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H-enrichments of topologies

Paul Bankston*

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53233, USA

Robert A. McCoy

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0123, USA

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Abstract

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An *H-enrichment* of a topology $\mathcal{T}$ on a set X is a topology $\mathcal{S}$ on X such that $\mathcal{S} \supseteq \mathcal{T}$ and every homeomorphism from X to itself with respect to $\mathcal{T}$ is also a homeomorphism with respect to $\mathcal{S}$. An H-enrichment is a *C-enrichment* if "homeomorphism" can be replaced by "continuous function" above. Generally in "nice" spaces, there is a scarcity of C-enrichments and an abundance of H-enrichments. We capitalize on the scarcity of C-enrichments to prove classification theorems for minimally free rings of continuous real-valued functions; with H-enrichments in general, we focus on separation and connectedness axioms.

Keywords: H-enrichments, C-enrichments, minimally free rings of continuous functions.

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Introduction

The notion "H-enrichment of a topology" came about during the investigation of minimally free rings of continuous real-valued functions (see [3, 4]), and was found to be of independent interest. Thus, although this notion has its origins firmly in algebra (see also [1, 2, 6]), our approach in this paper is altogether topological.

Given two topologies $\mathcal{S}$ and $\mathcal{T}$ on a set X, say $\mathcal{S}$ is an *enrichment* of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{S}$. Whenever $f: X \to X$ is continuous as a function from $\langle X, \mathcal{S}\rangle$ to $\langle X, \mathcal{T}\rangle$, we say f is *$\langle \mathcal{S}, \mathcal{T}\rangle$-continuous.* (Other related notions, e.g., "$\langle \mathcal{S}, \mathcal{T}\rangle$-homeomorphism", "$\mathcal{T}$-open set", are defined as one might expect.)

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Let $\mathscr{S}$ be an enrichment of $\mathscr{T}$ on the set X, with F a family of functions from X to itself. We say $\mathscr{S}$ is an *F-enrichment* of $\mathscr{T}$ if every member of F is $\langle\mathscr{S}, \mathscr{S}\rangle$-continuous. Three successively stronger instances of this notion are as follows: (i) *H-enrichment*, in which F is the family of $\langle\mathscr{T}, \mathscr{T}\rangle$-homeomorphisms; (ii) *C-enrichment*, in which F is the family of $\langle\mathscr{T}, \mathscr{T}\rangle$-continuous functions; and (iii) *coreflective enrichment*, in which F is the family of $\langle\mathscr{S}, \mathscr{T}\rangle$-continuous functions.

Historically, coreflective enrichments were the first F-enrichments to come to our attention [4], and are what arise when one applies a coreflective functor from the category of spaces and continuous functions to itself. We found in [4] that the "κ-free" unital (i.e., with a 1) rings of continuous real-valued functions on a topological space, i.e., those rings $C(\mathscr{X})$ possessing a subset P (a "pseudobasis") of cardinality κ such that every function from P into $C(\mathscr{X})$ extends uniquely to a (1-preserving) ring endomorphism on $C(\mathscr{X})$, are precisely those rings of the form $C(\langle\mathbb{R}^\kappa, \mathscr{T}\rangle)$, where $\mathbb{R}$ is the set of real numbers, and $\mathscr{T}$ is a realcompact coreflective enrichment of the κ-fold Tichonov power $\mathscr{U}^\kappa$ of the usual topology $\mathscr{U}$ on $\mathbb{R}$. (N.b. here we depart slightly from the terminology of [3, 4] by insisting that realcompact spaces automatically be completely regular (=Tichonov). Also, our separation axioms always assume the T_1 axiom without further comment.) A ring (or any algebraic system) that is κ-free for some cardinal κ is termed "minimally free". In [3] a complete classification of the κ-free unital rings $C(\mathscr{X})$ is given for countable κ; namely they are the rings $C(\langle\mathbb{R}^\kappa, \mathscr{U}^\kappa\rangle)$ and $C(\langle\mathbb{R}^\kappa, \mathscr{D}\rangle)$ (where $\mathscr{D}$ refers to the discrete topology on the appropriate underlying set). We are interested, *inter alia*, in the classification problem for uncountable κ in this paper. We present some partial results, but the issue is far from resolved.

0.1. Examples. (i) Let G be a coreflective functor on the category of topological spaces and continuous functions (so the image of G is a subcategory, and G is right-adjoint to the inclusion functor). Then for any space $\mathscr{X} = \langle X, \mathscr{T}\rangle$, we may view $G(\mathscr{X})$ as a space $\langle X, \mathscr{S}\rangle$ where $\mathscr{S}$ is a coreflective enrichment of $\mathscr{T}$. This phenomenon is very special; its best-known manifestations are: *discretization*, in which $\mathscr{S} = \mathscr{D}$; *$\kappa$-modification* ($\kappa$ an infinite cardinal), in which $\mathscr{S} = (\mathscr{T})_\kappa$, the smallest topology that includes all intersections of fewer than κ $\mathscr{T}$-open sets; *k-space modification*, in which $\mathscr{S} = k(\mathscr{T}) = \{A \subseteq X\colon A \cap K$ is open in K for each $\mathscr{T}$-compact $K \subseteq X\}$; and *sequential modification*, in which $\mathscr{S} = \sigma(\mathscr{T}) = \{A \subseteq X\colon$ whenever (x_n) is a sequence in X that converges to a point in A, then (x_n) is eventually in $A\}$.

(ii) Let $\langle X, \mathscr{T}\rangle$ be a topological space, $\mathscr{F}$ a collection of subsets of X. Define $\mathscr{T}_{\mathscr{F}}$ to be the topology on X with subbasis $\mathscr{T} \cup \{h(S)\colon S \in \mathscr{F}$ and h is a $\langle\mathscr{T}, \mathscr{T}\rangle$-homeomorphism$\}$. It is trivial to show that $\mathscr{T}_{\mathscr{F}}$ is always an H-enrichment of $\mathscr{T}$, and every H-enrichment of $\mathscr{T}$ may be obtained in this fashion. (A similar mechanism may be used to obtain the C-enrichments of $\mathscr{T}$: just adjoin the inverse images of members of $\mathscr{F}$ under $\langle\mathscr{T}, \mathscr{T}\rangle$-continuous functions. However, coreflective enrichments require a more complicated process involving transfinite induction.) Perhaps the most familiar example of this construction is the following: $X = \mathbb{R}$, $\mathscr{T} = \mathscr{U}$, and

$\mathcal{F} = \{A \subseteq \mathbb{R}: \mathbb{R}\backslash A \text{ is countable}\}$. Typical basic open sets in $\mathcal{U}_{\mathcal{F}}$ are of the form (open interval)\(countable set). ($\langle\mathbb{R}, \mathcal{U}_{\mathcal{F}}\rangle$ is hereditarily Lindelöf and nonseparable, but fails to be an L-space since it is not regular.)

The sequel is divided into two sections. Section 1 is about C-enrichments and gives conditions under which topologies have no proper nondiscrete C-enrichments. These topological results have direct applications to the classification problem for minimally free rings of continuous real-valued functions. Section 2 deals with H-enrichments in general. Usually a topology has many H-enrichments; we focus on conditions that permit or prohibit the existence of H-enrichments that satisfy certain separation and connectedness axioms.

1. *C*-enrichments

The main result of [3] is the classification theorem: Let $1 \leq \kappa \leq \omega$ (where ω always stands for the first infinite cardinal). A unital ring $C(\mathcal{X})$ is κ-free if and only if $C(\mathcal{X})$ is of the form $C(\langle\mathbb{R}^{\kappa}, \mathcal{U}^{\kappa}\rangle)$ or $C(\langle\mathbb{R}, \mathcal{D}\rangle)$. The topological lemma that powers this theorem is the following [3, Theorem 1.1]: *Let $\mathcal{X}$ be a normed linear space over the real field. Then any proper C-enrichment of the norm topology is discrete.*

This lemma has gone through several stages of generalization (with the same basic idea of proof). Originally proved by the second author for the case $\mathcal{X} = \langle\mathbb{R}, \mathcal{U}\rangle$, we soon realized that the argument goes through for the Euclidean topologies $\mathcal{U}^n$, $1 \leq n < \omega$; thence to the form that appears in [3]. The next stage of generalization was the locally convex metrizable topological vector space topologies; finally, in a private communication [8], Sanderson saw that the original argument could be adapted to work for all normal locally path-connected first countable topologies. This section is devoted to that generalization, as well as to related results.

Let $\langle X, \mathcal{T}\rangle$ be a topological space, with F a family of functions from X to itself. Define $\langle X, \mathcal{T}\rangle$ to be *F-filled* provided that for any nonisolated point $x \in X$ and each nonclosed $S \subseteq X$ there exists a finite subset $\{f_1, \ldots, f_n\} \subseteq F$ such that $f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S) = N\backslash\{x\}$, where N is a neighborhood of x (i.e., $x \in U \subseteq N$ for some $U \in \mathcal{T}$). The family F is called *monoidal* if: (i) every member of F is $\langle\mathcal{T}, \mathcal{T}\rangle$-continuous; (ii) F contains the identity function id_X on X; and (iii) F is closed under function composition. (F-enrichments for F monoidal include both C-enrichments and H-enrichments.)

Our first result is a characterization of when the topology $\mathcal{T}$ on X has a proper nondiscrete F-enrichment when F is monoidal.

1.1. Theorem. *Let $\mathcal{X} = \langle X, \mathcal{T}\rangle$ be a topological space, F a monoidal family. Then $\mathcal{T}$ has a proper nondiscrete F-enrichment if and only if $\mathcal{X}$ is not F-filled.*

Proof. First suppose $\mathcal{X}$ is F-filled, and suppose $\mathcal{S}$ is a proper F-enrichment of $\mathcal{T}$. Then there exists a subset $S \subseteq X$ that is $\mathcal{S}$-closed but not $\mathcal{T}$-closed. Let x be any nonisolated point of $\mathscr{X}$. Then there is a finite $\{f_1, \ldots, f_n\} \subseteq F$ such that $f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S) = N \backslash \{x\}$ for some $\mathscr{T}$-neighborhood of x. By definition, each f_i is $\langle \mathscr{S}, \mathscr{S} \rangle$-continuous, so $N\backslash\{x\}$ is $\mathscr{S}$-closed. Thus there is an $\mathscr{S}$-neighborhood M of x such that $M \cap N = \{x\}$; whence x is an $\mathscr{S}$-isolated point. Since x was arbitrarily chosen, we infer that $\mathscr{S}$ is discrete.

