $g \in \mathrm{Sym}(X)$ and $g \mid M = id_M$, then $g \in G$.*

4.8 THEOREM. *Let $\langle X, \mathcal{T} \rangle$ be a Hausdorff space such that $|\mathrm{Sym}(X) : H(\mathcal{T})| < 2^{|X|}$. Then $\mathcal{T}$ has $|X|$ isolated points.*

Proof. If X is finite, then $\mathcal{T} = \mathcal{D}$ immediately. So assume X is infinite. By 4.7 there is a subset $M \subseteq X$, with $|X \backslash M| = |X|$, such that every $h \in \mathrm{Sym}(X)$ that fixes the points of M is a $\mathcal{T}$-homeomorphism. So let $x, y \in X \backslash M$ be distinct, and let $h \in \mathrm{Sym}(X)$ exchange x and y, leaving everything else fixed. Then $h \in H(\mathcal{T})$, and hence $\{x, y\} = \mathrm{supp}(h) \in \mathcal{T}$ by 4.1 (since $\mathcal{T}$ is Hausdorff). Thus x is an isolated point. Since x was chosen arbitrarily from $X \backslash M$, a set of cardinality $|X|$, there are $|X|$ $\mathcal{T}$-isolated points.

4.9 Corollary. *Let* $1 \leq \lambda$, *with* $\mathcal{T} \in HE(\mathcal{U}^\lambda)$ *nondiscrete. Then* $|\mathrm{Sym}(\mathbb{R}^\lambda) : H(\mathcal{T})| = 2^{c^\lambda}$.

Proof. $\mathcal{T}$ is homogeneous and Hausdorff. Use 4.8.

In another sense, $H(\mathcal{T})$ is "large" in Sym $(\mathbb{R})$ whenever $\mathcal{T} \in HE(\mathcal{U})$ is nonconnected. We show that $H(\mathcal{T}/\mathcal{U})$ is dense in $\mathrm{Sym}^\omega(\mathbb{R})$, i.e., that $H(\mathcal{T}/\mathcal{U})$ is highly transitive.

4.10 Theorem. *Let* $\mathcal{T} \in HE(\mathcal{U})$ *be nonconnected, and let* $A \subseteq \mathbb{R}$ *be a* $\mathcal{U}$*-closed* $\mathcal{U}$*-discrete set. Then every permutation on* A *extends to a* $\mathcal{T}/\mathcal{U}$*-homeomorphism.*

Proof. Let $s \in \mathrm{Sym}(A)$. We need to find $h \in H(\mathcal{T}/\mathcal{U})$ such that $h \mid A = s$. Now symmetric groups are well known to be generated by involutions (i.e., elements of order 2), so we may as well assume $s^2 = id_A$. Then we have a partition $A = A_0 \cup A_1 \cup A_2$, where s takes A_0 to A_1 and vice versa, and fixes A_2. A is $\mathcal{U}$-discrete, hence countable since $w(\mathcal{U}) = \omega$. Because $\mathcal{U}$ is also a regular topology, there exist sets I_a, U_a, for $a \in A$, such that: (i) I_a is a closed interval containing a in its $\mathcal{U}$-interior, (ii) $I_a \subseteq U_a \in \mathcal{U}$; and (iii) for each $a, b \in A$ distinct, $U_a \cap I_b = \varnothing$. Since A is $\mathcal{U}$-closed, so is $B := \bigcup_{a \in A} I_a$. For each $a \in A_0$ pick, using 2.9, a $\mathcal{T}$-clopen neighborhood T_a of a, with $T_a \subseteq I_a$, and let $h_a \in H(\mathcal{U})$ take I_a monotonic increasingly onto $I_{s(a)}$, with $h_a(a) = s(a)$. Define

$$h(x) = \begin{cases} h_a(x) & \text{if } x \in T_a,\ a \in A_0 \\ h_a^{-1}(x) & \text{if } x \in h_a(T_a),\ a \in A_0 \\ x & \text{otherwise.} \end{cases}$$

The sets $\{T_a, h_a(T_a) \colon a \in A_0\}$ are pairwise disjoint and $\mathcal{T}$-clopen, so h is well defined. h is clearly a bijection on $\mathbb{R}$ extending s. To show $h \in H(\mathcal{T}/\mathcal{U})$ it remains to prove that $T = \bigcup_{a \in A_0} (T_a \cup h_a(T_a))$ is $\mathcal{T}$-closed. Then h and h^{-1} will have been shown to be $\mathcal{U}$-continuous when restricted to the elements of a $\mathcal{T}$-clopen partition.

