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EXPRESSIVE POWER IN FIRST ORDER TOPOLOGY

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Abstract. A *first order representation* (f.o.r.) in topology is an assignment of finitary relational structures of the same type to topological spaces in such a way that homeomorphic spaces get sent to isomorphic structures. We first define the notions "one f.o.r. is at least as expressive as another relative to a class of spaces" and "one class of spaces is definable in another relative to an f.o.r.", and prove some general statements. Following this we compare some well-known classes of spaces and first order representations. A principal result is that if X and Y are two Tichonov spaces whose posets of zero-sets are elementarily equivalent then their respective rings of bounded continuous real-valued functions satisfy the same positive-universal sentences. The proof of this uses the technique of constructing ultraproducts as direct limits of products in a category theoretic setting.

§1. Introduction and general discussion. We are concerned in this paper with "first order topology" in the sense of [10]; in particular with various ways in which one can assert that one topological space "satisfies the same sentences" (in a first order language) as another. Since both topological and model theoretic notions will be used extensively, we refer the reader to [7] and [12] for notation and terminology in the former arena and to [6] for same in the latter.

Let L be a lexicon of finitary predicate and function symbols (with equality). (We also use the symbol L to stand for the first order language $L_{\omega\omega}$.) By a *topological L-representation* we mean an assignment of L-structures to topological spaces in such a way that homeomorphic spaces get sent to isomorphic structures. In practice, L-representations will be "functorial" in the sense that they also assign L-homomorphisms (i.e. atomic relation preserving functions) to continuous maps in the time-honored way. A *first order representation* (f.o.r.) is an L-representation for some L. Given two f.o.r.'s R and S, and a class $\mathscr{X}$ of spaces (i.e. close under homeomorphic copies) define "S is at least as expressive as R, relative to $\mathscr{X}$" (in symbols $R \leq_{\mathscr{X}} S$) if for any $X, Y \in \mathscr{X}$, if $S(X)$ and $S(Y)$ are elementarily equivalent $(S(X) \equiv S(Y))$ then so are $R(X)$ and $R(Y)$. We write $R \leq S$ to mean $R \leq_{\mathscr{X}} S$, where $\mathscr{X}$ is the class of all spaces. Given an f.o.r. R and two classes of spaces $\mathscr{X}$ and $\mathscr{Y}$, define "$\mathscr{X}$ is R-definable in $\mathscr{Y}$" (in symbols $\mathscr{X} \leq_R \mathscr{Y}$) if there is a sentence ϕ in the language $L(R)$ of R such that, for any $Y \in \mathscr{Y}$, $R(Y) \models \phi$ iff $Y \in \mathscr{X}$. $\mathscr{X}$ is *definable* in $\mathscr{Y}$ $(\mathscr{X} \leq \mathscr{Y})$ if $\mathscr{X} \leq_R \mathscr{Y}$ for some R. We say $\mathscr{X}$ is *definable* if $\mathscr{X} \leq \mathscr{Y}$, where $\mathscr{Y}$ is the class of all spaces.

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The following is a small collection of trivial facts concerning these relations.

1.1 PROPOSITION. (i) *The relations* $\leq_R$ *and* $\leq_{\mathscr{X}}$ *are reflexive and transitive. In fact* $\mathscr{X} \leq_R \mathscr{Y}$ *if either* $\mathscr{Y} \subseteq \mathscr{X}$ *or* $\mathscr{Y} \cap \mathscr{X} = \varnothing$.

(ii) $R \leq_{\varnothing} S$.

(iii) *If* $R \leq_{\mathscr{X}} S$ *and* $\mathscr{Y} \subseteq \mathscr{X}$ *then* $R \leq_{\mathscr{Y}} S$.