For the converse, suppose $\mathscr{X}$ is not F-filled. Then there is a nonisolated $x \in X$ and a nonclosed $S \subseteq X$ such that for every finite subset $\{f_1, \ldots, f_n\}$ of $F, f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S)$ cannot be of the form $N\backslash\{x\}$ for some neighborhood N of x. Let $\mathscr{B}$ be the union of $\mathscr{T}$ with the family of all sets of the form $U \cap f_1^{-1}(X\backslash S) \cap \cdots \cap f_n^{-1}(X\backslash S)$, where $U \in \mathscr{T}$ and $f_1, \ldots, f_n \in F$. $\mathscr{B}$, being closed under finite intersections, is a basis for some topology $\mathscr{S} \supseteq \mathscr{T}$. To show $\mathscr{S}$ is an F-enrichment of $\mathscr{T}$, let $f \in F$ and $B = U \cap f_1^{-1}(X\backslash S) \cap \cdots \cap f_n^{-1}(X\backslash S) \in \mathscr{B}$. Then

$$f^{-1}(B) = f^{-1}(U) \cap (f_1 \circ f)^{-1}(X\backslash S) \cap \cdots \cap (f_n \circ f)^{-1}(X\backslash S) \in \mathscr{B}.$$

Thus f is $\langle \mathscr{S}, \mathscr{S} \rangle$-continuous. Since $\mathrm{id}_X \in F$, we have $X\backslash S \in \mathscr{S}$. But S is not $\mathscr{T}$-closed, so $\mathscr{S}$ is a proper F-enrichment of $\mathscr{T}$. It remains to show $\mathscr{S}$ is nondiscrete; we show x is not $\mathscr{S}$-isolated. Suppose to the contrary that $\{x\} \in \mathscr{S}$. Then we can write $\{x\} = U \cap f_1^{-1}(X\backslash S) \cap \cdots \cap f_n^{-1}(X\backslash S)$ for some $U \in \mathscr{T}$ and $f_1, \ldots, f_n \in F$. Then $X\backslash\{x\} = (X\backslash U) \cup f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S)$, so that $U\backslash\{x\} = U \cap (f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S))$. Let $N = \{x\} \cup f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S)$. Then $x \in U \subseteq N$, so N is a neighborhood of x; moreover $N\backslash\{x\} = f_1^{-1}(S) \cup \cdots \cup f_n^{-1}(S)$, a contradiction. Thus $\mathscr{S}$ is a proper nondiscrete F-enrichment of $\mathscr{T}$. □

1.2. Theorem. *Every normal, locally path-connected, first countable space is C-filled.*

Proof. Suppose $\langle X, \mathscr{T} \rangle$ satisfies the hypothesis, and let $x \in X$ be nonisolated, $S \subseteq X$ be nonclosed. Let x_0 be an accumulation point of S that is not in S. Since the space is locally path-connected and first countable, one can easily construct by induction a basis $\{V_n : 1 \leq n < \omega\}$ at x_0 consisting of path-connected open sets, such that $V_{n+1} \subseteq V_n$ for each n. Let $\{U_n : 1 \leq n < \omega\}$ be a basis at x consisting of open sets, such that $\overline{U_{n+1}} \subseteq U_n$ for each n (where overbar indicates topological closure). This we can do because the space is normal. We may assume $U_1 = X$; for each n define the "annulus" $A_n = \overline{U_n} \backslash U_{n+1}$. Also for each n, let $x_n \in V_n \cap S$, and let $p_n : [0, 1] \to V_n$ be a (continuous) path starting at x_n and ending at x_{n+1}.

Since $\langle X, \mathscr{T} \rangle$ is normal, there is for each n, a continuous $g_n : \overline{U_{2n-1}} \backslash U_{2n+2} \to [0, 1]$ and $h_n : \overline{U_{2n}} \backslash U_{2n+3} \to [0, 1]$ such that $g_n(A_{2n-1}) = \{0\}$, $g_n(A_{2n+1}) = \{1\}$, $h_n(A_{2n}) = \{0\}$, and $h_n(A_{2n+2}) = \{1\}$. Then define continuous f_1, f_2 on $\langle X, \mathscr{T} \rangle$ by:

$$f_1(y) = \begin{cases} p_n(g_n(y)) & \text{if } y \in \overline{U_{2n-1}} \backslash U_{2n+2} \text{ for some } n, \\ x_0 & \text{if } y = x, \end{cases}$$

$$f_2(y) = \begin{cases} p_n(h_n(y)) & \text{if } y \in \overline{U_{2n}} \backslash U_{2n+3} \text{ for some } n, \\ x_0 & \text{if } y = x. \end{cases}$$

Note that for each n, $f_1(A_{2n-1})=\{x_n\}$ and $f_2(A_{2n})=\{x_n\}$. Thus $\bigcup\{A_{2n-1}: 1\leq n<\omega\}\cup\bigcup\{A_{2n}: 1\leq n<\omega\}\subseteq f_1^{-1}(S)\cup f_2^{-1}(S)$; whence $f_1^{-1}(S)\cup f_2^{-1}(S)=X\backslash\{x\}$, so $\langle X, \mathcal{T}\rangle$ is C-filled. □

As mentioned earlier Theorems 1.1 and 1.2 are considerably more than adequate to establish the classification theorem for κ-free unital rings of continuous real-valued functions where $1\leq\kappa\leq\omega$ (the $\kappa=0$ case being trivial). The possibility of a complete classification when $\kappa\geq\omega_1$ (where ω_1 always stands for the first uncountable cardinal) seems remote; however we are able to give a continuum hypothesis (CH: $\omega_1=2^\omega=$ the power $|\mathbb{R}|$ of the continuum) classification for the ω_1-free rings $C(\mathscr{X})$ that are not connected.

Recall that a unital ring is *connected* just in case its only idempotents are 0 and 1. Thus $C(\mathscr{X})$ is a connected ring if and only if $\mathscr{X}$ is a connected space. In order to prove our CH-dependent classification results, we need two preliminary theorems. The first is similar in form to Theorem 1.2.

1.3. Theorem. *If $\langle X, \mathcal{T}\rangle$ is either zero-dimensional and first countable or regular and of character $\kappa\geq\omega_1$ such that $(\mathcal{T})_\kappa=\mathcal{T}$, then $\mathscr{X}$ is C-filled.*

Proof. Assume first that $\langle X, \mathcal{T}\rangle$ is zero-dimensional and first countable, and let $x\in X$ be a nonisolated point, $S\subseteq X$ nonclosed. Let $\{U_n: 1\leq n<\omega\}$ be a basis at x consisting of clopen sets such that $U_{n+1}\subseteq U_n$ for each n. We may assume $U_1=X$. For each n, let $A_n=U_n\backslash U_{n+1}$, with (x_n) a sequence in S converging to some $x_0\notin S$. Define $f: X\to X$ by

$$f(y)=\begin{cases}x_n & \text{if } y\in A_n \text{ for some } n,\\ x_0 & \text{if } y=x.\end{cases}$$

Then f is continuous and $f^{-1}(S)=X\backslash\{x\}$. Thus $\langle X, \mathcal{T}\rangle$ is C-filled.

Now assume $\langle X, \mathcal{T}\rangle$ is regular and of character $\kappa\geq\omega_1$, such that $(\mathcal{T})_\kappa=\mathcal{T}$. Then it is an easy exercise to show that $\langle X, \mathcal{T}\rangle$ is zero-dimensional. One can now mimic the proof in the first paragraph. We have a basis $\{U_\xi: 1\leq\xi<\kappa\}$ at x consisting of clopen sets such that $U_{\xi+1}\subseteq U_\xi$ for each ξ, and a κ-indexed sequence (x_ξ) converging to some $x_0\notin S$. □

The second theorem is an improvement upon Theorem 2.6 in [3].

1.4. Theorem. *Let $\mathcal{T}$ be a C-enrichment of $\mathscr{U}^\kappa$. Then the following are equivalent:*

(a) $(\mathscr{U}^\kappa)_{\omega_1}\subseteq\mathcal{T}$;

(b) *$\mathcal{T}$ is not a connected topology;*

(c) *$\mathcal{T}$ is not a path-connected topology;*

(d) *there exists a $\mathscr{U}^\kappa$-convergent sequence in $\mathbb{R}^\kappa$ that is not $\mathcal{T}$-convergent.*

Proof. The implications (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) are immediate.

For the implication (c)$\Rightarrow$(d), assume (c). Then there are points $x, y \in \mathbb{R}^\kappa$ such that the affine line segment A with endpoints x and y is not a path in $\langle \mathbb{R}^\kappa, \mathcal{T}\rangle$. Thus there exists a $V \in \mathcal{T}$ such that $V \cap A$ is not open in A with respect to the topology $\mathcal{U}^\kappa | A$ inherited from $\mathcal{U}^\kappa$. This means that there is a point $a \in V \cap A$ and a sequence (a_n) in $A \setminus V$ such that (a_n) $\mathcal{U}^\kappa$-converges to a. Such a sequence cannot $\mathcal{T}$-converge.

We note in passing that the chain of implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) uses nothing more than the fact that $\mathcal{U}^\kappa \subseteq \mathcal{T}$. None of the implications can be reversed, however, without the assumption of C-enrichment.

The implication (d)$\Rightarrow$(a) can be proved in a manner similar to the way we proved Theorem 1.2; except that $\mathcal{U}^\kappa$ is not a normal topology for uncountable κ, so the proof has to be modified. Assuming (d), let (x_n), $1 \leq n < \omega$, be a sequence in $\mathbb{R}^\kappa$ that $\mathcal{U}^\kappa$-converges to x_0 but does not $\mathcal{T}$-converge. Let $S = \{x_n : 1 \leq n < \omega\}$. Then S is $\mathcal{T}$-closed.

