H-ENRICHMENTS AND THEIR HOMEOMORPHISM GROUPS

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0. Introduction

The study of H-enrichments of topologies arose as a spin-off of the notion of minimal freeness in universal algebra [1], especially as it applies to rings of continuous real-valued functions [2]: If \( \langle X, \mathcal{T} \rangle \) is a real-compact topological space whose ring of continuous functions is minimally free with pseudobasis of cardinality \( \lambda \), then we may view \( \mathcal{T} \) as an H-enrichment of the usual (product) topology \( \mathcal{U}^\lambda \) on the cartesian power \( \mathbb{R}^\lambda \) of the real line. H-enrichments were recognized as interesting in their own right, and their study was initiated properly in [3]. The present paper is a continuation of that study.

Let \( \mathcal{T} \) and \( \mathcal{T}' \) be two topologies on a set \( X \), with \( \mathcal{T}' \) an enrichment of \( \mathcal{T} \), i.e., \( \mathcal{T} \subseteq \mathcal{T}' \). \( \mathcal{T}' \) is an H-enrichment of \( \mathcal{T} \) if every homeomorphism on \( \langle X, \mathcal{T} \rangle \) is also a homeomorphism on \( \langle X, \mathcal{T}' \rangle \). Letting \( H(\mathcal{T}) \) denote the group of all homeomorphisms on \( \langle X, \mathcal{T} \rangle \) (the underlying set \( X \) being understood), this condition simply says that the group \( H(\mathcal{T}) \) is a subgroup of the group \( H(\mathcal{T}') \), in symbols, \( H(\mathcal{T}) \leq H(\mathcal{T}') \). We are interested here in part in how \( H(\mathcal{T}') \) sits between \( H(\mathcal{T}) \) and the full symmetric group \( \text{Sym}(X) \).

For a topological space \( \langle X, \mathcal{T} \rangle \), let \( HE(\mathcal{T}) \) be the set of all H-enrichments of \( \mathcal{T} \). \( HE(\mathcal{T}) \) is a complete lattice under intersection and topological join; it is a bounded lattice, with \( \mathcal{T} \) at the bottom and the discrete topology (generically denoted \( \mathcal{D} \)) at the top. We are also interested in properties of the map \( H(\cdot) \) from \( HE(\mathcal{T}) \) into the lattice of subgroups of \( \text{Sym}(X) \).

Although certain of our results are stated in a general context, the major applications concern spaces related to the real line \( \mathbb{R} \) with its usual topology \( \mathcal{U} \) (e.g., \( \langle \mathbb{R}^\lambda, \mathcal{U}^\lambda \rangle \), also the rational line \( \langle \mathbb{Q}, \mathcal{U}' \rangle \)). Our notation is pretty standard regarding ordinal and cardinal numbers:

\( \omega := \{0, 1, \ldots \} \) is the first infinite ordinal (cardinal); each ordinal is the set of its predecessors; \( \kappa^+ \) is the cardinal successor of the cardinal \( \kappa \); the notation \( \kappa^\lambda \), when \( \kappa \) and \( \lambda \) are cardinals, indicates the set of functions from \( \lambda \) into \( \kappa \), as well as the cardinality of that set; \( \kappa^{<\lambda} := \sup \{ \kappa^\alpha : \alpha < \lambda \} \); \( |X| \) is the cardinality of the set \( X \); \( c := |\mathbb{R}| = 2^\omega \). A subset \( Y \subseteq X \) is small (resp. co-small, a moiety) if \( |Y| \leq |X| \) (resp. \( |X\setminus Y| < |X| \), \( |Y| = |X\setminus Y| \)). (The term "moiety" is the coinage of P. M. Neumann.)

We first collect some results about \( H \)-enrichments proved in [3]. The first one says they are easy to come by.

**0.1 Theorem ([3, Theorem 2.1]).** If \( \langle X, \mathcal{T} \rangle \) has a nonclosed set that is nowhere dense, then \( \mathcal{T} \) has a proper nondiscrete \( H \)-enrichment.

A natural way to form \( H \)-enrichments of \( \langle X, \mathcal{T} \rangle \) is to let \( \mathcal{A} \) be any family of subsets of \( X \) (in symbols, \( \mathcal{A} \subseteq \mathcal{P}(X) \)), and let \( \mathcal{T}_{\mathcal{A}} \) be the smallest \( H \)-enrichment of \( \mathcal{T} \) containing all the sets in \( \mathcal{A} \). A typical basic open set for \( \mathcal{T}_{\mathcal{A}} \) looks like \( U \cap h_1(A_1) \cap \cdots \cap h_n(A_n) \), where \( U \in \mathcal{T} \), \( \{h_1, \ldots, h_n\} \subseteq H(\mathcal{T}) \), and \( \{A_1, \ldots, A_n\} \subseteq \mathcal{A} \). (Equivalently, \( \mathcal{T} \cup \{h(A): h \in H(\mathcal{T}), A \in \mathcal{A} \} \) forms a subbasis for \( \mathcal{T}_{\mathcal{A}} \).) When \( \mathcal{A} = \{A\} \), we set \( \mathcal{T}_A := \mathcal{T}_{\{A\}} \). One problem is controlling the behavior of \( \mathcal{T}_{\mathcal{A}} \), given \( \mathcal{T} \) and \( \mathcal{A} \).

**0.2 Theorem ([3, Theorem 2.4]).** Suppose \( \langle X, \mathcal{T} \rangle \) is a space, and the complement of every member of \( \mathcal{A} \subseteq \mathcal{P}(X) \) is \( \mathcal{T} \)-nowhere dense. Then every \( \mathcal{T} \)-dense set in \( X \) is also \( \mathcal{T}_{\mathcal{A}} \)-dense. Thus if \( \mathcal{T} \) is nondiscrete, so also is \( \mathcal{T}_{\mathcal{A}} \).

**0.3 Proposition ([3, Proposition 2.5]).** Let \( A \subseteq \mathbb{R} \) be \( \mathcal{U} \)-closed but not \( \mathcal{U} \)-open. Then \( \mathcal{U}_A = \emptyset \).

Several of our main results involve Baire category arguments. Define a space \( \langle X, \mathcal{T} \rangle \) to be \( \kappa \)-Baire, where \( \kappa \) is an infinite cardinal, if the intersection of at most \( \kappa \) dense open subsets of \( \langle X, \mathcal{T} \rangle \) is dense. \( Y \subseteq X \) is \( \kappa \)-residual if \( Y \) contains such an intersection.

**0.4 Theorem ([3, Theorem 2.9]).** Let \( \langle X, \mathcal{T} \rangle \) be an \( \omega \)-Baire space, with \( \mathcal{A} \) a family of \( T \)-\( \omega \)-residual sets. Then:

(i) If \( \mathcal{T} \) is connected, so is \( \mathcal{T}_A \).

(ii) If \( \mathcal{T}_{\mathcal{A}} \neq \mathcal{T} \), then \( \mathcal{T}_{\mathcal{A}} \) is nonregular.

Preservation of regularity (complete regularity, normality, etc.) as one passes from \( \mathcal{T} \) to \( \mathcal{T}' \in HE(\mathcal{T}) \) is an important issue that was addressed in [3].

**0.5 Theorem ([3, Theorem 2.12]).** \( \langle \mathbb{Q}, \mathcal{U}' \rangle \) has no proper nondiscrete regular \( H \)-enrichment.

On the other hand:

**0.6 Theorem ([3, Theorem 2.13]).** \( \langle \mathbb{R}, \mathcal{U} \rangle \) has proper nondiscrete completely regular \( H \)-enrichment.

The topology constructed for 0.6 by R. A. McCoy, using a rather involved double induction on \( c \), is nonconnected. A question left over from that paper is whether there exists a proper connected completely regular \( H \)-enrichment of \( \mathcal{U} \).
0.7 Theorem ([3, Theorem 2.16]). Let $\mathcal{I} \in HE(\mathcal{U})$. Then either $\mathcal{I}$ is totally disconnected or the $\mathcal{I}$-connected subsets of $\mathbb{R}$ are precisely the intervals. In the latter case, $H(\mathcal{I}) = H(\mathcal{U}) = \{\text{the monotonic bijections on } \mathbb{R}\}$.

0.8 Remark. 0.7 suggests the question: For $\mathcal{I}_1, \mathcal{I}_2 \in HE(\mathcal{U})$, precisely when is it true that $H(\mathcal{I}_1) = H(\mathcal{I}_2)$? We consider this question in Section 4. Part of the answer is: $H(\mathcal{I}) = H(\mathcal{U})$ if and only if $\mathcal{I}$ is connected (see Proposition 4.2); $H(\mathcal{I}) = H(\mathcal{D}) = \text{Sym} (\mathbb{R})$ if and only if $\mathcal{I} = \mathcal{D}$. The situation in the latter case is quite general (and easy). Let $\langle X, \mathcal{I} \rangle$ be an infinite space such that there is some moiety $Y \subseteq X$ that is $\mathcal{I}$-open, and suppose $H(\mathcal{I}) = \text{Sym} (X)$. Let $x \in X$ be arbitrary, with $R, S$ two disjoint moieties on $X \setminus \{x\}$, and let $g, h \in \text{Sym} (X)$ take $Y$ to $R \cup \{x\}$ and $S \cup \{x\}$ respectively. Then $\{x\} = (R \cup \{x\}) \cap (S \cup \{x\}) \in \mathcal{I}$, so $\mathcal{I} = \mathcal{D}$.

The last preliminary fact we mention concern $H$-enrichments of the euclidean topologies $\mathcal{U}^n$, $1 \leq n < \omega$.

0.9 Theorem ([3, Theorem 2.19]). Let $1 \leq n < \omega$, with $\mathcal{I}$ a proper $H$-enrichment of $\mathcal{U}^n$. Then every $\mathcal{I}$-convergent sequence (i.e., $\omega$-sequence) in $\mathbb{R}^n$ is eventually constant. (Consequently: $\mathbb{R}^n$ has no infinite $\mathcal{I}$-compact subsets; no point of $\mathbb{R}^n$ has a countable neighborhood basis, unless $\mathcal{I} = \mathcal{D}$; and every metrizable subset of $\langle \mathbb{R}^n, \mathcal{I} \rangle$ is discrete in the subspace topology.)

0.10 Theorem ([3, Theorem 2.21]). Let $1 \leq n < \omega$, with $\mathcal{I}$ a nondiscrete $H$-enrichment of $\mathcal{U}^n$. Then every nonempty $\mathcal{I}$-open set has cardinality $c$.

A quick summary of the four sections that make up the sequel is as follows.

Section 1 uses a Baire category result from [4] to prove that, under certain conditions on $\langle X, \mathcal{I} \rangle$, we have that "almost all" (read "residually many") $A \subseteq X$ give rise to $H$-enrichments $\mathcal{I}_A$ (or $\mathcal{I}_{(A, X, A)}$) with a specific property (Theorem 1.8). This theorem is applied repeatedly in the next two sections, so can be fairly regarded as the "keystone" result of the paper.

Section 2 consists almost entirely of direct applications of 1.8 in the context of euclidean space. One such is a simple strengthening of 0.6 to read that the usual topology $\mathcal{I}$ on $\mathbb{R}$ has a nondiscrete 0-dimensional $H$-enrichment. The price to pay for avoiding a messy inductive argument, however, is the assumption that $c$ is a regular cardinal.