(iv) *If* $\mathscr{X} \leq_R \mathscr{Y}$ *and* $\mathscr{Z} \subseteq \mathscr{Y}$ *then* $\mathscr{X} \leq_R \mathscr{Z}$. □

Let R and S be f.o.r.'s, whose lexions are $L(R)$ and $L(S)$ respectively; and assume for the moment no nonconstant function symbols are present. We form the *join* $R \vee S$ in the following way. Let L consist of the disjoint union of $L(R)$, $L(S)$ and the two unary predicates U_R and U_S. This is the lexion for $R \vee S$; and, given a space X, the domain of $(R \vee S)(X)$ is the disjoint union of the domains of $R(X)$ and $S(X)$, the interpretation of U_R (resp. U_S) is the domain of $R(X)$ (resp. $S(X)$), and the interpretations of the symbols from $L(R)$ and $L(S)$ are given by the obvious inclusion maps into $(R \vee S)(X)$.

We now translate $L(R)$- and $L(S)$-formulas into $L(R \vee S)$-formulas in the obvious manner: atomic formulas are unchanged; the translation commutes with the logical connectives; and the $L(R)$-formula (resp. $L(S)$-formula) $\exists x\phi$ gets translated to $\exists x(U_R(x) \,\&\, \phi^t)$ (resp. $\exists x(U_S(x) \,\&\, \phi^t)$), where ϕ^t denotes the translate of ϕ. The following is then easy to check.

1.2. PROPOSITION. *Let R and S be f.o.r.'s, let X be a space, let $\phi(x_1,\dots,x_n)$ be an $L(R)$-formula (with free variables among $x_1,\dots,x_n$), and let $a_1,\dots,a_n \in R(X)$. Then* $R(X) \models \phi[a_1,\dots,a_n]$ *iff* $(R \vee S)(X) \models \phi^t[a_1,\dots,a_n]$. □

1.3. COROLLARY. $R, S \leq R \vee S$. □

1.4. COROLLARY. *If* $\mathscr{X} \leq_R \mathscr{Y}$ *and* $\mathscr{Y} \leq_S \mathscr{Z}$ *then* $\mathscr{X} \leq_{R \vee S} \mathscr{Z}$. *Thus the relation* $\leq$ *between classes of spaces is transitive.*

PROOF. Let ϕ (resp. ψ) be an $L(R)$-sentence (resp. $L(S)$-sentence) defining $\mathscr{X}$ in $\mathscr{Y}$ (resp. $\mathscr{Y}$ in $\mathscr{Z}$). Then $\phi^t \,\&\, \psi^t$ is an $L(R \vee S)$-sentence defining $\mathscr{X}$ in $\mathscr{Z}$. □

1.5. REMARK. In the above discussion, the stipulation that there be no function symbols is inessential. One could, for example, get around it by using multi-sorted logic.

§2. Some specific representations and classes compared. For the remainder of this paper, we will be dealing with the lexicons $L_{PO} = \{=, \leq\}$ and $L_R = \{=, +, \cdot, 0, 1\}$ of posets and unitary rings respectively, and will be concerned with the various comparisons of expressive power among the following first order representations of a space X: (i) $F(X) \supseteq Z(X) \supseteq B(X)$, the posets of closed, zero-, and clopen sets in X; and (ii) $C(X) \supseteq C^*(X)$, the rings of continuous and bounded continuous real-valued functions on X.

2.1. PROPOSITION. $B \leq Z$ *and* $B \leq F$.

PROOF. $B(X)$ is first order definable in $Z(X)$ and in $F(X)$ as the collection of complemented elements. □

2.2. THEOREM. *Let $\mathscr{X}$ be the class of Tichonov (= completely regular) spaces. Then* $Z \leq_{\mathscr{X}} C$ *and* $B \leq_{\mathscr{X}} C^*$.

PROOF. The technique is due to A. MacIntyre and is spelled out in [10, Theorem 5.1]. In particular one translates formulas of L_{PO} to formulas of L_R using the basic fact that if X is any Tichonov space and $f, g \in C(X)$ then their zero-sets $Z(f)$ and $Z(g)$ are disjoint iff $f^2 + g^2$ has an inverse. Thus $Z(f) \subseteq Z(g)$ iff for each