Let $U_1 = \mathbb{R}^\kappa$, and for $n = 2, 3, \ldots$ let $U_n = \prod_{\alpha<\kappa} I_\alpha$, where $I_\alpha = (-1/n, 1/n)$ if $\alpha \leq n$ and $I_\alpha = \mathbb{R}$ if $\alpha > n$. Also for each $n \geq 1$, define $A_n = \overline{U_n} \setminus U_{n+1}$ (where overbar indicates $\mathcal{U}^\kappa$-closure). Finally let $G = \bigcap_{n=1}^\infty U_n$. Given any G_δ set B and $x \in B$, there is a G' with $x \in G' \subseteq B$ and a $\langle \mathcal{U}^\kappa, \mathcal{U}^\kappa\rangle$-homeomorphism taking G onto G'. Consequently, if any H-enrichment $\mathcal{S}$ of $\mathcal{U}^\kappa$ contains G, then $(\mathcal{U}^\kappa)_{\omega_1} \subseteq \mathcal{S}$. It remains, then, to show $G \in \mathcal{T}$.

For each $1 \leq n < \omega$, define $g_n : \overline{U_{2n-1}} \setminus U_{2n+2} \to [0, 1]$ and $h_n : \overline{U_{2n}} \setminus U_{2n+3} \to [0, 1]$ as follows. First let $\pi : \mathbb{R}^\kappa \to \mathbb{R}^{2n+5}$ be the projection of $\mathbb{R}^\kappa$ onto the product of the first $2n+5$ factors of $\mathbb{R}^\kappa$ (starting with the 0th factor). Now $\pi(A_{2n-1})$ and $\pi(A_{2n+1})$ are disjoint closed subsets of $\mathbb{R}^{2n+5}$, as are $\pi(A_{2n})$ and $\pi(A_{2n+2})$. So there exist continuous $g'_n : \pi(\overline{U_{2n-1}} \setminus U_{2n+2}) \to [0, 1]$ and $h'_n : \pi(\overline{U_{2n}} \setminus U_{2n+3}) \to [0, 1]$ such that $g'_n(\pi(A_{2n-1})) = \{0\}$, $g'_n(\pi(A_{2n+1})) = \{1\}$, $h'_n(\pi(A_{2n})) = \{0\}$, and $h'_n(\pi(A_{2n+2})) = \{1\}$. Then define $g_n = g'_n \circ \pi | (\overline{U_{2n-1}} \setminus U_{2n+1})$ and $h_n = h'_n \circ \pi | (\overline{U_{2n}} \setminus U_{2n+3})$.

For each $1 \leq n < \omega$, let $p_n : [0, 1] \to \mathbb{R}^\kappa$ be the function defined by $p_n(t) = tx_{n+1} + (1-t)x_n$. Now define f_1 and f_2 on $\mathbb{R}^\kappa$ by:

$$f_1(x) = \begin{cases} p_n(g_n(x)) & \text{if } x \in \overline{U_{2n-1}} \setminus U_{2n+2} \text{ for some } n, \\ x_0 & \text{if } x \in G, \end{cases}$$

$$f_2(x) = \begin{cases} p_n(h_n(x)) & \text{if } x \in \overline{U_{2n}} \setminus U_{2n+3} \text{ for some } n, \\ x_0 & \text{if } x \in G, \\ x_1 & \text{if } x \in \mathbb{R}^\kappa \setminus U_2. \end{cases}$$

The functions f_1 and f_2 are $\langle \mathcal{U}^\kappa, \mathcal{U}^\kappa\rangle$-continuous, hence they are $\langle \mathcal{T}, \mathcal{T}\rangle$-continuous. Since $f_1^{-1}(S) \cup f_2^{-1}(S) = \mathbb{R}^\kappa \setminus G$ and S is $\mathcal{T}$-closed, we have $G \in \mathcal{T}$. Therefore $(\mathcal{U}^\kappa)_{\omega_1} \subseteq \mathcal{T}$. $\square$

The version of Theorem 1.4 proved in [3] is the equivalence (a)$\Leftrightarrow$(c). Just that and Theorem 1.3 can be used to get the desired classification. (The full strength of Theorem 1.4 comes into play later on.)

1.5. Corollary. (CH) *Every ω_1-free nonconnected ring $C(\mathscr{X})$ is of the form $C(\langle\mathbb{R}^{\omega_1}, (\mathscr{U}^{\omega_1})_{\omega_1}\rangle)$ or $C(\langle\mathbb{R}^{\omega_1}, \mathscr{D}\rangle)$.*

Proof. We know from [4] that every ω_1-free ring $C(\mathscr{X})$ is isomorphic to $C(\langle\mathbb{R}^{\omega_1}, \mathscr{T}\rangle)$, where $\mathscr{T}$ is a coreflective enrichment of $\mathscr{U}^{\omega_1}$ that is realcompact. Since $C(\mathscr{X})$ is a nonconnected ring, $\mathscr{T}$ is not a connected topology; hence by Theorem 1.4, $\mathscr{T} \supseteq (\mathscr{U}^{\omega_1})_{\omega_1}$. Now the weight of $(\mathscr{U}^{\omega_1})_{\omega_1}$ can be easily shown to lie between ω_1 and 2^{ω}. By the CH, the weight, hence the character, is exactly ω_1. By Theorem 1.3, then, $\langle\mathbb{R}^{\omega_1}, (\mathscr{U}^{\omega_1})_{\omega_1}\rangle$ is C-filled; by Theorem 1.1 there can be no proper nondiscrete C-enrichment. The topologies $(\mathscr{U}^{\omega_1})_{\omega_1}$ and $\mathscr{D}$ are well known to be realcompact coreflective enrichments of $\mathscr{U}^{\omega_1}$ (see [5, 11]), and the proof is complete. □

For $\kappa \leq \omega$, the only connected κ-free ring $C(\mathscr{X})$ is $C(\langle\mathbb{R}^{\kappa}, \mathscr{U}^{\kappa}\rangle)$. We conjecture that this is the case for all κ, and the remainder of this section is devoted to developing some evidence for our conjecture.

As pointed out in Examples 0.1(i), $k(\mathscr{U}^{\kappa})$ and $\sigma(\mathscr{U}^{\kappa})$ are both coreflective enrichments of $\mathscr{U}^{\kappa}$ (see, e.g., [9, 10] for more detailed discussions of this fact); they are connected, and $\mathscr{U}^{\kappa}, k(\mathscr{U}^{\kappa})$, and $\sigma(\mathscr{U}^{\kappa})$ are distinct for $\kappa > \omega$ (as we show presently). We do not know whether they, or any other connected coreflective enrichments of $\mathscr{U}^{\kappa}$, are realcompact, or even regular.

The following theorem collects what we know about $k(\mathscr{U}^{\kappa})$ and $\sigma(\mathscr{U}^{\kappa})$, and makes strong use of Theorem 1.4.

1.6. Theorem. (i) *$k(\mathscr{U}^{\kappa})$ and $\sigma(\mathscr{U}^{\kappa})$ are connected coreflective enrichments of $\mathscr{U}^{\kappa}$.*

(ii) *Every connected C-enrichment of $\mathscr{U}^{\kappa}$ is contained in $\sigma(\mathscr{U}^{\kappa})$.*

(iii) *If $\mathscr{T}$ is any C-enrichment of $\mathscr{U}^{\kappa}$, then either $\mathscr{T} \subseteq \sigma(\mathscr{U}^{\kappa})$ or $\mathscr{T} \supseteq (\mathscr{U}^{\kappa})_{\omega_1}$.*

(iv) *Let $\kappa > \omega$. Then the connected spaces $\langle\mathbb{R}^{\kappa}, \mathscr{U}^{\kappa}\rangle$, $\langle\mathbb{R}^{\kappa}, k(\mathscr{U}^{\kappa})\rangle$, and $\langle\mathbb{R}^{\kappa}, \sigma(\mathscr{U}^{\kappa})\rangle$ are topologically distinct.*

Proof. (i) As noted above, $k(\mathscr{T})$ and $\sigma(\mathscr{T})$ are well known to be coreflective enrichments of $\mathscr{T}$ for any topology $\mathscr{T}$. If $\mathscr{T}$ is path-connected, then so is $k(\mathscr{T})$ since both topologies share the same compact subsets (hence paths). $\sigma(\mathscr{U}^{\kappa})$ is connected by Theorem 1.4: Both $\sigma(\mathscr{U}^{\kappa})$ and $\mathscr{U}^{\kappa}$ share the same convergent sequences.

(ii) Suppose $\mathscr{T}$ is a connected C-enrichment of $\mathscr{U}^{\kappa}$, with $V \in \mathscr{T}$ and (x_n) a sequence in $\mathbb{R}^{\kappa}$ that $\mathscr{U}^{\kappa}$-converges to $x_0 \in V$. By Theorem 1.4, (x_n) also $\mathscr{T}$-converges to x_0; hence (x_n) is eventually in V. Thus $V \in \sigma(\mathscr{U}^{\kappa})$.

(iii) This is immediate from Theorem 1.4 and (ii) above.

(iv) Let $\kappa > \omega$. It is well known (see [10, Exercise 43H]) that $\langle\mathbb{R}^{\kappa}, \mathscr{U}^{\kappa}\rangle$ is not a k-space; hence $\langle\mathbb{R}, \mathscr{U}^{\kappa}\rangle$ and $\langle\mathbb{R}^{\kappa}, k(\mathscr{U}^{\kappa})\rangle$ are nonhomeomorphic. $\langle\mathbb{R}^{\kappa}, \mathscr{U}^{\kappa}\rangle$ is not a sequential space either, because otherwise $\mathscr{U}^{\kappa} = \sigma(\mathscr{U}^{\kappa})$, hence $\mathscr{U}^{\kappa} = k(\mathscr{U}^{\kappa})$. Therefore $\langle\mathbb{R}, \mathscr{U}^{\kappa}\rangle$ and $\langle\mathbb{R}^{\kappa}, \sigma(\mathscr{U}^{\kappa})\rangle$ are nonhomeomorphic. To show that $\langle\mathbb{R}^{\kappa}, k(\mathscr{U}^{\kappa})\rangle$ and $\langle\mathbb{R}^{\kappa}, \sigma(\mathscr{U}^{\kappa})\rangle$ are nonhomeomorphic, it suffices to show $k(\mathscr{U}^{\kappa})$ is not a sequential topology.

If $x \in \mathbb{R}^\kappa$ and $\alpha < \kappa$, we let x_α abbreviate $\pi_\alpha(x)$, the image of x under the αth projection map. Let $A = \{x \in \mathbb{R}^\kappa$: for some $\alpha < \kappa$, $x_\beta = 0$ for $\beta \leq \alpha$ and $x_\beta = 1$ for $\alpha < \beta < \kappa\}$.