In Section 3, 1.8 is used to study $H$-enrichments of $\mathcal{I}$ that are maximal (in the complete lattice $HE(\mathcal{I})$) with respect to having a given property. Theorem 3.4 says that if $c$ is a regular cardinal, $1 \leq n < \omega$, and $\mathcal{I} \in HE(\mathcal{U}^n)$ is maximal connected, then $\mathcal{I}$ has no open basis of cardinality $c$. 
Section 4 is concerned mainly with group-theoretic properties of $H(\mathcal{F})$ for $\mathcal{F} \in HE(\mathcal{U})$. For example, if $\mathcal{F}$ is nonconnected, then $H(\mathcal{F})$ is $k$-transitive for all $k < \omega$ (Corollary 4.11) ($H(\mathcal{U})$ is not 3-transitive). We also consider the relative positioning of $H(\mathcal{U})$ in $H(\mathcal{F})$ and $H(\mathcal{\overline{F}})$ in Sym ($\mathbb{R}$).

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1. $H$-enrichments via Baire category arguments

A very powerful, but highly nonconstructive, way of getting $H$-enrichments is to use Baire category arguments applied to power sets fitted out with appropriate topologies.

We first introduce an important mechanism (read "coreflection") for enriching topologies. Let $\langle X, \mathcal{F} \rangle$ be a space, $\kappa$ an infinite cardinal. Define the topology $(\mathcal{F})_{\kappa}$ to be the smallest enrichment $\mathcal{F}'$ of $\mathcal{F}$ such that intersections of $<\kappa$ $\mathcal{F}'$-open sets are still $\mathcal{F}'$-open. $(\mathcal{F})_{\kappa}$ is called the $\kappa$-modification of $\mathcal{F}$, and is a very special $H$-enrichment of $\mathcal{F}$. If $\mathcal{X} := \langle X, \mathcal{F} \rangle$, we set $(\mathcal{X})_{\kappa} := \langle X, (\mathcal{F})_{\kappa} \rangle$.

If $X$ is an infinite set, we identify the power set $\mathcal{P}(X)$ with the cartesian power $2^X$ in the standard way: $\langle $subset$\rangle \mapsto \langle $characteristic map$\rangle$. If $\kappa$ is an infinite cardinal, $\mathcal{P}^{\kappa}(X)$ is the space $\langle \mathcal{P}(X), \mathcal{F} \rangle$, where $\mathcal{F}$ is the "inclusion-exclusion" topology basically generated by sets $[F, G] := \{A \subseteq X : F \subseteq A \subseteq X \setminus G\}$, $F, G \subseteq X$ of cardinality $<\kappa$. When $\kappa$ is a regular cardinal, i.e., not the supremum of any increasing sequence $\langle \alpha_\xi : \xi < \lambda \rangle$ of ordinals where $\lambda < \kappa$ and $\alpha_\xi < \kappa$ for all $\xi < \lambda$, then it is easy to see that $\mathcal{P}^{\kappa}(X)$ is just $(2^X)_\kappa$ ($2^X$ being given the usual product topology).

The main results of this section are based on work of H. H. Hung and S. Negrepontis [4].

1.1 Theorem ([4, Theorem 15.8]). Let $\kappa$ be an infinite regular cardinal. Then: (i) $(2^X)_{\kappa}$ is a $\kappa$-Baire space; and (ii) if $\kappa$ is not weakly compact, then
any intersection of \( \leq \kappa \) dense open subsets of \((2^\kappa)^\kappa\) is homeomorphic to \((2^\kappa)^\kappa\).

1.2 Remark. Weak compactness is a "large cardinal" property which is larger than strong inaccessibility, and which we need not elaborate upon here; except to say that \( \omega \) has the property and every other cardinal we will be dealing with (e.g., \( \omega_1 \), \( \omega_3 \), \( \varepsilon \)) does not. It is proved in [4, Corollary 15.7] that a cardinal \( \kappa \) is weakly compact if and only if \((2^\kappa)^\kappa\) and \((\kappa^\kappa)^\kappa\) are nonhomeomorphic. Thus, when \( \kappa \) is regular and not weakly compact, \((\kappa^\kappa)^\kappa\) is \( \kappa \)-Baire. (This also holds when \( \kappa = \omega \), but in this case complete metrizability is the reason.)

Let \( \langle X, \mathcal{T} \rangle \) be a topological space. As noted above, one can parametrize all of \( HE(\mathcal{T}) \) using \( \mathcal{P}(\mathcal{P}(X)) \), i.e., for \( \mathcal{A} \subseteq \mathcal{P}(X) \), we have \( \mathcal{T}_\mathcal{A} \in HE(\mathcal{T}) \). Let \( \kappa = 2^{2^{\aleph_1}} \). Then appropriate basic open sets for \( \mathcal{P}^\kappa(\mathcal{P}(X)) \) look like \( [\mathcal{F}, \mathcal{G}] \), where \( \mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X) \) are small. Recall that a space is perfect if it has no isolated points.

1.3 Proposition. Let \( \langle X, \mathcal{T} \rangle \) be an infinite topological space. Then \( \{\mathcal{A} \subseteq \mathcal{P}(X); \mathcal{T}_\mathcal{A} \text{ is nonperfect}\} \) is dense open in \( \mathcal{P}^\kappa(\mathcal{P}(X)) \), where \( \kappa = 2^{2^{\aleph_1}} \).

Proof. Let \( [\mathcal{F}, \mathcal{G}] \neq \emptyset \) be a basic open set in \( \mathcal{P}^\kappa(\mathcal{P}(X)) \). For any infinite set \( S \), there are \( 2^{2^{\aleph_1}} \) moieties on \( S \). [Indeed, if \( |\{T \subseteq S: |T| < |S|\}| < 2^{2^{\aleph_1}} \), then the conclusion is immediate; if \( |\{T \subseteq S: |T| < |S|\}| = 2^{2^{\aleph_1}} \), then every such \( T \) can be expanded to a moiety on \( S \) in a one-one fashion.] So let \( A \subseteq X \) be a moiety that is not in \( \mathcal{G} \). Fix \( a \in A \). Since \( |X/A| = |X| \), there are \( \kappa \) sets \( B \cup \{a\} \), where \( B \subseteq X/A \). Find \( B \subseteq X/A \) with \( B \cup \{a\} \notin \mathcal{G} \), and set \( \mathcal{A} := \mathcal{F} \cup \{A, B \cup \{a\}\} \). Then \( \mathcal{A} \in [\mathcal{F}, \mathcal{G}] \) and \( \mathcal{T}_\mathcal{A} \) contains \( \{a\} \). This establishes density.

Next suppose \( \mathcal{A} \subseteq \mathcal{P}(X) \) is such that \( \mathcal{T}_\mathcal{A} \) is nonperfect. Then for some \( U \in \mathcal{T} \), \( h_1, \ldots, h_\kappa \in H(\mathcal{T}) \), and \( A_1, \ldots, A_\kappa \in \mathcal{A} \), we have \( |U \cap h_1(A_1) \cap \cdots \cap h_\kappa(A_\kappa)| = 1 \). If \( \mathcal{B} \) contains \( \{A_1, \ldots, A_\kappa\} \), then \( \mathcal{T}_\mathcal{B} \) has an isolated point. Thus \( \{\{A_1, \ldots, A_\kappa\}, \emptyset\} \) is an open neighborhood of \( \mathcal{A} \) (in \( \mathcal{P}^\kappa(\mathcal{P}(X)) \), in fact) such that for every member \( \mathcal{B} \), \( \mathcal{T}_\mathcal{B} \) has an isolated point. This establishes openness.

If \( \langle X, \mathcal{T} \rangle \) is a homogeneous topological space (i.e., \( H(\mathcal{T}) \) acts transitively on \( X \)), then clearly every \( H \)-enrichment of \( \mathcal{T} \) is also homogeneous. In the context of homogeneity, e.g., when we are talking about \( H \)-enrichments of \( \mathcal{U}^1 \) or of \( \mathcal{U}' \), "perfect" is synonymous with "nondiscrete."

1.4 Corollary. Let \( \langle X, \mathcal{T} \rangle \) be a homogeneous topological space. Then \( \{\mathcal{A} \subseteq \mathcal{P}(X); \mathcal{T}_\mathcal{A} = \emptyset\} \) is dense open in \( \mathcal{P}^\kappa(\mathcal{P}(X)) \), where \( \kappa = 2^{2^{\aleph_1}} \).
1.5 Proposition. Let \(\langle X, \mathcal{T}\rangle\) be a topological space, \(\kappa := |X| \geq \omega\). Then \(\{A \subseteq X : \mathcal{T}_A \text{ has a small nonempty set}\}\) is dense in \(\mathcal{P}^*(X)\).

Proof. Let \([F, G] \neq \emptyset\) be a basic open set in \(\mathcal{P}^*(X)\). If \(F \neq \emptyset\), we have \(F \in [F, G]\) and \(\mathcal{T}_F\) contains the small set \(F\). If \(F = \emptyset\), then since \(G\) is small, we can find some \(x \in X \setminus G\). Then \(\{x\} \in [F, G]\) and \(\mathcal{T}_{(x)}\) contains the small set \(\{x\}\).

1.6 Corollary. Let \(1 \leq n < \omega\). Then \(\{A \subseteq \mathbb{R}^n : \mathcal{U}_A = \emptyset\}\) is dense in \(\mathcal{P}(\mathbb{R}^n)\).

Proof. Immediate from 1.5 and 0.10.

The main result of this section stands in contrast to 1.3–1.6. Define the weight of a topological space \(\langle X, \mathcal{T}\rangle\) to be the least cardinal \(\kappa \geq \omega\) such that \(\mathcal{T}\) has an open basis of cardinality \(\kappa\); denote this number by \(w(\mathcal{T})\). (So \(w(\mathcal{U}^\lambda) = \omega \cdot \lambda\) for \(\lambda \geq 1\).) Let \(\mathcal{H} \leq H(\mathcal{T})\), \(\mathcal{A} \subseteq \mathcal{P}(X)\). The partial \(H\)-enrichment \(\mathcal{T}^\mathcal{A}\) is the smallest enrichment of \(\mathcal{T}\) in which each \(A \in \mathcal{A}\) is open and such that each \(h \in \mathcal{H}\) is a homeomorphism. It is easy to see that if \(\mathcal{B} \subseteq \mathcal{T}\) is an open basis for \(\mathcal{T}\), then \(\mathcal{B} \cup \{h(A) : h \in \mathcal{H}, A \in \mathcal{A}\}\) forms a subbasis for \(\mathcal{T}^\mathcal{A}\). When \(\mathcal{H} = H(\mathcal{T})\), we have \(\mathcal{T}^\mathcal{H} = \mathcal{T}\) as above. (We will have occasion to be given \(\mathcal{T}\) and \(\mathcal{T}_1 \in \mathcal{H}(\mathcal{T})\), and need to find \(\mathcal{T}_2 \in \mathcal{H}(\mathcal{T})\) with \(\mathcal{T}_1 \subseteq \mathcal{T}_2\). \(\mathcal{T}_2\) will be constructed to be of the form \((\mathcal{T}_1)\) where \(\mathcal{H} = H(\mathcal{T})\).)

Define the space \(\langle X, \mathcal{T}\rangle\) to be even if: (i) \(|U| = |X|\) for every nonempty \(U \in \mathcal{T}\); and (ii) \(w(\mathcal{T}) \leq |X|\).

1.7 Examples. Let \(\lambda \geq 1\). Then the generalized euclidean space \(\langle \mathbb{R}^\lambda, \mathcal{U}^\lambda\rangle\) is even. The space \(\langle \mathbb{Q}, \mathcal{U}'\rangle\) of rational numbers is also even. The argument in [3] to prove 0.9 also shows that if \(\mathcal{T} \in \mathcal{H}(\mathcal{U}')\) is proper, then every \(\mathcal{T}\)-convergent sequence in \(\mathbb{Q}\) is eventually constant. Thus if \(\mathcal{T}\) is also nondiscrete (a possibility guaranteed by 0.1), then \(w(\mathcal{T}) \geq \omega_1\); hence \(\langle \mathbb{Q}, \mathcal{T}\rangle\) is not even.