So suppose $x \in B \backslash T$, say $x \in I_a$. Then $U_a \backslash T_a$ is a $\mathcal{T}$-open neighborhood of x missing T, hence T is $\mathcal{T}$-closed in B. Since B is $\mathcal{U}$-closed in $\mathbb{R}$, we infer that T is indeed a $\mathcal{T}$-clopen set.

4.11 Corollary. *Let* $\mathcal{T} \in HE(\mathcal{U})$ *be nonconnected. Then* $H(\mathcal{T}/\mathcal{U})$ *is highly transitive, thus dense in* $\mathrm{Sym}^\omega(\mathbb{R})$.

Proof. Let $1 \leq k < \omega$, $a_1 < \cdots < a_k$, $b_1, \ldots, b_k$ distinct. Let s be a permutation on $\{1, \ldots, k\}$ such that $b_{s(1)} < \cdots < b_{s(k)}$. Let $g \in H(\mathcal{U})$ take a_i to $b_{s(i)}$, $1 \leq i \leq k$; and, by 4.10, let $h \in H(\mathcal{T}/\mathcal{U})$ take b_i to $b_{s(i)}$, $1 \leq i \leq k$. Then $h^{-1} \circ g \in H(\mathcal{T}/\mathcal{U})$ takes a_i to b_i, $1 \leq i \leq k$.

Another way in which $H(\mathcal{T}/\mathcal{U})$ is "like" Sym $(\mathbb{R})$ when $\mathcal{T} \in HE(\mathcal{U})$ is 0-dimensional is that elements of $H(\mathcal{T}/\mathcal{U})$ need not respect boundedness in $\mathbb{R}$.

4.12 THEOREM. *Let $\mathcal{T} \in HE(\mathcal{U})$ be 0-dimensional. Then there is some $h \in H(\mathcal{T}/\mathcal{U})$ and a bounded set $A \subseteq \mathbb{R}$ such that $h(A)$ is unbounded.*

Proof. Let $A = \left\{\frac{1}{n+2}\colon n < \omega\right\}$, $B = \{n+2\colon n < \omega\}$. We find $h \in H(\mathcal{T}/\mathcal{U})$ taking $\frac{1}{n+2}$ to $n+2$ for each $n < \omega$. Now the sequence $\left(\frac{1}{n+2}\right)$ cannot $\mathcal{T}$-converge, by 0.9, so there is a $T \in \mathcal{T}$ with $0 \in T$ and $T \cap A = \varnothing$. By 2.9 we can arrange that $T \subseteq (-1, 1)$. Since $\mathcal{T}$ is assumed to be 0-dimensional, we can arrange for T to be $\mathcal{T}$-closed as well. For each $a \in A \cup B$ choose I_a, U_a as in the proof of 4.10. Then $B := \{0\} \cup \bigcup_{a \in A \cup B} I_a$ is $\mathcal{U}$-closed. For each $\frac{1}{n} \in A$, let h_n take $I_{1/n}$ monotonic increasingly onto I_n, with $h_n\left(\frac{1}{n}\right) = n$, and let $T_n \subseteq I_{1/n} \backslash T$ be a $\mathcal{T}$-clopen neighborhood of $\frac{1}{n}$. Define

$$h(x) = \begin{cases} h_n(x) & \text{if } x \in T_n, \quad 2 \leqslant n < \omega \\ h_n^{-1}(x) & \text{if } x \in h_n(T_n), \quad 2 \leqslant n < \omega \\ x & \text{otherwise.} \end{cases}$$

Then h takes A to B as advertised. That $h \in H(\mathcal{T}/\mathcal{U})$ is proved as in the proof of 4.10; one needs to check that $T' = \bigcup_{2 \leqslant n < \omega} (T_n \cup h_n(T_n))$ is $\mathcal{T}$-closed. But T' is $\mathcal{T}$-closed in B (T is a $\mathcal{T}$-neighborhood of 0 missing T'), which in turn is $\mathcal{U}$-closed in $\mathbb{R}$.

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