Let $K = [0, 1]^\kappa$. Then K is a $\mathcal{U}^\kappa$-compact subset of $\mathbb{R}^\kappa$ containing A. Now A is not $\mathcal{U}^\kappa$-closed in A, since $0 \notin A$ is an accumulation point. Thus A is not $k(\mathcal{U}^\kappa)$-closed. But when $\kappa > \omega$, no sequence in A can $\mathcal{U}^\kappa$-converge without being eventually constant. Thus A is trivially $\sigma(\mathcal{U}^\kappa)$-closed. □

One way to show that $C(\langle \mathbb{R}^\kappa, \mathcal{U}^\kappa \rangle)$ is the only connected κ-free ring of continuous functions is to show that no proper connected C-enrichment of $\mathcal{U}^\kappa$ is regular. Toward this end, we define an enrichment $\mathcal{S}$ of a topology $\mathcal{T}$ on X to be *thick* if whenever $U \in \mathcal{S}$ is $\mathcal{T}$-dense, then U is also $\mathcal{S}$-dense. Clearly if $\mathcal{S}$ is a thick enrichment of $\mathcal{T}$ and $\mathcal{T}'$ is any topology with $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{S}$, then $\mathcal{T}'$ is also a thick enrichment of $\mathcal{T}$. We show presently that no proper thick C-enrichment of $\mathcal{U}^\kappa$ can be regular. This fact, together with the conjecture that $\sigma(\mathcal{U}^\kappa)$ is a thick enrichment of $\mathcal{U}^\kappa$, would gain us the desired result (in light of Theorem 1.6).

1.7. Examples. (i) For any $\kappa > 0$, $(\mathcal{U}^\kappa)_{\omega_1}$ is not a thick enrichment of $\mathcal{U}^\kappa$. Thus any enrichment of $\mathcal{U}^\kappa$ that contains $(\mathcal{U}^\kappa)_{\omega_1}$ fails to be thick.

(ii) Let X be an uncountable set, with $\mathcal{T}$ the cofinite topology on X. The cocountable topology $\mathcal{S}$ on X is just $(\mathcal{T})_{\omega_1}$. Every nonempty $U \in \mathcal{S}$ is $\mathcal{S}$-dense, so $\mathcal{S}$ is a thick coreflective enrichment of the T_1 topology $\mathcal{T}$.

1.8. Remarks. (i) If $\mathcal{S}$ contains $\mathcal{T}$ as a *π-basis* (i.e., every nonempty member of $\mathcal{S}$ contains a nonempty member of $\mathcal{T}$), then $\mathcal{S}$ is clearly a thick enrichment of $\mathcal{T}$. The converse holds if $\mathcal{S}$ happens to be regular, but not in general (see Examples 1.7(ii)).

(ii) The standard ways of constructing sets in $k(\mathcal{U}^\kappa) \backslash \mathcal{U}^\kappa$ for $\kappa > \omega$ (see [10, Exercise 43H]) fuel the belief that $k(\mathcal{U}^\kappa)$ may contain $\mathcal{U}^\kappa$ as a π-basis. Typically, a $\kappa(\mathcal{U}^\kappa)$-open set not already in $\mathcal{U}^\kappa$ can be found by adjoining a single accumulation point to a $\mathcal{U}^\kappa$-open set.

(iii) We do not know whether $\sigma(\mathcal{U}^\kappa)$ is a thick enrichment of $\mathcal{U}^\kappa$. $\sigma(\mathcal{U}^\kappa)$ does not contain $\mathcal{U}^\kappa$ as a π-basis, however; for let $A = \{x \in \mathbb{R}^\kappa: x_\alpha = 0$ for all but countably many indices $\alpha\}$. Then A is $\sigma(\mathcal{U}^\kappa)$-closed and $\mathcal{U}^\kappa$-dense.

In order to prove no proper thick C-enrichment of $\mathcal{U}^\kappa$ is regular, we first need a lemma. By way of notation, if $\langle X, \mathcal{T} \rangle$ is a space and $A \subseteq X$, let $\mathrm{Cl}_\mathcal{T}(A)$ (respectively $\mathrm{Int}_\mathcal{T}(A)$) denote the $\mathcal{T}$-closure (respectively $\mathcal{T}$-interior) of A in X.

1.9. Lemma. *Let $\mathcal{S}$ be a thick enrichment of $\mathcal{T}$ on the set X. Then for each $V \in \mathcal{S}$, $\mathrm{Int}_\mathcal{T}(\mathrm{Cl}_\mathcal{S}(V))$ is a $\mathcal{T}$-dense subset of $\mathrm{Cl}_\mathcal{S}(V)$.*

Proof. Suppose otherwise. Then there is a nonempty $V \in \mathcal{S}$ such that $\mathrm{Int}_\mathcal{T}(\mathrm{Cl}_\mathcal{S}(V))$ is not $\mathcal{T}$-dense in $\mathrm{Cl}_\mathcal{S}(V)$, so the set $W = V \backslash \mathrm{Cl}_\mathcal{T}(\mathrm{Int}_\mathcal{T}(\mathrm{Cl}_\mathcal{S}(V)))$ is a nonempty member of $\mathcal{S}$. Let $C = \mathrm{Cl}_{\mathcal{S}}(V) \cap \mathrm{Cl}_{\mathcal{S}}(X \backslash \mathrm{Cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{S}}(V))))$. Then it is easy to show that $\mathrm{Int}_{\mathcal{T}}(C) = \emptyset$. $\mathrm{Cl}_{\mathcal{S}}(W) \subseteq C$, so $W' = X \backslash \mathrm{Cl}_{\mathcal{S}}(W)$ is a $\mathcal{T}$-dense $\mathcal{S}$-open set. Since $\mathcal{S}$ is a thick enrichment of $\mathcal{T}$, W' is $\mathcal{S}$-dense as well. But $W' \cap W = \emptyset$, a contradiction. □

1.10. Theorem. *For every cardinal κ, $\mathcal{U}^\kappa$ has no proper regular thick C-enrichment.*

Proof. Suppose $\mathcal{T}$ is a proper regular thick C-enrichment of $\mathcal{U}^\kappa$. We derive a contradiction by showing $\mathcal{T} \supseteq (\mathcal{U}^\kappa)_{\omega_1}$ and appealing to Examples 1.7(i). By Theorem 1.4, it suffices to find a $\mathcal{U}^\kappa$-convergent sequence in $\mathbb{R}^\kappa$ that does not $\mathcal{T}$-converge.

Let $V \in \mathcal{T} \backslash \mathcal{U}^\kappa$. Since $\langle \mathbb{R}^\kappa, \mathcal{U}^\kappa \rangle$ is (point-)homogeneous, we may assume without loss of generality that $0 \in V \backslash \mathrm{Int}_{\mathcal{U}^\kappa}(V)$. Since $\mathcal{T}$ is a regular topology, there is some $W \in \mathcal{T}$ with $0 \in W \subseteq \mathrm{Cl}_{\mathcal{T}}(W) \subseteq V$. Set $W' = X \backslash \mathrm{Cl}_{\mathcal{T}}(W')$. By Lemma 1.9, we know that $U = \mathrm{Int}_{\mathcal{U}^\kappa}(\mathrm{Cl}_{\mathcal{T}}(W'))$ is $\mathcal{U}^\kappa$-dense in $\mathrm{Cl}_{\mathcal{T}}(W')$. Since $0 \in \mathrm{Cl}_{\mathcal{U}^\kappa}(X \backslash V)$ and $X \backslash V \subseteq W'$, we have $0 \in \mathrm{Cl}_{\mathcal{U}^\kappa}(W')$. But then $0 \in \mathrm{Cl}_{\mathcal{U}^\kappa}(\mathrm{Cl}_{\mathcal{T}}(W'))$ so $0 \in \mathrm{Cl}_{\mathcal{U}^\kappa}(U)$.

Using Zorn's lemma, let $\mathcal{B}$ be a maximal family of pairwise disjoint $\mathcal{U}^\kappa$-basic open subsets of U. Then $\bigcup \mathcal{B}$ is $\mathcal{U}^\kappa$-dense in U, so $0 \in \mathrm{Cl}_{\mathcal{U}^\kappa}(\bigcup \mathcal{B})$. Now $\mathcal{U}^\kappa$ satisfies the countable chain condition, so $\mathcal{B}$ must be countable, say $\mathcal{B} = \{B_n : 1 \leq n < \omega\}$. Each B_n is $\mathcal{U}^\kappa$-basic open, so we write $B_n = \bigcap \{\pi_\alpha^{-1}(U_{n,\alpha}) : \alpha \in A_n\}$, where $A_n \subseteq \kappa$ is finite and $U_{n,\alpha}$ is a $\mathcal{U}$-open subset of $\mathbb{R}$. Set $A = \bigcup_{n=1}^\infty A_n$, a countable set, let $\pi : \mathbb{R}^\kappa \to \mathbb{R}^A$ be the natural projection map, and define $\iota : \mathbb{R}^A \to \mathbb{R}^\kappa$ to be the natural injection that turns an A-sequence into a κ-sequence by filling in the missing coördinates with zeros. Let $\mathcal{U}^A = \mathcal{U}^\kappa | \mathbb{R}^A$, and let $U^A = \pi(\bigcup \mathcal{B})$. $U^A \in \mathcal{U}^A$ since π is a $\langle \mathcal{U}^\kappa, \mathcal{U}^A \rangle$-open map; $0 \in \mathrm{Cl}_{\mathcal{U}^A}(U^A)$ since π is $\langle \mathcal{U}^\kappa, \mathcal{U}^A \rangle$-continuous and $\pi(0) = 0$. Because $\mathcal{U}^A$ is a first countable topology, there is a sequence (x_n^A) in U^A that $\mathcal{U}^A$-converges to 0 in $\mathbb{R}^A$. For each n, let $x_n = \iota(x_n^A)$. Then (x_n) is a sequence in U that $\mathcal{U}^\kappa$-converges to 0 in $\mathbb{R}^\kappa$. But W is a $\mathcal{T}$-open neighborhood of 0 that is disjoint from U. Thus (x_n) fails to $\mathcal{T}$-converge. This establishes the contradiction, and hence the theorem. □

2. *H*-enrichments

In [3], as well as in the last section, we traded on the paucity of C-enrichments to obtain classification theorems for minimally free rings of continuous real-valued functions. With H-enrichments, however, the emphasis is on abundance.