1.8 Theorem. Assume \(\kappa\) is an infinite regular cardinal, \(\langle X, \mathcal{T}\rangle\) is an even space of cardinality \(\kappa\), and \(\mathcal{H} \leq H(\mathcal{T})\) has cardinality \(\leq \kappa\).

(i) If \(\mathcal{T}\) is a \(T_1\) topology, then \(\{A \subseteq X : \mathcal{T}_A^\mathcal{H}\text{ is perfect}\}\) is \(\kappa\)-residual in \(\mathcal{P}^*(X)\).

(ii) If \(\mathcal{T}\) is a connected topology, then \(\{A \subseteq X : \mathcal{T}_A^\mathcal{H}\text{ is connected}\}\) is \(\kappa\)-residual in \(\mathcal{P}^*(X)\).

(iii) If \(\mathcal{T}\) is a Hausdorff topology, then \(\{A \subseteq X : \mathcal{T}_{(A, X \setminus A)}^\mathcal{H}\text{ is perfect}\}\) is \(\kappa\)-residual in \(\mathcal{P}^*(X)\).

Proof. Ad (i). Let \(\mathcal{B} \subseteq \mathcal{T}\) be an open basis of cardinality \(\leq \kappa\). A typical basic open set for \(\mathcal{T}_A^\mathcal{H}\) then looks like \(U \cap \bigcap_{h \in H} h(A)\), where \(U \in \mathcal{B}\) and \(H \subseteq \mathcal{H}\) is finite. Let \(\mathcal{F} = \{(U, H) : U \in \mathcal{B}\setminus\{\emptyset\} \text{ and } H \subseteq \mathcal{H}\text{ is finite}\}\).
Define \( \mathcal{F}_{U,H} := \{ A \subseteq X : U \cap \bigcap_{h \in H} h(A) \neq \emptyset \} \) for \( \langle U, H \rangle \in \mathcal{F} \). By hypothesis, \( |\mathcal{F}| \leq \kappa \). By 1.1(i) it remains to show that each \( \mathcal{F}_{U,H} \) is dense open in \( \mathcal{P}^\kappa(X) \) and that \( \{ A \subseteq X : \mathcal{F}_A^\kappa \text{ is perfect} \} \supseteq \bigcap \{ \mathcal{F}_{U,H} : \langle U, H \rangle \in \mathcal{F} \} \).

First let \([F, G] \neq \emptyset\) be basic open in \( \mathcal{P}^\kappa(X) \), so \( F, G \subseteq X \) are disjoint and of cardinality \( < \kappa \). Set \( A := X \setminus G \). Then \( A \in [F, G] \). Because \( |U| = \kappa \), we have \( U \cap \bigcap_{h \in H} h(A) = U \setminus \left( \bigcup_{h \in H} h(G) \right) \neq \emptyset \). Thus \( A \in [F, G] \cap \mathcal{F}_{U,H} \) and \( \mathcal{F}_{U,H} \) is dense in \( \mathcal{P}^\kappa(X) \).

Next let \( A \in \mathcal{F}_{U,H} \) with \( r \in U \cap \bigcap_{h \in H} h(A) \). Let \( F := \{ h^{-1}(r) : h \in H \} \). Then \( F \subseteq A \) is finite, so \( A \in [F, \emptyset] \). Also if \( B \in [F, \emptyset] \) then clearly \( r \in U \cap \bigcap_{h \in H} h(B) \); whence \( B \in \mathcal{F}_{U,H} \). This says that \( \mathcal{F}_{U,H} \) is open in \( \mathcal{P}^\kappa(X) \) (even in \( \mathcal{P}^{\kappa(\alpha)}(X) \)).

Finally suppose \( A \in \bigcap \{ \mathcal{F}_{U,H} : \langle U, H \rangle \in \mathcal{F} \} \). We must show that for \( U \in \mathcal{B} \) and \( H \subseteq \mathcal{H} \) finite, \( \left| U \cap \bigcap_{h \in H} h(A) \right| \neq 1 \). If \( U = \emptyset \), the cardinality is 0; if \( U \neq \emptyset \), then \( \langle U, H \rangle \in \mathcal{F} \). Let \( r \in U \cap \bigcap_{h \in H} h(A) \). Then there is some \( V \in \mathcal{B} \setminus \{ \emptyset \} \), \( V \subseteq U \setminus \{ r \} \), since \( \mathcal{F} \) is a \( T_1 \) topology. Then \( \langle V, H \rangle \in \mathcal{F} \), so \( A \in \mathcal{F}_{V,H} \). This tells us that \( U \cap \bigcap_{h \in H} h(A) \) contains elements other than \( r \), and is hence not a singleton.

Ad(ii). Use the proof above to show that \( \mathcal{F}_{U,H} \) is dense open in \( \mathcal{P}^\kappa(X) \) for \( \langle U, H \rangle \in \mathcal{F} \). Assuming \( \mathcal{F} \) is connected and \( A \in \bigcap \{ \mathcal{F}_{U,H} : \langle U, H \rangle \in \mathcal{F} \} \), let \( \mathcal{V} \) be a cover of \( X \) by \( \mathcal{F}_A^\kappa \)-basic open sets, with \( a, b \in X \) arbitrary and \( V_a, V_b \in \mathcal{V} \) containing \( a \) and \( b \) respectively. It suffices to find \( V_1, \ldots, V_m \in \mathcal{V} \) with \( V_1 = V_a, V_m = V_b \), and \( V_i \cap V_{i+1} \neq \emptyset \) for \( 1 \leq i \leq m - 1 \). For each \( \mathcal{F}_A^\kappa \)-basic open set \( V \), let \( U_V \in \mathcal{B}, H_V \subseteq \mathcal{H} \) finite be such that \( V = U_V \cap \bigcap_{h \in H_V} h(A) \). Let \( \mathcal{V}' = \{ U_V : V \in \mathcal{V} \} \). Then \( \mathcal{V}' \) is a \( \mathcal{F} \)-open cover of \( X \). Since \( \mathcal{F} \) is a connected topology, there exist \( V_1, \ldots, V_m \in \mathcal{V} \) such that \( V_1 = V_a, V_m = V_b \), and setting \( U_i = U_{V_i}, H_i = H_{V_i}, 1 \leq i \leq m - 1 \), we have \( U_i \cap U_{i+1} \neq \emptyset \). Now for each \( i \), \( V_i \cap V_{i+1} = (U_i \cap U_{i+1}) \cap \bigcap_{h \in H_i \cup H_{i+1}} h(A) \). Since \( U_i \cap U_{i+1} \neq \emptyset \), we know that \( A \in \mathcal{F}_{U \cap U_{i+1}, H_i \cup H_{i+1}} \) for \( 1 \leq i \leq m - 1 \). Thus \( V_i \cap V_{i+1} \neq \emptyset \), and \( \mathcal{F}_A^\kappa \) is therefore connected.

Ad(iii). Let \( \mathcal{B} \) as be above. A typical basic open set for \( \mathcal{F}_{(A,X \setminus A)}^\kappa \) looks like \( U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A) \), where \( U \in \mathcal{B} \) and \( H_1, H_2 \subseteq \mathcal{H} \) are finite. \( \mathcal{F}_{(A,X \setminus A)}^\kappa \) is perfect if and only if no such set is a singleton.

Let \( \mathcal{F} = \{ \langle U, H_1, H_2 \rangle : U \in \mathcal{B}, H_1, H_2 \subseteq \mathcal{H} \text{ are finite, and there is some } r \in U \text{ with } \{ h^{-1}(r) : h \in H_1 \} \cap \{ h^{-1}(r) : h \in H_2 \} = \emptyset \} \). For each
\( \langle U, H_1, H_2 \rangle \in \mathcal{F} \), let \( \mathcal{S}_{U,H_1,H_2} := \left\{ A \subseteq X : \bigcup_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A) \neq \emptyset \right\} \). We know \( |\mathcal{F}| \leq \kappa \), so it suffices to show that each such \( \mathcal{S}_{U,H_1,H_2} \) is dense open in \( \mathcal{P}^*(X) \) and that \( \{ A \subseteq X : \mathcal{F}_{U,H_1,H_2}(A) \text{ is perfect} \} \supseteq \bigcap \{ \mathcal{S}_{U,H_1,H_2} : \langle U, H_1, H_2 \rangle \in \mathcal{F} \} \).

First fix \( \mathcal{S}_{U,H_1,H_2} \) for \( \langle U, H_1, H_2 \rangle \in \mathcal{F} \), and let \( [F, G] \neq \emptyset \) be basic open in \( \mathcal{P}^*(X) \). Let \( V := \{ r \in U : \{ h^{-1}(r) : h \in H_1 \} \cap \{ h^{-1}(r) : h \in H_2 \} = \emptyset \} \). Because \( \mathcal{F} \) is a Hausdorff topology and \( H_1 \cup H_2 \) is finite, there are disjoint \( \mathcal{F} \)-open sets \( W_1 \supseteq \{ h^{-1}(r) : h \in H_1 \} \) and \( W_2 \supseteq \{ h^{-1}(r) : h \in H_2 \} \), where \( r \in V \) is arbitrary. Let \( W = U \cap \bigcup_{h \in H_1} h(W_1) \cap \bigcup_{h \in H_2} h(W_2) \). Then \( W \in \mathcal{F} \) and \( r \in W \in V \); whence \( V \in \mathcal{F} \). By definition of \( \mathcal{F} \), \( V \neq \emptyset \); hence \( |V| = \kappa \). Let \( r \in V \setminus \left( \bigcup_{h \in H_1} h(G) \cup \bigcup_{h \in H_2} h(F) \right) \). Then we have

\[
\{ h^{-1}(r) : h \in H_1 \} \cap \{ h^{-1}(r) : h \in H_2 \} = \{ h^{-1}(r) : h \in H_1 \} \cap G = \{ h^{-1}(r) : h \in H_2 \} \cap F = \emptyset.
\]

Set \( A := F \cup \{ h^{-1}(r) : h \in H_1 \} \). Then \( A \in [F, G] \). Also \( \{ h^{-1}(r) : h \in H_2 \} \subseteq X \setminus A \), so we have \( r \in U \cap \bigcup_{h \in H_1} h(A) \cap \bigcup_{h \in H_2} h(X \setminus A) \). Thus \( A \in \mathcal{S}_{U,H_1,H_2} \), establishing density of \( \mathcal{S}_{U,H_1,H_2} \) in \( \mathcal{P}^*(X) \).

Next suppose \( A \in \mathcal{S}_{U,H_1,H_2} \), where \( U \in \mathcal{B} \) and \( H_1, H_2 \subseteq \mathcal{H} \) are finite. Let \( r \in U \cap \bigcup_{h \in H_1} h(A) \cap \bigcup_{h \in H_2} h(X \setminus A) \), with \( F := \{ h^{-1}(r) : h \in H_1 \} \), \( G := \{ h^{-1}(r) : h \in H_2 \} \). Then \( [F, G] \) is a basic open set in \( \mathcal{P}^*(X) \), and \( A \in [F, G] \). If \( B \in [F, G] \), then we have \( r \in U \cap \bigcup_{h \in H_1} h(B) \cap \bigcup_{h \in H_2} h(X \setminus B) \), so \( B \in \mathcal{S}_{U,H_1,H_2} \). This establishes openness of \( \mathcal{S}_{U,H_1,H_2} \).