$h \in C(X)$, if $Z(g)$ and $Z(h)$ are disjoint then so are $Z(f)$ and $Z(h)$. This gives the clue to how the atomic L_{PO}-formula $x \leq y$ should be translated. Complex formulas are translated by commuting with the logical operations; hence the translate $\phi^t(x_1,\dots,x_n)$ of any L_{PO}-formula $\phi(x_1,\dots,x_n)$ has the same free variables and satisfies the condition that if X is any Tichonov space and $f_1,\dots,f_n \in C(X)$ then $C(X) \models \phi^t[f_1,\dots,f_n]$ iff $Z(X) \models \phi[Z(f_1),\dots,Z(f_n)]$. This clearly gets us $Z \leq_{\mathscr{X}} C$. To get $B \leq_{\mathscr{X}} C^*$, one modifies the above a bit. First note that the elements of $B(X)$ are the zero-sets of the idempotents of $C^*(X)$. Thus if f, g are idempotents then $Z(f)$ and $Z(g)$ are disjoint iff $f + g = 2$, and $Z(f) \subseteq Z(g)$ iff for every idempotent $h \in C^*(X)$, if $Z(g)$ and $Z(h)$ are disjoint then so are $Z(f)$ and $Z(h)$. □

Thus by 2.1 and 2.2 we see that B is of minimal expressive power among the first order representations considered here. To show that the minimality is proper, we prove the following.

2.3. THEOREM. *Let $\mathscr{X}$ be the class of Boolean* (=*totally disconnected compact Hausdorff*) *spaces. Then* $F, Z, C, C^* \not\leq_{\mathscr{X}} B$.

PROOF. In view of 2.2 and the fact that Boolean spaces are pseudocompact (i.e. $C = C^*$), it will suffice to find Boolean spaces X and Y such that $B(X) \equiv B(Y)$ but $F(X) \not\equiv F(Y)$ and $Z(X) \not\equiv Z(Y)$. Now it is well known that if X and Y are two self-dense (i.e. with no isolated points) Boolean spaces then $B(X)$ and $B(Y)$, being atomless Boolean algebras, are elementarily equivalent. So pick X extremally disconnected (i.e. interiors of closed sets are clopen), say $X =$ (the Stone space of the regular-open algebra for the real line); and pick Y so that Y is not basically disconnected (i.e. interiors of zero-sets are clopen), say $Y =$ (the Cantor discontinuum). We show that there is an L_{PO}-sentence ϕ such that, for any space W, W is extremally disconnected iff $F(W) \models \phi$ and W is basically disconnected iff $Z(W) \models \phi$. To get ϕ, we translate the English definitions above first into "pseudocode" and then into L_{PO}. The pseudocode is:

$$\forall x \exists y(\text{“}y \text{ clopen''} \,\&\, y \leq x \,\&\, \forall z(\text{“}z \cup x = 1\text{''} \rightarrow \text{“}z \cup y = 1\text{''})).$$

Now "$z \cup x = 1$" translates to

$$\forall u((z \leq u \,\&\, x \leq u) \rightarrow \forall v(v \leq u))$$

(similarly translate "$z \cap x = 0$"). It is easy now to translate "y clopen" to $\exists u(\text{“}y \cap u = 0\text{''} \,\&\, \text{“}y \cup u = 1\text{''})$. □

As an immediate consequence of the above proof, we have the following.

2.4. COROLLARY. *The class of extremally* (*resp. basically*) *disconnected spaces is F-definable* (*resp. Z-definable*). □

If X is any locally compact Tichonov space, let αX denote its (Aleksandrov) one-point compactification. Theorem 3.3 of [10] says that $F(\alpha X) \equiv F(\alpha Y)$ whenever X and Y are infinite discrete. Analogous results fail, however, when F is replaced by Z, C, or C^*. Theorem 5.2 of [10], due to J. R. Isbell, asserts that $Z(\alpha X) \not\equiv Z(\alpha Y)$ whenever X and Y are discrete, X is countable, and Y is uncountable. This proves the following theorem.

2.5. THEOREM. *Let $\mathscr{X}$ be the class of Boolean spaces. Then* $Z, C, C^* \not\leq_{\mathscr{X}} F$. □

2.6. THEOREM. *Let $\mathscr{X}$ be the class of pseudocompact Tichonov spaces. Then* $F \not\leq_{\mathscr{X}} Z, C, C^*$.