2.1. Theorem. *If $\mathcal{X} = \langle X, \mathcal{T} \rangle$ has a nonclosed set that is nowhere dense, then $\mathcal{X}$ is not H-filled. Hence $\mathcal{T}$ has proper nondiscrete H-enrichments.*

Proof. Let $S \subseteq X$ be a nonclosed nowhere dense subset. Then X is nondiscrete, so we pick $x \in X$ a nonisolated point. Let $\{h_1, \ldots, h_n\}$ be any finite set of $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphisms. Then $h_n^{-1}(S) \cup \cdots \cup h_n^{-1}(S)$ is nowhere dense in $\mathcal{X}$. Let N be any neighborhood of x. Then $x \in \overline{N\backslash\{x\}}$, so $N \subseteq \overline{N\backslash\{x\}}$. Thus $N\backslash\{x\}$ fails to be nowhere dense in $\mathscr{X}$, and so $h_1^{-1}(S) \cup \cdots \cup h_n^{-1}(S) \neq N\backslash\{x\}$. Therefore $\mathscr{X}$ is not H-filled, and, by Theorem 1.1, $\mathscr{T}$ has a proper nondiscrete H-enrichment. □

2.2. Corollary. *If $\mathscr{X}$ is any first countable Hausdorff space whose derived set (= set of nonisolated points) is nondiscrete, then $\mathscr{X}$ is not H-filled.*

Proof. Let X' be the derived set of $\mathscr{X}$. By hypothesis, we have in X' a sequence (x_n) converging to some $x \in X'$ with $x \neq x_n$ for all n. Let $S = \{x_n : n < \omega\}$, a nonclosed subset of X whose closure, because of the Hausdorff axiom, is $S \cup \{x\}$. $\bar{S}$ thus does not contain any nonempty open subset of X, so S is a nonclosed nowhere dense subset of X. By Theorem 2.1, $\mathscr{X}$ is not H-filled. □

2.3. Examples. (i) Let $X = \{0\} \cup \{1/n : 0 < n < \omega\} \subseteq \mathbb{R}$ inherit the usual topology $\mathscr{T} = \mathscr{U}|X$. Then $\mathscr{T}$ is a nondiscrete metrizable topology that is H-filled.

(ii) Let X be countably infinite, with $\mathscr{T}$ the cofinite topology on X. Then $\mathscr{T}$ is a first countable T_1 topology with no isolated points that is H-filled.

Recall the notation of Examples 0.1(ii). $\mathscr{T}$ is a topology on X, $\mathscr{F}$ is a collection of subsets of X, and $\mathscr{T}_{\mathscr{F}}$ is the H-enrichment of $\mathscr{T}$ with subbasis $\mathscr{T} \cup \{h(S) : S \in \mathscr{F}$ and h is a $\langle\mathscr{T}, \mathscr{T}\rangle$-homeomorphism$\}$.

2.4. Theorem. *Suppose $\langle X, \mathscr{T}\rangle$ is a space and the complement of every member of $\mathscr{F}$ is $\mathscr{T}$-nowhere dense. Then every $\mathscr{T}$-dense subset of X is also $\mathscr{T}_{\mathscr{F}}$-dense. Thus, if $\mathscr{T}$ is nondiscrete, so also is $\mathscr{T}_{\mathscr{F}}$.*

Proof. Let $D \subseteq X$ be any $\mathscr{T}$-dense subset of X, and let T be any nonempty $\mathscr{T}_{\mathscr{F}}$-open set. Since $\langle\mathscr{T}, \mathscr{T}\rangle$-homeomorphic images of $\mathscr{T}$-nowhere dense sets are $\mathscr{T}$-nowhere dense, and finite unions of $\mathscr{T}$-nowhere dense sets are $\mathscr{T}$-nowhere dense, T is of the form $U\backslash A$ where $U \in \mathscr{T}$ and A is $\mathscr{T}$-nowhere dense. Since $T \neq \emptyset$, T must contain a nonempty $\mathscr{T}$-open set. Thus $T \cap D \neq \emptyset$. If $\mathscr{T}$ is nondiscrete, there is a $\mathscr{T}$-dense set $D \neq X$. Thus $\mathscr{T}_{\mathscr{F}}$ is also nondiscrete. □

One way to construct "minimal" H-enrichments of $\mathscr{T}$ on a set X is to take $\mathscr{T}_{\mathscr{F}}$ where $\mathscr{F}$ consists of just one $\mathscr{T}$-nonopen set. Lack of care in the choice of this set can result in the discrete topology.

2.5. Proposition. *Let $A \subseteq \mathbb{R}$ be $\mathscr{U}$-closed but not $\mathscr{U}$-open. Then $\mathscr{U}_{\{A\}}$ is discrete.*

Proof. Since A is $\mathscr{U}$-closed and not $\mathscr{U}$-open, A is not $\mathscr{U}$-dense. Let $<$ be the natural ordering on the real line, with $a < b < c$ such that $A \cap (a, c) = \emptyset$. Then either $A \cap (-\infty, b) \neq \emptyset$ or $A \cap (b, \infty) \neq \emptyset$. Suppose $A \cap (-\infty, b) \neq \emptyset$. Then $B = A \cap (-\infty, b)$ is $\mathscr{U}_{\{A\}}$-open. Also $B = A \cap (-\infty, b]$, so B is $\mathscr{U}$-closed. Let d be the least upper bound of B. Then $d \in B$. Let $h: \mathbb{R} \to \mathbb{R}$ be reflection through d. Then h is a $\langle \mathcal{U}, \mathcal{U} \rangle$-homeomorphism, hence a $\mathcal{U}_{\{A\}}$-homeomorphism, and $h(B) \in \mathcal{U}_{\{A\}}$. But then $\{d\} = B \cap h(B) \in \mathcal{U}_{\{A\}}$. It is a triviality that an H-enrichment of a homogeneous topology is also homogeneous. Thus every point of $\mathbb{R}$ is $\mathcal{U}_{\{A\}}$-isolated. □

The remainder of this paper is concerned with how separation and connectedness axioms tie in with H-enrichments. We begin with a well-known class of spaces which have all but one point isolated. Let X be an uncountable set, $x_0 \in X$, and κ a cardinal such that $\omega \leq \kappa < |X|$. The topology $\mathcal{T}(x_0, \kappa)$ has as open basis all singleton sets $\{x\}$ for $x \in X \setminus \{x_0\}$ and all sets U containing x_0 such that $|X \setminus U| \leq \kappa$. Clearly $\mathcal{T}(x_0, \kappa)$ is a zero-dimensional Hausdorff topology, hence regular, and x_0 is the only nonisolated point of X. The following useful fact is trivial to show.

2.6. Lemma. *A bijection $f: X \to X$ is a $\mathcal{T}(x_0, \kappa)$-homeomorphism if and only if $f(x_0) = x_0$.*

2.7. Proposition. *Let X, x_0, and κ be as above. The following are equivalent*:

(a) *There is a cardinal λ with $\kappa < \lambda < |X|$*;

(b) *$\mathcal{T}(x_0, \kappa)$ has a proper nondiscrete regular H-enrichment*;

(c) *$\mathcal{T}(x_0, \kappa)$ has a proper nondiscrete H-enrichment.*

Proof. (a)⇒(b) Let $\kappa < \lambda < |X|$. Then $\mathcal{T}(x_0, \lambda)$ is a proper nondiscrete regular H-enrichment of $\mathcal{T}(x_0, \kappa)$.

(b)⇒(c) Trivial.

(c)⇒(a) Suppose $\mathcal{T}$ is a proper nondiscrete H-enrichment of $\mathcal{T}(x_0, \kappa)$. Let $U \in \mathcal{T} \setminus \mathcal{T}(x_0, \kappa)$. Then $x_0 \in U$ and $|X \setminus U| > \kappa$. Let $\lambda = |X \setminus U|$. Suppose, by way of contradiction, that $\lambda = |X|$, and let $A \subseteq X \setminus U$ have cardinality $|U|$. Let $f: X \to X$ be any bijection such that $f(x_0) = x_0$ and $f(A) = U \setminus \{x_0\}$. Then, by Lemma 2.6, f is a $\mathcal{T}(x_0, \kappa)$-homeomorphism, hence a $\mathcal{T}$-homeomorphism. Thus $A \cup \{x_0\} = f^{-1}(U) \in \mathcal{T}$, so $\{x_0\} = (A \cup \{x_0\}) \cap U \in \mathcal{T}$, and $\mathcal{T}$ is discrete. □

2.8. Remark. Note that the hypothesis of Theorem 2.1 is never satisfied for the spaces $\langle X, \mathcal{T}(x_0, \kappa) \rangle$. However, the question of whether $\langle X, \mathcal{T}(x_0, \kappa) \rangle$ is H-filled is equivalent to that of whether the cardinal interval $(\kappa, |X|)$ is empty.

The following result illustrates how attempts to preserve connectedness in passing to H-enrichments by adding "large" sets can preclude regularity. Recall that a space is *Baire* if countable intersections of dense open sets are dense; a subset of a Baire space is *residual* if it contains such a countable intersection. In Baire spaces, the family of all residual subsets forms a countably complete filter of sets.

2.9. Theorem. *Let $\langle X, \mathcal{T} \rangle$ be a Baire space with $\mathcal{F}$ a family of $\mathcal{T}$-residual subsets. Then*:

(i) *If $\mathcal{T}$ is connected, so is $\mathcal{T}_{\mathcal{F}}$*;

(ii) *If $\mathcal{T}_{\mathcal{F}}$ is proper, then it is nonregular.*

Proof. (i) Let $\mathscr{C}$ be a $\mathscr{T}_{\mathscr{F}}$-open cover of X, which we may take to be basic; set $\mathscr{C}=\{U_\alpha \cap A_\alpha : \alpha<\kappa\}$, where each U_α is $\mathscr{T}$-open and each A_α is $\mathscr{T}$-residual. Suppose $x \in U_\alpha \cap A_\alpha$, $y \in U_\beta \cap A_\beta$. $\langle X, \mathscr{T}\rangle$ is connected and $\mathscr{C}'=\{U_\alpha : \alpha<\kappa\}$ is a $\mathscr{T}$-open cover of X, so there is a finite "chain" $U_1, \ldots, U_n$ from $\mathscr{C}'$ such that $U_1=U_\alpha$, $U_n=U_\beta$, and for $1 \leq i<n$, $U_i \cap U_{i+1} \neq \emptyset$. Let $A_1, \ldots, A_n$ be chosen such that $A_1=A_\alpha$, $A_n=A_\beta$, and each $U_i \cap A_i$ is in $\mathscr{C}$. Then for $1 \leq i<n$, $(U_i \cap A_i) \cap(U_{i+1} \cap A_{i+1})=(U_i \cap U_{i+1}) \cap(A_i \cap A_{i+1})$ is clearly nonempty. Thus $\mathscr{T}_{\mathscr{F}}$ is a connected topology.