To finish, suppose \( A \in \mathcal{S}_{U,H_1,H_2} \) for all \( \langle U, H_1, H_2 \rangle \in \mathcal{F} \). We need to show that for all \( U \in \mathcal{B} \) and finite

\[
H_1, H_2 \subseteq \mathcal{H}, \quad \left| U \cap \bigcap_{h \in H_1} h(A) \cup \bigcap_{h \in H_2} h(X \setminus A) \right| \neq 1.
\]

So pick \( \langle U, H_1, H_2 \rangle \) arbitrary. If \( \langle U, H_1, H_2 \rangle \in \mathcal{F} \), then let \( r_0 \in U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A) \). We know that

\[
V := \{ r \in U : \{ h^{-1}(r) : h \in H_1 \} \cap \{ h^{-1}(r) : h \in H_2 \} = \emptyset \}
\]

is nonempty and \( \mathcal{F} \)-open. So let \( U' \in \mathcal{B} \) be nonempty and contained in \( V \setminus \{ r_0 \} \). Then \( \langle U', H_1, H_2 \rangle \in \mathcal{F} \), so \( A \in \mathcal{S}_{U',H_1,H_2} \). Thus \( U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A) \) contains elements other than \( r_0 \), and is hence not a singleton. If
\[\langle U, H_1, H_2 \rangle \notin \mathcal{F},\] then for all \( r \in U, \) \( \{h^{-1}(r) : h \in H_1\} \cap \{h^{-1}(r) : h \in H_2\} \neq \emptyset.\] Thus for each \( r \in U, \) \( r \notin \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A);\) consequently \( U \cap \bigcap_{h \in H_1} h(A) \cap \bigcap_{h \in H_2} h(X \setminus A) = \emptyset,\) and is not a singleton in this case either.

2. Applications of 1.8

2.1 Theorem. Assume \( \kappa \) is an infinite regular cardinal, \( \langle X, \mathcal{T} \rangle \) an even space of cardinality \( \kappa, \) and \( |H(\mathcal{T})| \leq \kappa.\)

(i) If \( \mathcal{T} \) is a connected topology, then there is a proper nonempty \( A \subset X \) such that both \( \mathcal{F}_A \) and \( \mathcal{F}_{X \setminus A} \) are connected. If \( \mathcal{T} \) is also a Hausdorff topology, then \( A \) can be found as above so that \( \mathcal{T}_{\{A, X \setminus A\}} \) is perfect.

(ii) Under the hypothesis that \( \kappa = 2^{<\kappa}, \) the set \( A \) above may be taken to be a moiety of \( X.\)

Proof. Ad(i). The map \( B \mapsto X \setminus B, \) complementation in \( \mathcal{P}(X), \) is a homeomorphism on \( \mathcal{P}(X) \) (taking \([F, G] \) to \([G, F]\)). Thus the sets \( \mathcal{S}_1 := \{A \subset X : \mathcal{F}_A \text{ is connected}\} \) and \( \mathcal{S}_2 := \{A \subset X : \mathcal{F}_{X \setminus A} \text{ is connected}\} \) are \( \kappa \)-residual in \( \mathcal{P}(X), \) by 1.8(ii). Thus \( \mathcal{S}_1 \cap \mathcal{S}_2 = \{A \subset X : \text{both } \mathcal{F}_A \text{ and } \mathcal{F}_{X \setminus A} \text{ are connected}\} \) is \( \kappa \)-residual in \( \mathcal{P}(X), \) hence infinite, and \( A \) can be found as claimed so that \( \emptyset \neq A \neq X.\) If \( \mathcal{T} \) is also Hausdorff, we set \( \mathcal{S}_3 := \{A \subset X : \mathcal{T}_{\{A, X \setminus A\}} \text{ is perfect}\}. \) \( \mathcal{S}_3 \) is \( \kappa \)-residual in \( \mathcal{P}(X) \) by 1.8(iii), so \( \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \) is \( \kappa \)-residual. We then find \( A \) proper nonempty in this intersection.

Ad(ii). Assume \( \kappa = 2^{<\kappa}, \) and set \( \mathcal{S}_4 := \{A \subset X : A \text{ is a moiety}\}. \) Then \( |\mathcal{P}(X) \setminus \mathcal{S}_4| = \kappa. \) For each pair \( F, G \) of small subsets of \( X, \) let \( \mathcal{S}_{F, G} = \{A \subset X : F \neq A \neq X \setminus G\}. \) It is easy to show each \( \mathcal{S}_{F, G} \) is dense open in \( \mathcal{P}(X), \) and \( \mathcal{S}_4 = \bigcap \{\mathcal{S}_{F, G} : F, G \text{ small in } X\}. \) Because \( \kappa = 2^{<\kappa}, \) \( \mathcal{S}_4 \) is \( \kappa \)-residual by 1.1(i). Thus the set \( A \) we want comes from \( \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_4 \) (or from \( \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \cap \mathcal{S}_4, \) depending).

2.2 Theorem. Let \( \lambda \geq 1, \) and assume \( c^\lambda \) is a regular cardinal. Then there is a proper nonempty subset \( A \subset R^\lambda, \) which can be taken to be a moiety if either \( \lambda < \omega \) or \( c^\lambda \) satisfies \( 2^{<\kappa} = \kappa, \) such that \( \mathcal{U}_A^\lambda \) and \( \mathcal{U}_{R^\lambda \setminus A}^\lambda \) are both connected, and such that \( \mathcal{U}_{\{A, R^\lambda \setminus A\}}^\lambda \) is perfect.

Proof. \( \langle R^\lambda, \mathcal{U}^\lambda \rangle \) is an even connected Hausdorff space, and \( |H(\mathcal{U}^\lambda)| \leq (c^\lambda)^\lambda = c^\lambda \) (because \( w(\mathcal{U}^\lambda) = \omega \cdot \lambda, \) so there is a dense subset of cardinality \( \omega \cdot \lambda). \) Thus 2.1 applies. This takes care of the theorem, except for the case \( \lambda < \omega \) and we want \( A \) to be a moiety. But then 0.10 is applicable: If \( A \) is not a moiety, then one of \( \mathcal{U}_A^\lambda, \mathcal{U}_{R^\lambda \setminus A}^\lambda \) contains a small nonempty open set, forcing discreteness.
The following application of 1.8 shows that the map $H(\cdot)$ from $HE(\mathcal{U})$ to the subgroup lattice of $\text{Sym}(\mathbb{R})$ is not join-preserving.

2.3 Theorem. Assume $\mathfrak{c}$ is a regular cardinal. There are $H$-enrichments $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{U}$ such that $H(\mathcal{F}_1) \vee H(\mathcal{F}_2) = H(\mathcal{U})$, but $H(\mathcal{F}_1 \vee \mathcal{F}_2) \neq H(\mathcal{U})$. Thus $H(\cdot)$ does not preserve lattice joins.

Proof. By 2.2 there is a moiety $A \subseteq \mathbb{R}$ such that both $\mathcal{U}_A$ and $\mathcal{U}_{R \setminus A}$ are connected. Let $\mathcal{F}_1 = \mathcal{U}_A$, $\mathcal{F}_2 = \mathcal{U}_{R \setminus A}$. Then $H(\mathcal{F}_1) = H(\mathcal{F}_2) = H(\mathcal{U})$, by 0.7. Thus $H(\mathcal{F}_1) \vee H(\mathcal{F}_2) = H(\mathcal{U})$. However, $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{U}_{(A, R \setminus A)}$ is non-connected. Hence, by 4.2(i), $H(\mathcal{F}_1 \vee \mathcal{F}_2) \neq H(\mathcal{U})$. (Section 4 deals with how much larger $H(\mathcal{F})$ is than $H(\mathcal{U})$ when $\mathcal{F} \in H(\mathcal{U})$ is nonconnected.)

2.4 Remark. The hard work in proving 0.6 went in constructing a set $A \subseteq \mathbb{R}$ such that $\mathcal{U}_{(A, R \setminus A)}$ is nondiscrete (which one can do without assuming $\mathfrak{c}$ is a regular cardinal). Showing complete regularity is easy.

2.5 Proposition. Let $\mathcal{T}$ be a topology on a set $X$, let $\mathcal{H} \subseteq H(\mathcal{T})$, and suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is closed under complementation. If $\mathcal{T}$ is regular (resp. completely regular, 0-dimensional), then so is $\mathcal{T}_{\mathcal{A}}^\mathcal{H}$.

Proof. Set $\mathcal{T}' := \mathcal{T}_{\mathcal{A}}^{\mathcal{H}}$, and let $x \in T \in \mathcal{T}'$ be given. Assume first that $\mathcal{T}$ is regular. Then we may assume that $T$ is basic open, so that $T = U \cap B$, where $U \in \mathcal{T}$ and $B := h_1(A_1) \cap \cdots \cap h_n(A_n)$ for some $\{h_1, \ldots, h_n\} \subseteq \mathcal{H}$ and $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$. Since each $A \in \mathcal{A}$ is $\mathcal{T}'$-clopen, so is $B$. Since $\mathcal{T}$ is regular, there is a $\mathcal{T}$-open neighborhood $V$ of $x$ with $cl_\mathcal{T}(V)$, the $\mathcal{T}$-closure of $V$, contained in $U$. Then we have $x \in V \cap B \subseteq cl_\mathcal{T}(V \cap B) \subseteq cl_\mathcal{T}(V) \cap cl_\mathcal{T}(B) = cl_\mathcal{T}(V) \cap B \subseteq U \cap B = T$. Thus $\mathcal{T}'$ is regular.

Next let $\mathcal{T}$ be completely regular, $x \in T$ as above, and $T = U \cap B$. Then there is a continuous $f$: $(X, \mathcal{T}) \to (\mathbb{R}, \mathcal{U})$ taking $x$ to 0 and $X \setminus U$ to $\{1\}$. Define $g$: $X \to \mathbb{R}$ to agree with $f$ on the $\mathcal{T}'$-clopen set $B$ and to be constantly 1 on $X \setminus B$. Then $g$: $(X, \mathcal{T}') \to (\mathbb{R}, \mathcal{U})$ is clearly continuous, and takes $x$ to 0 and $X \setminus T$ to $\{1\}$.

Finally let $\mathcal{T}$ be 0-dimensional, $x \in T = U \cap B$. Let $C$ be a $\mathcal{T}$-clopen set with $x \in C \subseteq U$. Then $C \cap B$ is a $\mathcal{T}$-clopen set with $x \in C \cap B \subseteq T$.

2.6 Theorem. Assume $\kappa$ is an infinite regular cardinal, $(X, \mathcal{T})$ is an even space of cardinality $\kappa$, and $|H(\mathcal{T})| = \kappa$. If $\mathcal{T}$ is regular (resp. completely regular, 0-dimensional) Hausdorff, then $(\{A \subseteq X: \mathcal{T}_{(A, X \setminus A)} \text{ is perfect regular (resp. perfect completely regular, perfect 0-dimensional)\}}) \text{ is } \kappa$-residual in $\mathcal{P}^*(X)$.

Proof. Immediate from 1.8(iii) and 2.5.

2.7 Corollary. Let $\lambda > 1$, and assume $\mathfrak{c}^\lambda$ is a regular cardinal. Then $(\{A \subseteq \mathbb{R}^\lambda: \mathcal{U}_{(A, R \setminus A)} \text{ is a perfect completely regular topology\})$ is $\mathfrak{c}^\lambda$-residual in $\mathcal{P}^*(\mathbb{R}^\lambda)$. 
2.8 Remark. We have achieved in 2.7 a result with more generality than that of 0.6, but at the cost of having to assume cardinal regularity. (It is known that the regularity of $c$ is independent of the usual, i.e., $ZFC$, axioms of set theory.) An honest improvement on 0.6 is given in 2.12 below.

2.9 Proposition. Let $\mathcal{T} \in HE(\mathcal{U})$ be nonconnected. For each $x \in \mathbb{R}$ and $\mathcal{U}$-neighborhood $U$ of $x$, there is a $\mathcal{T}$-clopen set $T$ with $x \in T \subseteq U$.