PROOF. We recall the definition of the "deleted Tichonov plank" $X = ((\omega + 1) \times (\omega_1 + 1)) \backslash \{\langle \omega, \omega_1 \rangle\}$ (see [7] or [12]).

The relevant facts are these:

(i) X is pseudocompact, hence $C(X) = C^*(X)$.

(ii) Letting β and υ denote respectively the Stone-Čech compactification and the Hewitt realcompactification operators, and setting $Y = \upsilon X$ (so $C(X) \cong C(Y)$), we have $Y \cong \beta X \cong \alpha X \cong (\omega + 1) \times (\omega_1 + 1)$. (Here, $\cong$ denotes isomorphism of relational structures, while $\simeq$ denotes homeomorphism of topological spaces.) By 2.2, $Z(X) \equiv Z(Y)$ (of course $Z(X) \ncong Z(Y)$ since $X \not\simeq Y$).

(iii) X is not normal. The "top" $\omega \times \{\omega_1\}$ and the "right-hand side" $\{\omega\} \times \omega_1$ are disjoint closed sets which cannot be separated by open sets. Of course Y, being compact Hausdorff, is normal; and it is an easy exercise to generate a sentence ϕ of L_{PO} such that a space W is normal iff $F(W) \models \phi$. Thus $F(X) \not\equiv F(Y)$. □

Using the above example, we have immediately:

2.7. COROLLARY. *The class of normal spaces is F-definable; however it is not R-definable in the class of pseudocompact Tichonov spaces for $R = Z, B, C, C^*$.* □

2.8. THEOREM. *Let $\mathscr{X}$ be the class of Tichonov spaces. Then $Z, C \not\leq_{\mathscr{X}} C^*$.*

PROOF. A *P-space* (see [7] or [12]) is a space all of whose G_δ-sets (i.e. countable intersections of open sets) are open. Now it is a triviality to see that no infinite compact Hausdroff space can also be a P-space, so let X be any infinite Tichonov P-space (say an infinite discrete space) and let $Y = \beta X$. Then of course $C^*(X) \cong C^*(Y)$. In [7, Exercise 4J] several conditions equivalent to "P-space" for Tichonov spaces are given; notably "every zero-set has a complement", and "the function ring is (von Neumann) regular" (i.e. for all f *there is a* g such that $f^2 \cdot g = f$). Thus $Z(X) \not\equiv Z(Y)$ and $C(X) \not\equiv C(Y)$. □

2.9. COROLLARY. *The class of P-spaces is both Z- and C-definable (but not C^*- or B-definable) in the class of Tichonov spaces.* □

In [10] several questions of the form "$\mathscr{X} \leq_R \mathscr{Y}$" are treated. Some of the notable ones are collected in the following assertion.

2.10. THEOREM ([10]). (i) (Corollary 3.8) *The class of compact spaces is not F-definable in the class of metric spaces.*

(ii) (Theorems 4.1 and 4.2) *Both of the homeomorphism types of the closed unit interval and the closed unit disk are F-definable in the class of metric spaces.*

(iii) (Theorem 5.4) *The homeomorphism type of the closed unit interval is R-definable in the class of Tichonov spaces, where R takes a space to its* $\mathbf{R}$*-algebra of real-valued continuous functions* (*i.e.* $L(R) = L_R \cup \{$*constant symbols for real numbers* $r \in \mathbf{R}$ *plus unary function symbols denoting scalar multiplication*$\}$).

(iv) (Theorem 5.5) *The class of Boolean spaces is Z-definable in the class of compact Hausdorff spaces.* □

We add to this list with the following negative result.

2.11. THEOREM. (i) *The class of metric spaces is not F-definable in the class of compact Hausdorff spaces.*

(ii) *The class of metric spaces is neither C^*-definable nor B-definable in the class of extremally disconnected normal Tichonov spaces.*

(iii) *If there exists an uncountable measurable cardinal then the class of metric spaces is not C-definable (and hence not Z-definable) in the class of Tichonov spaces.*

PROOF. (i) Theorem 3.3 of [10] says, as mentioned earlier, that $F(\alpha X) \equiv F(\alpha Y)$ whenever X and Y are infinite discrete. So let X be countable discrete and let Y be discrete of cardinal ω_1. Then αX is metric, but αY is not.