(ii) Assume $\mathscr{T}_{\mathscr{F}} \neq \mathscr{T}$. Then there is some $A \in \mathscr{F} \backslash \mathscr{T}$. Let $x \in A \cap \mathrm{Cl}_{\mathscr{T}}(X \backslash A)$. We show x cannot be separated via $\mathscr{T}_{\mathscr{F}}$ from the $\mathscr{T}_{\mathscr{F}}$-closed set $X \backslash A$. For choose any $\mathscr{T}_{\mathscr{F}}$-basic open $U \cap B$ containing x (where $U \in \mathscr{T}$ and B is $\mathscr{T}$-residual), and let W contain $X \backslash A$ and be $\mathscr{T}_{\mathscr{F}}$-open. Let $y \in U \cap(X \backslash A)$, with $\mathscr{T}_{\mathscr{F}}$-basic open $V \cap C$ in W containing y. Then $(U \cap B) \cap W \supseteq(U \cap B) \cap(V \cap C)=(U \cap V) \cap(B \cap C) \neq \emptyset$, since $y \in U \cap V$ and $B \cap C$ is $\mathscr{T}$-residual. □

Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational numbers, with $\mathscr{U}'=\mathscr{U}|\mathbb{Q}$, the usual topology on $\mathbb{R}$ restricted to $\mathbb{Q}$ (well known to be the order topology on $\mathbb{Q}$). We prove below that $\mathscr{U}'$ has no proper nondiscrete regular H-enrichments (hence, because of Corollary 2.2, $\mathscr{U}'$ has proper nondiscrete H-enrichments, but they must all be nonregular). We first need two lemmas.

2.10. Lemma. *Let $\mathscr{T}$ be a nondiscrete H-enrichment of $\mathscr{U}'$. If $T \in \mathscr{T}$ and I is an open interval in $\mathbb{R}$ such that $T \cap I$ is nonempty, then $(\mathbb{Q} \backslash T) \cap I$ is not $\mathscr{U}'$-dense in I.*

Proof. Suppose that $(\mathbb{Q} \backslash T) \cap I$ is $\mathscr{U}'$-dense in I, with $x \in T \cap I$. Write $(\mathbb{Q} \backslash T) \cap I=A \cup B$, where A and B are disjoint $\mathscr{U}'$-dense subsets of I, and set $C=A \cup(T \cap I)$, $D=B \cup\{x\}$. Then C and D are countable $\mathscr{U}'$-dense subsets of I having only x in common, $C \cup D=\mathbb{Q} \cap I$. We obtain an increasing bijection (hence a $\langle\mathscr{U}', \mathscr{U}'\rangle$-homeomorphism) $h: \mathbb{Q} \rightarrow \mathbb{Q}$ fixing x and all elements of $\mathbb{Q} \backslash I$, and interchanging C and D by use of a simple "back-and-forth" order-theoretic construction. h is thus a $\langle\mathscr{T}, \mathscr{T}\rangle$-homeomorphism, and $\{x\}=h(T \cap I) \cap(T \cap I) \in \mathscr{T}$. Since $\langle\mathbb{Q}, \mathscr{T}\rangle$ is homogeneous, $\mathscr{T}$ must be discrete. We have a contradiction, therefore $(\mathbb{Q} \backslash T) \cap I$ cannot be $\mathscr{U}'$-dense in I. □

2.11. Lemma. *Let $\mathscr{T}$ be a nondiscrete H-enrichment of $\mathscr{U}'$. If $T \in \mathscr{T}$ and I is an open interval in $\mathbb{R}$ such that $T \cap I$ is $\mathscr{U}'$-dense in I, then $(\mathbb{Q} \backslash T) \cap I$ is $\mathscr{U}'$-nowhere dense in I.*

Proof. Suppose, to the contrary, that there is a nonempty open interval $J \subseteq \mathbb{R}$ such that $J \subseteq I$ and $(\mathbb{Q} \backslash T) \cap J$ is $\mathscr{U}'$ dense in J. By Lemma 2.10, $T \cap J=\emptyset$; so $T \cap I$ cannot be $\mathscr{U}'$-dense in I. □

2.12. Theorem. *$\mathscr{U}'$ has no proper nondiscrete regular H-enrichment.*

Proof. Suppose that $\mathscr{T}$ is a proper nondiscrete regular H-enrichment of $\mathscr{U}'$, and let $T \in \mathscr{T} \backslash \mathscr{U}'$. Then there is an $x \in T$ and a sequence (x_n) in $\mathbb{Q} \backslash T$ that $\mathscr{U}'$-converges to x. We may assume that $x_1 > x_2 > \cdots$, and we let (p_n) be a sequence of irrational numbers such that $x_{n+1} < p_{n+1} < x_n < p_n$ for each n. Also for each n, let I_n be the interval (p_{n+1}, p_n). Since $\mathcal{T}$ is regular, there is an $S \in \mathcal{T}$ such that $x \in S \subseteq \bar{S} = \mathrm{Cl}_{\mathcal{T}}(S) \subseteq T$.

Suppose, for the sake of contradiction, that for each n, $S \cap I_n$ is not $\mathcal{U}'$-dense in I_n. Then for each n there is a nonempty open interval $J_n \subseteq I_n$ such that $S \cap J_n = \emptyset$. Each J_n may be chosen to be of the form (q_{2n}, q_{2n-1}) where (q_k) is a decreasing sequence of irrational numbers. Let $J = (x, r)$, and for each n, let $K_n = (q_{2n+1}, q_{2n})$. Then there is a $\langle \mathcal{U}', \mathcal{U}' \rangle$-homeomorphism h such that $h(t) = t$ for each $t \notin J$ and $h(K_n) = J_{n+1}$ for each n. Let $R = h(S) \cap S \cap (2x - r, r)$. $R \in \mathcal{T}$ since h is a $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphism. Now let f be the reflection $t \mapsto 2x - t$ about x in $\mathbb{Q}$. Then f is also a $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphism, and $\{x\} = f(R) \cap R \in \mathcal{T}$. This contradicts the assumption $\mathcal{T}$ is nondiscrete, so in fact there exists some n such that $S \cap I_n$ is $\mathcal{U}'$-dense in I_n.

By Lemma 2.11, then, $(\mathbb{Q} \backslash S) \cap I_n$ is $\mathcal{U}'$-nowhere dense in I_n. But $(\mathbb{Q} \backslash \bar{S}) \cap I_n$ is also $\mathcal{U}'$-nowhere dense in I_n; whence $\bar{S} \cap I_n$ is $\mathcal{U}'$-dense in I_n. Now $\bar{S} \cap I_n \subseteq T \cap I_n$, so that $x_n \in (\mathbb{Q} \backslash \bar{S}) \cap I_n$. Therefore, by Lemma 2.10, $\bar{S} \cap I_n$ is not $\mathcal{U}'$-dense in I_n. This contradiction establishes the theorem. □

We next turn to H-enrichments of Euclidean topologies (i.e., the topologies $\mathcal{U}^n$, $1 \leq n < \omega$). Our next result stands in contrast with Theorem 2.12.

2.13. Theorem. *The usual topology $\mathcal{U}$ on $\mathbb{R}$ has a proper nondiscrete completely regular H-enrichment.*

Proof. We define a topology $\mathcal{T} = \mathcal{U}_{\{A,B\}}$, where A and B are complementary $\mathcal{U}$-dense subsets of $\mathbb{R}$. $\mathcal{T}$ is *prima facie* a proper H-enrichment of $\mathcal{U}$; it remains to define the sets A, B judiciously so that $\mathcal{T}$ is nondiscrete and completely regular.

Let $\mathcal{H}$ be the $\langle \mathcal{U}, \mathcal{U} \rangle$-homeomorphisms, with $\mathcal{H}^*$ the set of finite subsets of $\mathcal{H}$. Both $\mathcal{H}$ and $\mathcal{H}^*$ have continuum cardinality c, and we let $\varphi : c \to \mathcal{U} \backslash \{\emptyset\}$, $\psi : c \to \mathcal{H}^*$ be bijections.

Define $\{A_\alpha\, \alpha < c\}$ and $\{B_\alpha : \alpha < c\}$ by induction on c as follows. First choose distinct points a, b from $\varphi(0)$ and define $A_0 = \{a\}$, $B_0 = \{b\}$. Next suppose $0 < \alpha < c$, and that $\{A_\beta : \beta < \alpha\}$ and $\{B_\beta : \beta < \alpha\}$ have been defined. Then define A_α and B_α as follows.

First define $\{C_\beta : \beta < \alpha\}$ and $\{D_\beta : \beta < \alpha\}$ by induction. Let $\beta < \alpha$. If $\beta = 0$, then take $C_\beta = \bigcup \{A_\gamma : \gamma < \alpha\}$ and $D_\beta = \bigcup \{B_\gamma : \gamma < \alpha\}$. If $\beta > 0$, then, assuming that $\{C_\gamma : \gamma < \beta\}$ and $\{D_\gamma : \gamma < \beta\}$ have been defined, let $C = \bigcup \{C_\gamma : \gamma < \beta\}$ and $D = \bigcup \{D_\gamma : \gamma < \beta\}$. Define C_β and D_β as follows.

Now $\psi(\beta) = \{h_i : i < n\}$ for some $n < \omega$. If there are $i, j < n$ with $i \neq j$ and $\varphi(\alpha) \cap \{x \in \mathbb{R}: h_i^{-1}(x) = h_j^{-1}(x)\}$ is $\mathcal{U}$-somewhere dense, then take $D_\beta = D$ and $C_\beta = C$. Otherwise, let $\xi : 2^n \to \varphi(\alpha) \backslash \bigcup \{h_i(C \cup D) : i < n\}$ be an injection such that for each $t \in 2^n$, $\{h_i^{-1}(\xi(t)) : i < n\}$ consists of n distinct points. Then define $C_\beta = C \cup \{h_i^{-1}(\xi(t)) : i < n, t \in 2^n, t(i) = 0\}$ and $D_\beta = D \cup \{h_i^{-1}(\xi(t)) : i < n, t \in 2^n, t(i) = 1\}$.