Proof. Suppose $A \subseteq \mathbb{R}$ is a proper nonempty $\mathcal{T}$-clopen set, $a \in \mathbb{R} \setminus A$, and that $A' := (a, \infty) \cap A$ is nonempty. Then $A' = [a, \infty) \cap A$ is $\mathcal{T}$-clopen and bounded below. Pick $x \in A'$ and let $h: \mathbb{R} \to \mathbb{R}$ take $y \in \mathbb{R}$ to $2x - y$. Then $h \in H(\mathcal{T})$, hence $h(A')$ is $\mathcal{T}$-clopen, contains $x$, and is bounded above. Therefore $A'' := A' \cap h(A')$ is a $\mathcal{T}$-clopen neighborhood of $x$ that is bounded. We can shrink $A''$ as small as we like, while keeping $x$ fixed, using affine maps. This gives us the set $T$ that we want.

2.10 Corollary. Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ be closed under complementation, where $\mathcal{A} \setminus \mathcal{U} \neq \emptyset$. Then $\mathcal{U}_{\mathcal{A}}$ is 0-dimensional.

Proof. Let $\mathcal{T} := \mathcal{U}_{\mathcal{A}}$, and suppose $x \in T \in \mathcal{T}$ is given. As in the proof of 2.5, $T = U \cap B$, where $U \in \mathcal{U}$ and $B$ is $\mathcal{T}$-clopen. Because $\mathcal{A} \setminus \mathcal{U} \neq \emptyset$, $\mathcal{T}$ is nonconnected. Thus, by 2.9, there is a $\mathcal{T}$-clopen $T'$ with $x \in T' \subseteq U$. Consequently, $T' \cap B$ is a $\mathcal{T}$-clopen neighborhood of $x$ contained in $T$.

2.11 Corollary. Assume $c$ is a regular cardinal. Then $\{A \subseteq \mathbb{R}: \mathcal{U}_{(A, R \setminus A)}\}$ is a perfect 0-dimensional topology is $c$-residual in $\mathcal{P}^c(\mathbb{R})$.

Proof. Immediate from 1.8(iii) and 2.10.

The following is an immediate consequence of 2.10 and the proof of 0.6 in [3].

2.12 Corollary. $\mathcal{U}$ has a nondiscrete 0-dimensional $H$-enrichment.

An interesting consequence of 0.5 and 2.5, concerning the rational line $\langle \mathbb{Q}, \mathcal{U}' \rangle$, is the following.

2.13 Corollary. Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$ be closed under complementation, where $\mathcal{A} \setminus \mathcal{U}' \neq \emptyset$. Then $\mathcal{U}'_{\mathcal{A}} = \mathcal{D}$.

2.14 Remark. A question remaining from [3] is whether $\mathcal{T} \in HE(\mathcal{U})$ can be both connected and (completely) regular. One easy fact, proved (in more generality) in [3], is that if $A \in \mathcal{U}$ is either residual or co-small, then $\mathcal{U}_A$ is both connected and nonregular. One might try to prove a version of "almost every $\mathcal{U}_A$ is connected and nonregular" by showing that $\{A \subseteq \mathbb{R}: A$ is residual$\}$ is $c$-residual in $\mathcal{P}^c(\mathbb{R})$. But this is false, since otherwise we would also have the same for $\{A \subseteq \mathbb{R}: \mathbb{R} \setminus A$ is residual$\}$; so
\[ \emptyset = \{ A \subseteq \mathbb{R} : \text{both } A \text{ and } \mathbb{R} \setminus A \text{ are residual} \} \] would be c-residual in \( \mathcal{P}(\mathbb{R}) \), an impossibility. The problem looks quite difficult, and at this point we cannot even guess as to the outcome.

3. Maximal H-enrichments

Another way of obtaining H-enrichments is via Zorn’s lemma. Let \( P \) be any topological property. Call \( P \) inductive if whenever each member of a chain of topologies on a set \( X \) satisfies \( P \), then the join of those topologies satisfies \( P \) as well. Clearly the conjunction of any number of inductive properties is inductive; examples of inductive properties include: perfect; \( T_i \); Hausdorff; (completely) regular; totally disconnected; 0-dimensional.

Let \( (X, \mathcal{T}) \) be a topological space. Then, because of Zorn’s lemma and the fact that the partially ordered set \( HE(\mathcal{T}) \) is closed under topological joins, every member of \( HE(\mathcal{T}) \) satisfying the inductive property \( P \) can be enriched to a member of \( HE(\mathcal{T}) \) that is \( H \)-maximal \( P \); i.e. maximal in \( HE(\mathcal{T}) \) with respect to satisfying \( P \).

3.1 Remark. Obvious examples of noninductive properties include: non-discrete; compact; non-Hausdorff. Less obvious is the property of connectedness. J. A. Guthrie and H. E. Stone have shown (see [6]) that there are connected topologies that cannot be enriched to maximal connected topologies. However, as is shown in [6] via two (nonconstructive) methods (one due to Guthrie–Stone, the other to M. Wage), \( \mathcal{U} \) can be so enriched. Wage’s enrichment is not an H-enrichment since it contains countably infinite open sets (viz. 0.10). It does not appear that an H-enrichment can be obtained using the Guthrie–Stone method either, so it seems to be an open question whether the euclidean topologies have H-maximal connected H-enrichments.

3.2 Proposition. Let \( 1 \leq n < \omega \), with \( \mathcal{T} \in HE(\mathcal{U}^n) \) either H-maximal connected or H-maximal nondiscrete. (N.B.: In this instance, “nondiscrete” is synonymous with “perfect,” since \( \mathcal{U}^n \) is homogeneous.) Then every small subset of \( \mathbb{R}^n \) is \( \mathcal{T} \)-closed.

Proof. Suppose \( \mathcal{T} \) is an H-maximal connected H-enrichment of \( \mathcal{U}^n \), and let \( \mathcal{T}' \in HE(\mathcal{U}^n) \) be obtained from \( \mathcal{T} \) by adding in all co-small subsets of \( \mathbb{R}^n \). Because nonempty \( \mathcal{T} \)-open sets are not small, \( \mathcal{T}' \) must be connected. (Use an argument similar to that in 1.8(ii).) By maximality of \( \mathcal{T} \), we have \( \mathcal{T} = \mathcal{T}' \). Similarly we have the assertion for \( \mathcal{T} \) H-maximal nondiscrete.

The next results concern \( w(\mathcal{T}) \) when \( \mathcal{T} \in HE(\mathcal{U}^n) \), \( 1 \leq n < \omega \). By 0.9 we know that \( w(\mathcal{T}) \geq \omega_1 \) when \( \mathcal{T} \) is proper. By putting conditions on \( \mathcal{T} \),
and perhaps on the set-theoretic universe, we can improve this lower bound. It is well known that the hypothesis $2^{<\aleph_0} < 2^\aleph_0$ is a consequence of Martin's Axiom, and is hence strictly weaker than the Continuum Hypothesis.

3.3 Theorem. Assume $\aleph_0$ is a regular cardinal and $2^{<\aleph_0} < 2^\aleph_0$. If $1 \leq n < \omega$ and $\mathcal{T} \in HE(\mathcal{W}^n)$ is $H$-maximal regular nondiscrete (resp. completely regular non-discrete, $0$-dimensional non-discrete), then $w(\mathcal{T}) \geq \aleph_0$.

Proof. Suppose $\mathcal{T}$ is $H$-maximal regular nondiscrete (resp. completely regular non-discrete, $0$-dimensional non-discrete), and assume $w(\mathcal{T}) < \aleph_0$. By 0.10, $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is even. By 1.8(iii), then, with $\mathcal{H} = H(\mathcal{W}^n)$, we have that $\mathcal{T} := \{ A \subseteq \mathbb{R}^n : \mathcal{T}_{\langle A, \mathbb{R}^n, A \rangle} \text{ is nondiscrete} \}$ is $\mathcal{C}$-residual in $\mathcal{P}(\mathbb{R}^n)$. By 1.1(ii), $|\mathcal{H}| = 2^\aleph_0$. Since $w(\mathcal{T}) < \aleph_0$ and $2^{<\aleph_0} < 2^\aleph_0$, we know $|\mathcal{T}| < 2^\aleph_0$; hence there is some $A \in \mathcal{P}(\mathcal{T})$, and $\mathcal{T}_{\langle A, \mathbb{R}^n, A \rangle}$ is a nondiscrete $H$-enrichment of $\mathcal{W}^n$ properly containing $\mathcal{T}$. By 2.5, $\mathcal{T}_{\langle A, \mathbb{R}^n, A \rangle}$ is regular (resp. completely regular, $0$-dimensional). This contradicts the maximality of $\mathcal{T}$.

Under other maximality assumptions, we can get better bounds, as well as dispence with the hypothesis $2^{<\aleph_0} < 2^\aleph_0$.

3.4 Theorem. Assume $\aleph_0$ is a regular cardinal, $1 \leq n < \omega$. If $\mathcal{T} \in HE(\mathcal{W}^n)$ is $H$-maximal connected, then $w(\mathcal{T}) \geq \aleph_+^+$.

Proof. Assume $w(\mathcal{T}) \leq \aleph_0$ and $\mathcal{T}$ is $H$-maximal connected. $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is even by 0.10; so by 1.8(ii), with $\mathcal{H} = H(\mathcal{W}^n)$, there is a set $A \subseteq \mathbb{R}^n$ such that both $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{R}^n, \mathcal{A}}$ are connected. (A is, of course, a moiety, by 0.10.) Each of these topologies is in $HE(\mathcal{W}^n)$; so since $\mathcal{T}$ is maximal connected in $HE(\mathcal{W}^n)$, we have $\mathcal{T} = \mathcal{T}_{\mathcal{A}}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{R}^n, \mathcal{A}}$. This means that the moiety $A$ is clopen in $\mathcal{T}$, a contradiction. Thus $w(\mathcal{T}) \geq \aleph_+^+$.

3.5 Corollary. Assume $\aleph_0$ is a regular cardinal, $1 \leq n < \omega$. If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^n)$ has cardinality $\leq \aleph_0$, then $\mathcal{W}_\alpha$ is not $H$-maximal connected.

Aside from 0.7, we know very little about homeomorphism groups of $H$-enrichments. As we see in the sequel, one way of specifying $\mathcal{T}'$-homeomorphisms when $\mathcal{T}' \in HE(\mathcal{T})$ is via piecewise definition. Let $\langle X, \mathcal{T} \rangle$ be a topological space, with $\mathcal{T}' \in HE(\mathcal{T})$. Define $h \in \text{Sym}(X)$ to be a $\mathcal{T}'/\mathcal{T}$-homeomorphism if there exists a family $\{ \langle T_i, h_i \rangle : i \in I \}$ where $\{ T_i : i \in I \}$ is a $\mathcal{T}'$-open cover of $X$ and $\{ h_i : i \in I \}$ is a family of $\mathcal{T}$-homeomorphisms such that for each $i \in I$, $h \mid T_i = h_i \mid T_i$. The family $\{ \langle T_i, h_i \rangle : i \in I \}$ is said to be a witness for $h$; the set of $\mathcal{T}'/\mathcal{T}$-homeomorphisms is denoted $H(\mathcal{T}'/\mathcal{T})$.

In the case $X = \mathbb{R}$ and $\mathcal{T} = \mathcal{U}$, homeomorphisms in $H(\mathcal{T}'/\mathcal{U})$ may be appropriately described as "piecewise monotonic."
3.6 Proposition. Let \( \langle X, \mathcal{T} \rangle \) be a topological space, with \( T' \in HE(\mathcal{T}) \). Then \( H(\mathcal{T}) \leq H(\mathcal{T}'/\mathcal{T}) \leq H(\mathcal{T}') \).

Proof. If \( h \in H(\mathcal{T}) \), then \( \{\langle x, h \rangle\} \) is a witness for \( h \); so \( h \in H(\mathcal{T}'/\mathcal{T}) \).

If \( h \in H(\mathcal{T}'/\mathcal{T}) \) has witness \( \{\langle T_i, h_i \rangle : i \in I\} \) and \( T \in \mathcal{T}' \), then \( h(T) = \bigcup_{i \in I} h_i(T \cap T_i) = \bigcup_{i \in I} h_i(T \cap T_i) \in \mathcal{T}' \). Also \( h^{-1} \) is witnessed by \( \{\langle h_i(T_i), h_i^{-1} \rangle : i \in I\} \), so \( h^{-1}(T) \in \mathcal{T}' \) as well. Thus \( h \in H(\mathcal{T}') \).

We must show \( H(\mathcal{T}'/\mathcal{T}) \) is a group. From the last paragraph, we know \( H(\mathcal{T}'/\mathcal{T}) \) is closed under inverses; so let \( g, h \in H(\mathcal{T}'/\mathcal{T}) \), say \( g \) (resp. \( h \))

is witnessed by \( \{\langle S_i, g_i \rangle : i \in I\} \) (resp. \( \{\langle T_j, h_j \rangle : j \in J\} \)). Then \( g \circ h \) is witnessed by \( \{\langle h_i^{-1}(S_i) \cap T_j, g_i \circ h_j \rangle : \langle i, j \rangle \in I \times J\} \).

3.7 Corollary. Suppose \( \langle X, \mathcal{T} \rangle \) is a space with \( \mathcal{T}_1, \mathcal{T}_2 \in HE(\mathcal{T}) \) and \( \mathcal{T}_1 \subseteq \mathcal{T}_2 \). Then \( H(\mathcal{T}_1/\mathcal{T}) \leq H(\mathcal{T}_2/\mathcal{T}) \).

Proof. It suffices to show \( H(\mathcal{T}_1/\mathcal{T}) \subseteq H(\mathcal{T}_2) \), in view of 3.6 and the hypothesis that \( \mathcal{T}_1 \subseteq \mathcal{T}_2 \). Let \( h \in H(\mathcal{T}_1/\mathcal{T}) \) be witnessed by \( \{\langle T_i, h_i \rangle : i \in I\} \). Then \( h(T) = \bigcup_{i \in I} h_i(T \cap T_i) \in \mathcal{T}_2 \) whenever \( T \in \mathcal{T}_2 \). Similarly \( h^{-1}(T) \in \mathcal{T}_2 \), so \( h \in H(\mathcal{T}_2) \).

3.8 Remark. 3.7 says that the operator \( H(\cdot/\mathcal{T}) \) is order-preserving as a mapping between lattices; 2.3 says that \( H(\cdot) \) is not join-preserving. In fact, if one picks \( \mathcal{T}_1 = \mathcal{U}_A \) and \( \mathcal{T}_2 = \mathcal{U}_{R \setminus A} \) as in 2.3, then clearly \( H(\mathcal{T}_1/\mathcal{U}) \vee H(\mathcal{T}_2/\mathcal{U}) = H(\mathcal{U}) \). However, by 4.2, \( H(\mathcal{T}_1 \vee \mathcal{T}_2/\mathcal{U}) = H(\mathcal{U}) \). Thus \( H(\cdot/\mathcal{T}) \) is not generally join-preserving. The question of whether \( H(\cdot) \) is generally order-preserving is more difficult and remains open.

Our next result concerns the cardinality of \( H(\mathcal{T}/\mathcal{U}) \) in 0-dimensional \( H \)-enrichments.

Call a set \( A \subseteq \mathbb{R} \) symmetric if \( -x \in A \) whenever \( x \in A \). Clearly the set of symmetric subsets of \( \mathbb{R} \) is a complete Boolean subalgebra of \( \mathcal{P}(\mathbb{R}) \). If \( A \subseteq \mathbb{R} \), then we set \( -A := \{-x : x \in A\} \), and observe that \( A \cup (-A) \) is symmetric. The operation \( A \mapsto A \cup (-A) \) is not one-one in general, but is so once we restrict attention to sets lying in \([0, \infty) \) (or in \((-\infty, 0]\)).

Let \( A \subseteq \mathbb{R} \) be symmetric, and define the function \( h_A \) on \( \mathbb{R} \) to fix \( x \in A \) and to take \( x \in \mathbb{R} \setminus A \) to \(-x\). Then clearly \( h_A \in \text{Sym} (\mathbb{R}) \). Also, if \( A, B \subseteq \mathbb{R} \) are symmetric sets and \( A \setminus \{0\} = B \setminus \{0\} \), then \( h_A \neq h_B \). Finally, if \( \mathcal{T} \in HE(\mathcal{U}) \) and \( T \subseteq \mathbb{R} \) is a symmetric \( \mathcal{T} \)-clopen set, then \( h_T \in H(\mathcal{T}/\mathcal{U}) \).

The character of a topology, \( \chi(\mathcal{T}) \), is defined to be the smallest cardinality of a \( \mathcal{T} \)-neighborhood basis of a point. If \( \langle X, \mathcal{T} \rangle \) is topological space, then \( \chi(\mathcal{T}) \leq w(\mathcal{T}) \leq \chi(\mathcal{T}) \cdot |X| \). Because of 2.11, the hypothesis of the next result is reasonably general.
3.9 Proposition. Assume $\mathcal{F} \in HE(\mathcal{U})$ is 0-dimensional. Then $\chi(\mathcal{F}) \leq |H(\mathcal{F}/\mathcal{U})|$.

Proof. Let $\chi(\mathcal{F}) = \kappa$. Because of homogeneity of $\mathcal{U}$ and the fact that $\mathcal{F}$ is 0-dimensional, there is a family $\{T_{\xi}^+: \xi < \kappa\}$ of distinct $\mathcal{F}$-clopen sets, each lying in the interval $(0, 1)$. Let $T_{\xi}^- := T_{\xi} \cup (-T_{\xi})$. Then $T_{\xi}^-$ is symmetric $\mathcal{F}$-clopen, and the sets $T_{\xi}^+, \xi < \kappa$ are all distinct. Thus the $\mathcal{F}/\mathcal{U}$-homeomorphisms $h_{T_{\xi}}$ are all distinct, so $|H(\mathcal{F}/\mathcal{U})| \geq \kappa$.

The following result is a companion to 3.3 and 3.4.

3.10 Theorem. Assume $\mathcal{C}$ is a regular cardinal and that $\kappa$ is any regular cardinal with $\mathcal{C}^+ \leq \kappa \leq 2^{\mathcal{C}}$. If $\mathcal{F} \in HE(\mathcal{U})$ is $H$-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete) and $w(\mathcal{F}) \leq \mathcal{C}$, then $|H(\mathcal{F}/\mathcal{U})| \geq \kappa$.

Proof. Assume $w(\mathcal{F}) \leq \mathcal{C}$ and $\mathcal{F} \in HE(\mathcal{U})$ is $H$-maximal regular nondiscrete (resp. completely regular nondiscrete, 0-dimensional nondiscrete). By 1.8(iii), with $\mathcal{K} = H(\mathcal{U})$, and 0.10, we have that $\mathcal{F} := \{A \subseteq \mathbb{R} : \mathcal{F}^{\mathcal{K}}_{A, R_A} \text{ is nondiscrete}\}$ is $\mathcal{C}$-residual in $\mathcal{P}^{\mathcal{K}}(\mathbb{R})$. By 2.5, each $\mathcal{F}^{\mathcal{K}}_{A, R_A}$ is regular (resp. completely regular, 0-dimensional); whence by maximality of $\mathcal{F}$, we have $\mathcal{F} = \mathcal{F}^{\mathcal{K}}_{A, R_A}$ for all $A \in \mathcal{F}$. Thus $\mathcal{F}$ contains a $\mathcal{C}$-residual family of clopen sets. By 1.1, $|\mathcal{F}| = 2^{\mathcal{C}}$, so there is a family $\{T_{\xi}^- : \xi < 2^{\mathcal{C}}\}$ of distinct $\mathcal{F}$-clopen sets. Now $|H(\mathcal{U})| = \mathcal{C}$, hence we may assume that for $\xi < \nu < 2^{\mathcal{C}}$, there is no $\eta \in H(\mathcal{U})$ taking $T_{\xi}^-$ to $T_{\nu}^-$. i.e., the sets $T_{\xi}^-$ are all distinct in $H(\mathcal{U})$-orbits. We may also assume, by 0.10, that each $T_{\xi}^-$ is a moiety in $\mathcal{F}$. For each $\xi < 2^{\mathcal{C}}$, there is a linear shift $s_{\xi}$ (i.e., of the form $x \mapsto x + b$) taking $T_{\xi}^-$ to a $\mathcal{F}$-clopen set that does not contain 0. All the sets $s_{\xi}(T_{\xi}^-)$ are distinct since the sets $T_{\xi}^-$ lie in distinct $H(\mathcal{U})$-orbits.

Thus, without loss of generality, we may assume each $T_{\xi}^-$ is a $\mathcal{F}$-clopen moiety that does not contain 0, all the sets $T_{\xi}^-$ being distinct, $\xi < 2^{\mathcal{C}}$. Let $T_{\xi}^+ := T_{\xi}^- \cap (0, \infty)$ and $T_{\xi}^- := T_{\xi}^- \cap (-\infty, 0)$. Then $T_{\xi}^+ = T_{\xi}^- \cap [0, \infty)$ and $T_{\xi}^- = T_{\xi}^- \cap (-\infty, 0]$, so the sets $T_{\xi}^+, T_{\xi}^-$ are all $\mathcal{F}$-clopen. Now suppose that $|\{T_{\xi}^+ : \xi < 2^{\mathcal{C}}\}| < \kappa$, where $\kappa$ is any regular cardinal with $\mathcal{C}^+ \leq \kappa \leq 2^{\mathcal{C}}$. Then there is a set $X \subseteq 2^{\mathcal{C}}$, $|X| \geq \kappa$, such that for $\xi, \nu \in X$, $T_{\xi}^+ = T_{\nu}^+$. Since for all $\xi \neq \nu$ in $X$ we know $T_{\xi} \neq T_{\nu}$, it follows that the sets $T_{\xi}^-$ are all distinct for $\xi \in X$. These observations allow us to assume without loss of generality that there is a family $\{T_{\xi}^- : \xi < \kappa\}$ of distinct $\mathcal{F}$-clopen moieties, each lying in the interval $(0, \infty)$. From this we conclude that the symmetric $\mathcal{F}$-clopen moieties $T_{\xi}^- := T_{\xi} \cup (-T_{\xi})$ are also distinct. Thus the corresponding $\mathcal{F}/\mathcal{U}$-homeomorphisms $h_{T_{\xi}^-}$ are distinct as well, and we conclude $|H(\mathcal{F}/\mathcal{U})| \geq \kappa$.

3.11 Corollary. Assume $\mathcal{C}$ is a regular cardinal. If $\mathcal{F} \in HE(\mathcal{U})$ contains an $H$-maximal 0-dimensional nondiscrete $H$-enrichment of $\mathcal{U}$, then $|H(\mathcal{F}/\mathcal{U})| \geq \mathcal{C}^+$. 
4. Permutations of the Real Line

In this section we concern ourselves with properties of the various homeomorphism groups $H(F)$ and $H(F/U)$, for $F \in HE(U)$, and their interrelationships.

For $h \in \text{Sym}(X)$, we let $\text{supp}(h) := \{x \in X : h(x) \neq x\}$, the support of $h$. Note that if $\langle X, S \rangle$ and $\langle Y, T \rangle$ are topological spaces, $F$ is a Hausdorff topology, and $f, g : \langle X, F \rangle \rightarrow \langle Y, T \rangle$ are continuous, then $\{x \in X : f(x) \neq g(x)\}$ is $F$-open. In particular, we have:

4.1 Proposition. Let $\langle X, F \rangle$ be a Hausdorff space, $h \in H(F)$. Then $\text{supp}(h)$ is $F$-open.

Let $F \in HE(U)$. By 0.7, $H(F) = H(U)$ if $F$ is connected; by 0.8, $H(F) = \text{Sym}(\mathbb{R})$ if and only if $F = \emptyset$. To answer further the question (raised in 0.8) of exactly when $H(F_1) = H(F_2)$ is true for $F_1, F_2 \in HE(U)$, we offer the following.