This completes the inner induction and defines $\{C_\beta : \beta < \alpha\}$ and $\{D_\beta : \beta < \alpha\}$. Now define $A_\alpha = \bigcup\{C_\beta : \beta < \alpha\}$ and $B_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$. This completes the outer induction and defines $\{A_\alpha : \alpha < c\}$ and $\{B_\alpha : \alpha < c\}$. Finally, define $A = \bigcup\{A_\alpha : \alpha < c\}$ and $B = \mathbb{R}\setminus A$.

Now define $\mathcal{T}$ to be $\mathcal{U}_{\{A,B\}}$. The fact that $\mathcal{T}$ is nondiscrete follows from the lemma below.

Lemma. *Let $X = h_1(A) \cap \cdots \cap h_m(A) \cap h_{m+1}(B) \cap \cdots \cap h_n(B)$, where $h_1, \ldots, h_n \in \mathcal{H}$. If $x \in X$, then there exists a $\mathcal{U}$-open neighborhood U of x such that $U \cap X$ is $\mathcal{U}$-dense in U.*

Proof. Define $U = \mathbb{R}\setminus\bigcup\{\{y \in \mathbb{R}: h_i^{-1}(y) = h_j^{-1}(y)\}: 1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant n\}$, which is a $\mathcal{U}$-open neighborhood of x. To show that $U \cap X$ is $\mathcal{U}$-dense in U, let V be a nonempty $\mathcal{U}$-open subset of U.

If $m > 1$, let $\{p_1, \ldots, p_s\}$ be an ordering of $\{\langle i,j\rangle: 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m,\ i \neq j\}$; and for each $p_k = \langle i,j\rangle$ let $C_k = \{y \in \mathbb{R}:\ h_i^{-1}(y) = h_j^{-1}(y)\}$. Likewise if $n > m+1$, let $\{q_1, \ldots, q_t\}$ be an ordering of $\{\langle i,j\rangle: m+1 \leqslant i \leqslant n,\ m+1 \leqslant j \leqslant n,\ i \neq j\}$; and for each $q_k = \langle i,j\rangle$, let $D_k = \{y \in \mathbb{R}:\ h_i^{-1}(y) = h_j^{-1}(y)\}$.

Define V' as follows. If $m = 1$, then take $V' = V$. If $m > 1$, then since $C_1, \ldots, C_s$ are $\mathcal{U}$-closed subsets of $\mathbb{R}$, there exists a nonempty $\mathcal{U}$-open subset V' of V such that for each $1 \leqslant k \leqslant s$, either $V' \subseteq C_k$ or $V' \cap C_k = \emptyset$.

Likewise define W as follows. If $n = m+1$, then take $W = V'$. If $n > m+1$, then choose W to be a nonempty $\mathcal{U}$-open subset of V' such that for each $1 \leqslant k \leqslant t$, either $W \subseteq D_k$ or $W \cap D_k = \emptyset$.

Now take H to be a subset of $\mathcal{H}$ having the following properties.

(1) If $m = 1$, then $h_1 \in H$.

(2) If $n = m+1$, then $h_n \in H$.

(3) If $m > 1$ and $W \subseteq C_k$, where $p_k = \langle i,j\rangle$, then H contains exactly one of h_i or h_j.

(4) If $m > 1$ and $W \cap C_k = \emptyset$, where $p_k = \langle i,j\rangle$, then H contains both h_i and h_j.

(5) If $n > m+1$ and $W \subseteq D_k$, where $q_k = \langle i,j\rangle$, then H contains exactly one of h_i or h_j.

(6) If $n > m+1$ and $W \cap D_k = \emptyset$, where $q_k = \langle i,j\rangle$, then H contains both h_i and h_j.

(7) H contains no other members of $\mathcal{H}$ besides those listed in properties (1)-(6).

To be specific, let $H = \{h'_1, \ldots, h'_a, h'_{a+1}, \ldots, h'_b\}$, where for $1 \leqslant i \leqslant a$, $h'_i \in \{h_1, \ldots, h_m\}$, and for $a+1 \leqslant j \leqslant b$, $h'_j \in \{h_{m+1}, \ldots, h_n\}$. Now $H = \psi(\beta)$ for some $\beta < c$. Choose some $\alpha > \beta$ such that $\varphi(\alpha) \subseteq W$. It follows from the construction of A and B that $\varphi(\alpha) \cap h'_1(A) \cap \cdots h'_a(A) \cap h'_{a+1}(B) \cap \cdots \cap h'_b(B) \neq \emptyset$. Since the only h'_j which are not in H come from the pairs of the form $p_k = \langle i,j\rangle$ where $W \subseteq C_k$ or $W \subseteq D_k$, we have $\varphi(\alpha) \cap X \neq \emptyset$ as desired. This establishes the Lemma.

To show that $\mathcal{T}$ is completely regular, let $x \in U \cap X$, where $U \in \mathcal{U}$ and $X = h_1(A) \cap \cdots \cap h_m(A) \cap h_{m+1}(B) \cap \cdots \cap h_n(B)$. Now X is both open and closed in $\langle \mathbb{R}, \mathcal{T}\rangle$. Choose $f: \langle \mathbb{R}, \mathcal{U}\rangle \to \langle \mathbb{R}, \mathcal{U}\rangle$ to be continuous so that $f(x) = 0$ and $f(\mathbb{R}\setminus U) = \{1\}$. Now define $g: \langle \mathbb{R}, \mathcal{T}\rangle \to \langle \mathbb{R}, \mathcal{U}\rangle$ by $g(y) = f(y)$ if $y \in X$ and $g(y) = 1$ if $y \in \mathbb{R}\setminus X$. Clearly

$g(x)=0$ and $g(\mathbb{R}\backslash(U\cap X))=\{1\}$. To see that g is continuous, let $V\in\mathscr{U}$. If $1\notin V$, then $g^{-1}(V)=f^{-1}(V)\cap X\in\mathscr{T}$. If $1\in V$, then $g^{-1}(V)=\mathbb{R}\backslash g^{-1}(\mathbb{R}\backslash V)=\mathbb{R}\backslash(f^{-1}(\mathbb{R}\backslash V)\cap X)\in\mathscr{T}$. □

2.14. Question. Is the topology constructed in Theorem 2.13 normal? If not, is there a proper nondiscrete normal H-enrichment of $\mathscr{U}$? (Nb.: The Sorgenfrey (or half-open interval) topology on $\mathbb{R}$ is well known to be a normal enrichment of $\mathscr{U}$. It is not an H-enrichment, however, since $-\mathrm{id}_{\mathbb{R}}$ is not a Sorgenfrey homeomorphism.)

The topology constructed in Theorem 2.13 is totally disconnected. This suggests the following.

2.15. Question. Is there a proper regular connected H-enrichment of $\mathscr{U}$ on $\mathbb{R}$?

2.16. Theorem. *Let $\mathscr{T}$ be an H-enrichment of $\mathscr{U}$. Then either $\mathscr{T}$ is totally disconnected or the $\mathscr{T}$-connected subsets of $\mathbb{R}$ are precisely the intervals. In the latter case, the $\langle\mathscr{T},\mathscr{T}\rangle$-homeomorphisms coincide with the $\langle\mathscr{U},\mathscr{U}\rangle$-homeomorphisms (i.e., the monotonic bijections).*

Proof. Clearly if $A\subseteq\mathbb{R}$ is not an interval then A is $\mathscr{U}$-disconnected, hence $\mathscr{T}$-disconnected. Thus the only $\mathscr{T}$-connected subsets lie among the intervals of $\mathbb{R}$. Suppose $C\subseteq\mathbb{R}$ is a nontrivial $\mathscr{T}$-connected set. We specify the *type* of the interval C in the usual way: if bounded, how many included endpoints; if unbounded, whether it is a ray and, if so, whether it includes its endpoint. Clearly, given any two intervals of the same type, there is a $\langle\mathscr{U},\mathscr{U}\rangle$-homeomorphism taking the first to the second. Thus every interval of the same type as C is $\mathscr{T}$-connected. Since $\mathbb{R}$ is a chain union of intervals of the same type as C, we see that $\mathscr{T}$ is a connected topology. Now let $[a,b]$ be any closed bounded interval. If $\{U, V\}$ is a $\mathscr{T}$-disconnection of $[a,b]$, then U and V are both $\mathscr{T}$-closed sets. Suppose $a\in U$ and $b\in V$. Then $\{(-\infty,a]\cup U, V\cup[b,\infty)\}$ is a $\mathscr{T}$-disconnection of $\mathbb{R}$, a contradiction. (If both a and b are in U, say, then we use $\{(-\infty,a]\cup[b,\infty)\cup U, V\}$.) Thus $[a,b]$ is $\mathscr{T}$-connected. Since every interval is a chain union of closed bounded intervals, we have our result. The classic intermediate value theorem then tells us that the $\langle\mathscr{T},\mathscr{T}\rangle$-homeomorphisms and the $\langle\mathscr{U},\mathscr{U}\rangle$-homeomorphisms are precisely the monotonic bijections. □

2.17. Corollary. *There is no proper nondiscrete locally connected H-enrichment of $\mathscr{U}$.*

Proof. If $\mathscr{T}$ is a proper locally connected H-enrichment of $\mathscr{U}$ with a basis of connected open sets, then this basis must consist of intervals by Theorem 2.16. Since $\mathscr{T}$ is proper, some of these intervals must have endpoints. Reflection about such an endpoint gives rise to two $\mathscr{T}$-open sets with a single point in common. By homogeneity, $\mathscr{T}$ must be discrete. □

We do not see how to extend Theorem 2.16 and Corollary 2.17 to higher powers of $\mathcal{U}$. The following lemma allows us to lift certain other arguments about H-enrichments of $\mathcal{U}$ to higher finite dimensions.