4.2 Proposition. 

(i) Let $F \in HE(U)$. Then $H(F) = H(U)$ if and only if $H(F/U) = H(U)$, if and only if $F$ is connected.

(ii) Let $F_1, F_2 \in HE(U)$, where $F_1$ is 0-dimensional and $F_1 \neq F_2$. Then $H(F_1/U) \neq H(F_2/U)$.

Proof. Ad(i). If $F$ is connected, then $H(F) = H(F/U) = H(U)$, by 0.7. Suppose $F$ is nonconnected. It suffices to show $H(F/U) \neq H(U)$. By 2.9, there is a $F$-clopen moiety $T \subseteq (0, 1)$. As in the proof of 3.9, $T' := T \cup (-T)$ is a symmetric $F$-clopen moiety, and $h_T \in H(F/U) \setminus H(U)$.

Ad(ii). Suppose $F_1, F_2 \in HE(U)$, $F_1$ is 0-dimensional, and $F_1 \neq F_2$. Since the bounded $F_1$-clopen sets form a basis for $F_1$, there is a $F_1$-clopen set $T \subseteq (0, 1)$ with $T \notin F_2$. Then $T' := T \cup (-T)$ is also $F_1$-clopen. However, if $T' \in F_2$, then $T = T' \cap (0, 1) \in F_2$ also. Consequently $T' \notin F_2$. However, if $h = -h_T$, i.e.,

$$h(x) = \begin{cases} 
-x & \text{if } x \in T' \\
 x & \text{if } x \in \mathbb{R} \setminus T',
\end{cases}$$

then $h \in H(F_1/U)$. But $\text{supp}(h) = T' \notin F_2$, so $h \notin H(F_2)$, by 4.1.

One can improve on 4.2(i) to show that when $F \in HE(U)$ is not connected, $H(F/U)$ is "significantly larger" than $H(U)$. By way of
notation, if $H$ is a subgroup of the group $G$, then $|G:H|$ is the index of $H$
 in $G$.

4.3 Theorem. Let $\mathcal{T} \in HE(\mathcal{U})$. Then there is a subgroup $K$ of $H(\mathcal{T}/\mathcal{U})$
 such that:

(i) $K$ has exponent 2 (i.e., $K$ consists of involutions), so is abelian.
(ii) $|K \cap H(\mathcal{U})| = 2$.
(iii) No $H(\mathcal{U})$-coset in $H(\mathcal{T}/\mathcal{U})$ contains more than 2 elements from $K$.
(iv) If $\mathcal{T}$ is nonconnected, then $|K| \geq c$; hence $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq c$.

Proof. For $A \subseteq \mathbb{R}$ symmetric, define $h_A \in \text{Sym}(\mathbb{R})$ as before:

$$h_A(x) = \begin{cases} 
  x & \text{if } x \in A \\
 -x & \text{if } x \in \mathbb{R} \setminus A
\end{cases}.$$ 

Clearly $h_A$ is an involution on $\mathbb{R}$, so has order 2 in $\text{Sym}(\mathbb{R})$. Also note that if $\min \{|A|, |\mathbb{R}\setminus A|\} \geq 2$, then $h_A$ is nonmonotonic, hence not in
$H(\mathcal{U})$. Also $h_{\mathbb{R}\setminus A} = -h_A$, and $h_A \circ h_B = h_{(A \cap B) \cup (\mathbb{R} \setminus A \cup B)}$ whenever $A, B \subseteq \mathbb{R}$ are both symmetric.

Suppose $\mathcal{T} \in HE(\mathcal{U})$ is nondiscrete. Then every nonempty proper
$\mathcal{T}$-clopen set is a moiety by 0.10. Define $K := \{h_T: T$ is a symmetric
$\mathcal{T}$-clopen set\}. Then clearly $K \cap H(\mathcal{U}) = \{id_{\mathbb{R}}, -id_{\mathbb{R}}\}$. K is a subgroup
since if $S, T$ are symmetric and $\mathcal{T}$-clopen, then so is $(S \cap T) \cup (\mathbb{R} \setminus (S \cup T))$. Also $K$ clearly has exponent 2. This establishes (i) and (ii) above. To
establish (iii), suppose $h_S, h_T \in K$ are distinct. Then $S \neq T$. $h_S$ and $h_T$ lie
in the same $H(\mathcal{U})$-coset if and only if $h_S \circ h_T^{-1} = h_S \circ h_T =
 h_{(S \cap T) \cup (\mathbb{R} \setminus (S \cup T))} \in H(\mathcal{U})$, if and only if $(S \cap T) \cup (\mathbb{R} \setminus (S \cup T)) \in \{\mathbb{R}, \emptyset\}$, if
and only if $S = T$ or $S = \mathbb{R} \setminus T$. Thus no $H(\mathcal{U})$-coset in $H(\mathcal{T}/\mathcal{U})$ contains
more than two elements of $K$.

Now suppose $\mathcal{T}$ is nonconnected, and let $n \in \{1, 2, \ldots\}$. By 2.9 there is
a $\mathcal{T}$-clopen neighborhood $S_n$ of $n$ diameter $<\frac{1}{2^n}$. Let $T_n := S_n \cup (-S_n)$.
For each $X \subseteq \{1, 2, \ldots\}$, let $T_X = \bigcup_{n \in X} T_n$. Then $T_X$ is a symmetric
$\mathcal{T}$-clopen moiety. Of course if $X$ and $Y$ are distinct subsets of $\{1, 2, \ldots\}$,
then $T_X \neq T_Y$, hence $h_{T_X} \neq h_{T_Y}$. Thus $|K| \geq c$. Since no $H(\mathcal{U})$-coset in
$H(\mathcal{T}/\mathcal{U})$ can contain more than two elements of $K$, we infer that
$|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq c$.

In the event $\mathcal{T}$ is discrete, we replace $\mathcal{T}$ with a nondiscrete $H$-
enrichment of $\mathcal{U}$. Of course, in this case $|H(\mathcal{T}/\mathcal{U})| = |H(\mathcal{U})| = 2^c$;
consequently $|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| = 2^c$.

4.4 Remarks.

(i) Of course, under the hypothesis of 3.11, we have
$|H(\mathcal{T}/\mathcal{U}):H(\mathcal{U})| \geq c^+$. Thus the problem of when one can improve
the lower bound in 4.3(iv) is inevitable.
(ii) Let Sym$^\kappa(X)$ denote Sym$(X)$ viewed as a subspace of $(X^X)_\kappa$. It is easy to show that $H(\mathcal{U})$ is closed in Sym$^\omega(\mathbb{R})$, as well as nowhere dense in Sym$^\kappa(\mathbb{R})$. For a subgroup $G$ of Sym$(X)$, to be closed in Sym$^\omega(X)$ means that there is a first order relational structure $\mathcal{R}$ on $X$ such that $G = \text{Aut}(X, \mathcal{R})$, the group of $\mathcal{R}$-automorphisms.

In the particular instance of $\mathcal{U}$, the structure $\mathcal{R}$ is the ternary relation of betweenness.

(iii) The subgroup $G$ of Sym$(X)$ is $\alpha$-transitive, for $\alpha$ a cardinal, if whenever $f \in$ Sym$(X)$ and $F \subseteq X$ has cardinality $\alpha$, there is some $g \in G$ such that $g \circ F = f \circ F$. So to say that $G$ is dense in Sym$^\kappa(X)$, for $\kappa$ an infinite cardinal, is to say $G$ is $\alpha$-transitive for all $\alpha < \kappa$. $H(\mathcal{U})$ is 2-transitive but not 3-transitive. By contrast, $H(\mathcal{U}^n)$, $n \geq 2$, is $k$-transitive for all $k < \omega$; i.e., $H(\mathcal{U}^n)$ is "highly transitive." Equivalently, $H(\mathcal{U}^n)$ is dense in Sym$^\omega(\mathbb{R}^n)$.

4.5 PROPOSITION. Suppose $\mathcal{I} \in HE(\mathcal{U})$ is nonconnected. Then $H(\mathcal{U})$ is closed nowhere dense in $H(\mathcal{I}/\mathcal{U}) \subseteq$ Sym$^\omega(\mathbb{R})$.

Proof. $H(\mathcal{U})$ is closed in Sym$^\omega(\mathbb{R})$, so it is closed in the subspace $H(\mathcal{I}/\mathcal{U})$. To show $H(\mathcal{U})$ is nowhere dense, it suffices to show $H(\mathcal{I}/\mathcal{U}) \setminus H(\mathcal{U})$ is dense in $H(\mathcal{I}/\mathcal{U})$. Let $h \in H(\mathcal{U})$, with $F \subseteq \mathbb{R}$ finite. Let $N := \{g \in H(\mathcal{I}/\mathcal{U}); g \circ F = h \circ F \}$. We show $N \cap (H(\mathcal{I}/\mathcal{U}) \setminus H(\mathcal{U})) \neq \emptyset$.

By the argument in 4.3 (iv) there is a $\mathcal{I}$-clopen symmetric moiety $T$ with $F \subseteq T$. Let $g = h \circ h_T$. Since $H(\mathcal{I}/\mathcal{U})$ is a group (by 3.6), $g \in H(\mathcal{I}/\mathcal{U})$. If $x \in F$, then we have $g(x) = h(x)$, so $g \in N$. $T$ is a moiety, and $g$ is monotonic in opposite directions on $T$ and $\mathbb{R} \setminus T$. Thus $g \notin H(\mathcal{U})$.

Another way of showing $H(T/\mathcal{U})$ is "significantly larger" than $H(\mathcal{U})$ for $\mathcal{I} \in HE(\mathcal{U})$ nonconnected is to show $H(\mathcal{U})$ to be nonmaximal in $H(\mathcal{I}/\mathcal{U})$, i.e., there is some $h \in H(\mathcal{I}/\mathcal{U}) \setminus H(\mathcal{U})$ such that the subgroup of $H(\mathcal{I}/\mathcal{U})$ generated by $H(\mathcal{U}) \cup \{h\}$ is a proper subgroup.

4.6 THEOREM. Suppose $\mathcal{I} \in HE(\mathcal{U})$ is nonconnected. Then $H(\mathcal{U})$ is not maximal in $H(\mathcal{I}/\mathcal{U})$.

Proof. Define $g \in H(\mathcal{I}/\mathcal{U})$ to be pleasant if: (i) for all $A \subseteq \mathbb{R}$, $A$ is bounded if and only if $g(A)$ is bounded; and (ii) there is a bounded set $R_1$ and a cobounded set $R_2$ such that $R_1 \cap R_2 = \emptyset$ and $g$ is monotonic on both $R_1$ and $R_2$, but with opposite parity (e.g., $g$ is increasing on $R_1$ and decreasing on $R_2$). Let $P := \{g \in H(\mathcal{I}/\mathcal{U}); g$ is pleasant $\}$.

LEMMA. $P$ is a subgroup of $H(\mathcal{I}/\mathcal{U})$ that contains $H(\mathcal{U})$.

Proof (Lemma). Clearly $H(\mathcal{U}) \subseteq P$. If $g \in P$ then obviously $g^{-1}$ satisfies (i) above. Suppose $g$ is increasing on $R_1$ (bounded) and decreasing on $R_2$ (cobounded). Then $g^{-1}$ is increasing on $g(R_1)$ (bounded)
and decreasing on \(g(R_2)\) (cobounded). Also \(g(R_1) \cap g(R_2) = g(R_1 \cap R_2) = g(\emptyset) = \emptyset\). The other cases are treated similarly, and we conclude \(P\) is closed under inverses.

Now suppose \(g, h \in P\). Then clearly \(g \circ h\) satisfies (i) above. Suppose \(g\) is increasing on \(R_1\) (bounded), decreasing on \(R_2\) (cobounded), \(R_1 \cap R_2 = \emptyset\); and suppose \(h\) is increasing on \(S_1\) (cobounded), decreasing on \(S_2\) (bounded), \(S_1 \cap S_2 = \emptyset\). Then \(g \circ h\) is increasing on \(S_1 \cap h^{-1}(R_1)\) as well as on \(S_2 \cap h^{-1}(R_2)\); \(g \circ h\) is decreasing on \(S_1 \cap h^{-1}(R_2)\), as well as on \(S_2 \cap h^{-1}(R_1)\). Thus \(g \circ h\) is increasing on \(S_1 \cap h^{-1}(R_1)\) (bounded) and decreasing on \(S_1 \cap h^{-1}(R_2)\) (cobounded). These two sets have empty intersection, so we infer that \(g \circ h \in P\). The remaining cases are treated similarly, so we conclude that \(P\) is a subgroup of \(H(\mathcal{T}/\mathcal{U})\). (Lemma)

Suppose \(T\) is a symmetric \(\mathcal{T}\)-clopen set. Then for any \(A \subseteq \mathbb{R}\),
\[
h_T(A) = h_T((A \cap T) \cup (A \setminus T)) = h_T(A \cap T) \cup h_T(A \setminus T) = (A \cap T) \cup (- (A \setminus T))
\]
(where \(h_T\) is defined as in 4.3). Thus \(h_T\) satisfies clause (i) in the definition of "pleasant." Suppose now \(T\) is a bounded symmetric \(\mathcal{T}\)-clopen set that is also a moiety. Then \(h_T \in P \setminus H(\mathcal{U})\). By 2.9 we know such sets \(T\) are in abundance; in particular \(P \neq H(\mathcal{U})\). We can also find symmetric \(\mathcal{T}\)-clopen sets \(T\) such that both \(T\) and \(\mathbb{R} \setminus T\) are unbounded. For such a \(T\), \(h_T \notin P\). Thus \(H(\mathcal{U})\) fails (quite dramatically) to be maximal in \(H(\mathcal{T}/\mathcal{U})\).

The results 4.3, 4.5 and 4.6 discuss ways in which we can say \(H(\mathcal{U})\) is "small" (or at least "not very large") in \(H(\mathcal{T}/\mathcal{U})\) whenever \(\mathcal{T} \in H(\mathcal{U})\) is nonconnected. Now we shift perspective and look at how \(H(\mathcal{T})\) sits in \(\text{Sym}(\mathbb{R})\). In one sense, that of index, we show that \(H(\mathcal{T})\) is "small" whenever \(\mathcal{T} \neq \emptyset\). We first cite an obvious consequence of the Lemma on p. 581 in [5].

4.7 Theorem ([5, Lemma for Theorem 2]). Let \(X\) be an infinite set of cardinality \(\kappa\), with \(G \leq \text{Sym}(X)\) a subgroup of index \(< 2^\kappa\) (i.e., of "small index"). Then there is a set \(M \subseteq X\), with \(|X\setminus M| = \kappa\), such that if \(g \in \text{Sym}(X)\) and \(g \upharpoonright M = \text{id}_M\), then \(g \in G\).

4.8 Theorem. Let \(\langle X, \mathcal{T} \rangle\) be a Hausdorff space such that \(|\text{Sym}(X)\cdot H(\mathcal{T})| < 2^{|X|}\). Then \(\mathcal{T}\) has \(|X|\) isolated points.

Proof. If \(X\) is finite, then \(\mathcal{T} = \emptyset\) immediately. So assume \(X\) is infinite. By 4.7 there is a subset \(M \subseteq X\), with \(|X\setminus M| = |X|\), such that every \(h \in \text{Sym}(X)\) that fixes the points of \(M\) is a \(\mathcal{T}\)-homeomorphism. So let \(x, y \in X \setminus M\) be distinct, and let \(h \in \text{Sym}(X)\) exchange \(x\) and \(y\), leaving everything else fixed. Then \(h \in H(\mathcal{T})\), and hence \(\{x, y\} = \text{supp}(h) \in \mathcal{T}\) by 4.1 (since \(\mathcal{T}\) is Hausdorff). Thus \(x\) is an isolated point. Since \(x\) was chosen arbitrarily from \(X \setminus M\), a set of cardinality \(|X|\), there are \(|X|\) \(\mathcal{T}\)-isolated points.
4.9 Corollary. Let $1 \leq \lambda$, with $\mathcal{F} \in \text{HE}^\lambda(\mathcal{U})$ nondiscrete. Then $|\text{Sym}(\mathcal{R}) : H(\mathcal{F})| = 2^{\omega \lambda}$.

Proof. $\mathcal{F}$ is homogeneous and Hausdorff. Use 4.8.

In another sense, $H(\mathcal{F})$ is "large" in $\text{Sym}(\mathcal{R})$ whenever $\mathcal{F} \in \text{HE}(\mathcal{U})$ is nonconnected. We show that $H(\mathcal{F}/\mathcal{U})$ is dense in $\text{Sym}^\omega(\mathcal{R})$, i.e., that $H(\mathcal{F}/\mathcal{U})$ is highly transitive.

4.10 Theorem. Let $\mathcal{F} \in \text{HE}(\mathcal{U})$ be nonconnected, and let $A \subseteq \mathcal{R}$ be a $\mathcal{U}$-closed $\mathcal{U}$-discrete set. Then every permutation on $A$ extends to a $\mathcal{F}/\mathcal{U}$-homeomorphism.

Proof. Let $s \in \text{Sym}(A)$. We need to find $h \in H(\mathcal{F}/\mathcal{U})$ such that $h|A = s$. Now symmetric groups are well known to be generated by involutions (i.e., elements of order 2), so we may as well assume $s^2 = id$, i.e., $s$ is an involution. Then we have a partition $A = A_0 \cup A_1 \cup A_2$, where $s$ takes $A_0$ to $A_1$ and vice versa, and fixes $A_2$. $A$ is $\mathcal{U}$-discrete, hence countable since $\omega(\mathcal{U}) = \omega$. Because $\mathcal{U}$ is also a regular topology, there exist sets $I_a$, $U_a$, for $a \in A$, such that: (i) $I_a$ is a closed interval containing $a$ in its $\mathcal{U}$-interior, (ii) $I_a \subseteq U_a \in \mathcal{U}$; and (iii) for each $a$, $b \in A$ distinct, $U_a \cap U_b = \emptyset$. Since $A$ is $\mathcal{U}$-closed, so is $B := \bigcup_{a \in A} I_a$. For each $a \in A_0$, pick, using 2.9, a $\mathcal{F}$-clopen neighborhood $T_a$ of $a$, with $T_a \subseteq I_a$, and let $h_a \in H(\mathcal{U})$ take $I_a$ monotonic increasingly onto $I_{s(a)}$, with $h_a(a) = s(a)$. Define

$$h(x) = \begin{cases} h_a(x) & \text{if } x \in T_a, a \in A_0 \\ h_a^{-1}(x) & \text{if } x \in h_a(T_a), a \in A_0 \\ x & \text{otherwise.} \end{cases}$$

The sets $\{T_a, h_a(T_a) : a \in A_0\}$ are pairwise disjoint and $\mathcal{F}$-clopen, so $h$ is well defined. $h$ is clearly a bijection on $\mathcal{R}$ extending $s$. To show $h \in H(\mathcal{F}/\mathcal{U})$ it remains to prove that $T = \bigcup_{a \in A_0} (T_a \cup h_a(T_a))$ is $\mathcal{F}$-closed.

Then $h$ and $h^{-1}$ will have been shown to be $\mathcal{U}$-continuous when restricted to the elements of a $\mathcal{F}$-clopen partition.

So suppose $x \in B \setminus T$, say $x \in I_a$. Then $U_a \setminus T_a$ is a $\mathcal{F}$-open neighborhood of $x$ missing $T$, hence $T$ is $\mathcal{F}$-closed in $B$. Since $B$ is $\mathcal{U}$-closed in $\mathcal{R}$, we infer that $T$ is indeed a $\mathcal{F}$-clopen set.

4.11 Corollary. Let $\mathcal{F} \in \text{HE}(\mathcal{U})$ be nonconnected. Then $H(\mathcal{F}/\mathcal{U})$ is highly transitive, thus dense in $\text{Sym}^\omega(\mathcal{R})$.

Proof. Let $1 \leq k < \omega$, $a_1 < \cdots < a_k$, $b_1, \ldots, b_k$ distinct. Let $s$ be a permutation on $\{1, \ldots, k\}$ such that $b_{s(1)} < \cdots < b_{s(k)}$. Let $g \in H(\mathcal{U})$ take $a_i$ to $b_{s(i)}$, $1 \leq i \leq k$; and, by 4.10, let $h \in H(\mathcal{F}/\mathcal{U})$ take $b_i$ to $b_{s(i)}$, $1 \leq i \leq k$. Then $h^{-1} \circ g \in H(\mathcal{F}/\mathcal{U})$ takes $a_i$ to $b_i$, $1 \leq i \leq k$. 
Another way in which $H(\mathcal{F}/\mathcal{U})$ is "like" Sym ($\mathbb{R}$) when $\mathcal{F} \in HE(\mathcal{U})$ is 0-dimensional is that elements of $H(\mathcal{F}/\mathcal{U})$ need not respect boundedness in $\mathbb{R}$.

**4.12 Theorem.** Let $\mathcal{F} \in HE(\mathcal{U})$ be 0-dimensional. Then there is some $h \in H(\mathcal{F}/\mathcal{U})$ and a bounded set $A \subseteq \mathbb{R}$ such that $h(A)$ is unbounded.

**Proof.** Let $A = \left\{ \frac{1}{n+2} : n < \omega \right\}$, $B = \{ n+2 : n < \omega \}$. We find $h \in H(\mathcal{F}/\mathcal{U})$ taking \( \frac{1}{n+2} \) to $n+2$ for each $n < \omega$. Now the sequence \( \left( \frac{1}{n+2} \right) \) cannot $\mathcal{F}$-converge, by 0.9, so there is a $T \in \mathcal{T}$ with $0 \in T$ and $T \cap A = \emptyset$. By 2.9 we can arrange that $T \subseteq (-1, 1)$. Since $\mathcal{F}$ is assumed to be 0-dimensional, we can arrange for $T$ to be $\mathcal{F}$-closed as well. For each $a \in A \cup B$ choose $I_a$, $U_a$ as in the proof of 4.10. Then $B := \{0\} \cup \bigcup_{a \in A \cup B} I_a$ is $\mathcal{U}$-closed. For each $\frac{1}{n} \in A$, let $h_n$ take $I_{1/n}$ monotonically increasing onto $I_n$, with $h_n(\frac{1}{n}) = n$, and let $T_n \subseteq I_{1/n} \setminus T$ be a $\mathcal{F}$-clopen neighborhood of $\frac{1}{n}$. Define

\[
h(x) = \begin{cases} 
  h_n(x) & \text{if } x \in T_n, \\
  h_n^{-1}(x) & \text{if } x \in h_n(T_n), \\
  x & \text{otherwise.}
\end{cases}
\]

Then $h$ takes $A$ to $B$ as advertised. That $h \in H(\mathcal{F}/\mathcal{U})$ is proved as in the proof of 4.10; one needs to check that $T' = \bigcup_{2 \leq n < \omega} (T_n \cup h_n(T_n))$ is $\mathcal{F}$-closed. But $T'$ is $\mathcal{F}$-closed in $B$ ($T$ is a $\mathcal{F}$-neighborhood of 0 missing $T'$), which in turn is $\mathcal{U}$-closed in $\mathbb{R}$.

**References**


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