2.18. Lemma. *Let (x_m) and (y_m) be two sequences in $\mathbb{R}^n$, $n \geq 2$, that $\mathcal{U}^n$-coverge to x and y respectively. Suppose further that the distances $\|x_m - x\|$ and $\|y_m - y\|$ are strictly decreasing with increasing m. Then there is a $\langle \mathcal{U}^n, \mathcal{U}^n \rangle$-homeomorphism taking x_m to y_m for each m.*

Proof. This assertion is false for $n = 1$ without a monotonicity assumption. We sketch a proof to show that if (x_m) is a sequence satisfying the hypothesis of the lemma, then (x_m) can be moved to a monotonic sequence (x'_m) on the (positive) first axis. We can then homeomorph one monotonic sequence to another on that axis and extend to a homeomorphism on all of $\mathbb{R}^n$ by crossing with the identity map on the orthogonal complement of the axis.

Without loss of generality, we may assume the sequence $x_1, x_2, \ldots$ converges to 0 in $\mathbb{R}^n$, with $\|x_1\| > \|x_2\| > \cdots$. For $m = 1, 2, \ldots$, let S_m be the $(n-1)$-sphere of radius $\|x_m\|$ centered at 0, with $A_1 = \{x: \|x\| \geq \|x_1\|\}$ and $A_m = \{x: \|x_m\| \leq \|x\| \leq \|x_{m-1}\|\}$, $m > 1$. For each m, let x'_m be the point of intersection of S_m with the positive first axis, and let $r_m : S_m \to S_m$ be a rotation of S_m which takes x_m to x'_m. One extends r_1 to a rotation h_1 on A_1 in the obvious way. Also in a straightforward manner one extends the rotations r_m and r_{m+1} to a homeomorphism h_{m+1} on A_{m+1} in such a way that $\|h_{m+1}(x)\| = \|x\|$ for all $x \in A_{m+1}$. (R. Mullins has written a proof in which each h_m is a linear map on $\mathbb{R}^n$ restricted to A_m.) We then define $h : \mathbb{R}^n \to \mathbb{R}^n$ as the obvious extension of the maps h_m, $h(0) = 0$; and it is a triviality to show h is a suitable homeomorphism. □

2.19. Theorem. *Let $\mathcal{T}$ be a proper H-enrichment of $\mathcal{U}^n$, $1 \leq n < \omega$. Then every $\mathcal{T}$-convergent sequence in $\mathbb{R}^n$ is eventually constant.*

Proof. First assume $n = 1$, and suppose (x_m) is a sequence of distinct terms that $\mathcal{T}$-converges to x. Then there is a monotonic subsequence (x_{m_k}) also $\mathcal{T}$-converging to x. Assume $\mathcal{T} \neq \mathcal{U}$. By homogeneity, the open intervals about x do not form a $\mathcal{T}$-neighborhood basis at x, so let $T \in \mathcal{T} \setminus \mathcal{U}$ be such that $x \in T$ and for $l = 1, 2, \ldots$ there is some $y_l \in (x - 1/l, x + 1/l) \setminus T$. The sequence (y_l) is not eventually constant, so there is a monotonic subsequence (y_{l_k}). This subsequence $\mathcal{U}$-converges, thus there is a monotonic bijection f on $\mathbb{R}$ taking x_{m_k} to y_{l_k}, $k = 1, 2, \ldots$. However, (y_{l_k}) does not $\mathcal{T}$-converge. Because f is a $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphism, (x_{m_k}) cannot $\mathcal{T}$-converge either, a contradiction. Thus our original $\mathcal{T}$-convergent sequence must be eventually constant.

Now assume $n \geq 2$. We argue as above using Lemma 2.18. The only change we make is in the subsequences (x_{m_k}) and (y_{l_k}): instead of monotonicity, we may assume monotonically decreasing distances to the points of $\mathcal{U}^n$-convergence. □

2.20. Corollary. *Let $1 \leq n < \omega$, with $\mathcal{T}$ a proper H-enrichment of $\mathcal{U}^n$. Then:*

(i) *$\mathbb{R}^n$ has no infinite $\mathcal{T}$-compact subsets (i.e., $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is "anticompact", or a "cf-space").*

(ii) *If $\mathcal{T}$ is nondiscrete, no point of $\mathbb{R}^n$ has a countable $\mathcal{T}$-neighborhood basis.*

(iii) *If $\mathcal{T}$ is nondiscrete, $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is not locally compact.*

(iv) *Every metrizable subset of $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is discrete in the subspace topology.*

Proof. (i) Let $C \subseteq \mathbb{R}^n$. If C is $\mathcal{T}$-compact, then C is $\mathcal{U}^n$-compact. Since $\mathcal{T} \supseteq \mathcal{U}^n$, and both topologies, when restricted to C, are compact Hausdorff, they must agree on C. But infinite compact subsets of $\langle \mathbb{R}^n, \mathcal{U}^n \rangle$ contain convergent sequences that are not eventually constant. This contradicts Theorem 2.19, so C must be finite.

(ii) If $\mathcal{T}$ is nondiscrete, then no point of $\mathbb{R}^n$ is isolated. A countable $\mathcal{T}$-neighborhood basis at x would give rise to a sequence of distinct terms $\mathcal{T}$-converging to x, contrary to Theorem 2.19.

(iii) This is an immediate consequence of (i) above.

(iv) This follows immediately from Theorem 2.19. □

2.21. Theorem. *Let $\mathcal{T}$ be a nondiscrete H-enrichment of $\mathcal{U}^n$, $1 \leq n < \omega$. Then every nonempty $\mathcal{T}$-open set has cardinality continuum.*

Proof. Fix $n > 0$, suppose $T \in \mathcal{T}$ is nonempty and of cardinality $< c$. Without loss of generality, we may assume $0 \in T$. For each nonsingular $n \times n$ matrix H over $\mathbb{R}$, let $\Gamma_H = \{\langle v, H_v \rangle : v \in \mathbb{R}^n\}$. Then Γ_H is an n-dimensional vector subspace of $\mathbb{R}^{2n}$. Now one can find c such matrices H such that any $n+1$ subspaces Γ_H have trivial intersection. Since $|T \times T| < c$, there is some such H with $\Gamma_H \cap (T \times T) = \{\langle 0, 0 \rangle\}$. Let h be the $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphism whose graph is Γ_H. Then $h(T) \cap T = \{0\}$. Since $h(T) \in \mathcal{T}$, we have $\{0\} \in \mathcal{T}$; hence $\mathcal{T}$ is discrete. □

Our last topic concerns whether certain well-known homogeneous enrichments of $\mathcal{U}^\omega$ are H-enrichments. The following lemma is taken from [3].

2.22. Lemma [3, Proposition 2.5(ii)]. *If $\mathcal{T}$ is an H-enrichment of $\mathcal{U}^\kappa$, then all straight lines in $\mathbb{R}^\kappa$ (viewed as an affine space) are equivalent via $\langle \mathcal{T}, \mathcal{T} \rangle$-homeomorphisms on $\mathbb{R}^\kappa$.*

2.23. Theorem. *The following homogeneous enrichments of $\mathcal{U}^\omega$ are not H-enrichments:*

(i) *the box topology;*

(ii) *the uniform topology; and*

(iii) *$\mathcal{D}^\omega$, the ω-fold power of the discrete topology.*

Proof. (i) In this topology, each axis in $\mathbb{R}^\omega$ inherits the usual topology. However, the diagonal in $\mathbb{R}^\omega$ inherits the discrete topology. By Lemma 2.22, the box topology cannot be an H-enrichment of $\mathcal{U}^\omega$.

(ii) If $\bar{\rho}(x, y) = \min\{|x-y|, 1\}$ denotes the truncated usual metric on $\mathbb{R}$, then $\bar{\rho}^\omega(s, t) = \sup_n \bar{\rho}(s(n), t(n))$ gives a metric for the uniform topology on $\mathbb{R}^\omega$. Two points $s, t \in \mathbb{R}^\omega$ lie in the same (connectedness) component of $\mathbb{R}^\omega$ relative to the uniform topology if and only if the sequence $(s(n) - t(n))$ is bounded in $\mathbb{R}$. Thus, if s and t lie in different components, then the line connecting s and t inherits a disconnected topology. But, as in (i) above, each axis in $\mathbb{R}^\omega$ inherits the usual topology. Again we resort to Lemma 2.22.

(iii) As for $\mathscr{D}^\omega$, all straight lines in $\mathbb{R}^\omega$ inherit the discrete topology, so Lemma 2.22 is useless here. Let $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be defined by its coördinate functions:

$$(\pi_k \circ h)(s) = \begin{cases} s(0) + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|s(n)|}{1+|s(n)|} & \text{if } k=0, \\ s(k) & \text{if } k>0. \end{cases}$$

Clearly h is a bijection; its inverse is given by:

$$(\pi_k \circ h^{-1})(s) = \begin{cases} s(0) - \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|s(n)|}{1+|s(n)|} & \text{if } k=0, \\ s(k) & \text{if } k>0. \end{cases}$$

The functions h and h^{-1} are $\langle \mathscr{U}^\omega, \mathscr{U}^\omega \rangle$-continuous since their coördinate functions are continuous onto $\langle \mathbb{R}, \mathscr{U} \rangle$. Thus, assuming $\mathscr{D}^\omega$ is an H-enrichment of $\mathscr{U}^\omega$, we infer that h is a $\langle \mathscr{D}^\omega, \mathscr{D}^\omega \rangle$-homeomorphism; hence that $f = \pi_0 \circ h : \mathbb{R}^\omega \to \mathbb{R}$ is $\langle \mathscr{D}^\omega, \mathscr{D} \rangle$-continuous. However, f takes the zero sequence to 0, and $\{0\}$ is a $\mathscr{D}$-open set. If U is a typical $\mathscr{D}^\omega$-basic open neighborhood of 0, then U is of the form $\prod_{n=0}^{\infty} U_n$, where $U_n = \{0\}$ for finitely many indices n, and $U_n = \mathbb{R}$ for the remaining indices. Thus $f(U) \not\subseteq \{0\}$, and f is not $\langle \mathscr{D}^\omega, \mathscr{D} \rangle$-continuous at 0, a contradiction. □

2.24. Corollary. *The relation of H-enrichment between topologies is not preserved under the taking of countable Tichonov powers.*

Proof. $\mathscr{D}$ is an H-enrichment of $\mathscr{U}$, but $\mathscr{D}^\omega$ is not an H-enrichment of $\mathscr{U}^\omega$ by Theorem 2.23(iii). □

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