(ii) Again let X be countable discrete, and let $p \in \beta X$ be a free ultrafilter. Then $Y = X \cup \{p\} \subseteq \beta X$ is an extremally disconnected normal Tichonov space, like X, which is nonmetric (see [7]). It is also easy to check that $C^*(X) \cong C^*(Y)$ and $B(X) \cong B(Y)$. (However, $F(X) = Z(X) = B(X)$ and $F(Y) = Z(Y)$. But Y is not a P-space, so $Z(Y)$ is not complemented. Thus $F(X) \not\equiv F(Y)$, $Z(X) \not\equiv Z(Y)$ and $C(X) \not\equiv C(Y)$, making this a rather limited example.)

(iii) Let X be discrete of uncountable measurable cardinality. Then $Y = \upsilon X$ is a proper extension of X in βX and is hence nonmetric. But $C(X) \cong C(Y)$. (This tack does not work if there are no uncountable measurable cardinals. Theorem 15.24 of [7] says that if W is a metric space whose cardinality is less than the first uncountable measurable cardinal then W is realcompact.) $\square$

2.12. QUESTIONS. The reader can no doubt generate a vast number of questions relating to the above discussion, as well as to the treatment in [10]. The following are ones which we found interesting.

(i) Is it true that $C^* \leq_{\mathscr{X}} Z$ for $\mathscr{X}$ the class of Tichonov spaces? (One place to look for a counterexample is to take two self-dense P-spaces X and Y such that $C^*(X) \not\equiv C^*(Y)$ (if that is possible). For then $Z(X) = B(X) \equiv B(Y) = Z(Y)$.)

(ii) Give an example of an undefinable (for any f.o.r. R) class of spaces. In particular, is the class of compact Hausdorff spaces definable?

Although we do not believe that $Z(X) \equiv Z(Y)$ implies the same for $C^*(X)$ and $C^*(Y)$ for Tichonov spaces X and Y, we can prove a limited version of this. Given a lexicon L, define the set of *positive-universal* formulas of L to be the closure of the set of atomic formulas under conjunction, disjunction and universal quantification. If A and B are two L-structures define $A \equiv_0 B$ to mean that A and B satisfy the same positive-universal sentences. The next result brings in this new semantic notion, and the following section is devoted to the development of the machinery which we have found necessary for the proof.

2.13. THEOREM. (i) *If X and Y are two Tichonov (resp. normal) spaces such that $Z(X) \equiv Z(Y)$ (resp. $F(X) \equiv F(Y)$), then $C^*(X) \equiv_0 C^*(Y)$.*

(ii) *There exist Boolean spaces X and Y such that $C(X) \equiv_0 C(Y)$ but $Z(X) \not\equiv Z(Y)$.*

(iii) *The class of basically disconnected spaces is not C-definable, via either a positive-universal or an existential sentence, in the class of Boolean spaces.*

2.14. REMARK. The spaces we use in (ii) above are those from 2.5; namely we let X (resp. Y) be the one-point compactification of a countable (resp. uncountable) discrete space. Isbell noted that in $Z(Y)$ every atom is complemented, whereas in $Z(X)$ the atom corresponding to the unique point at infinity of X is not complemented. This condition, when translated to L_{PO}, is far from positive-universal; in fact its quantifier prenex (when put in normal form) has three alternations, and its quantifier-free matrix has apparently essential negations. We do not know whether $Z(X) \equiv_0 Z(Y)$, as MacIntyre's translation (see 2.2) translates the atomic formula $x \leq y$ to an L_R-formula with two quantifier alternations, and we do not know how to simplify this.

§3. The proof of Theorem 2.13. The proof technique we use involves constructing ultraproducts as direct limits of products in a category theoretic setting (see [11] for background in category theory), and is hence quite nonconstructive. For a sketch of the proof, with details to be filled in later, let X and Y be two Tichonov spaces such that $Z(X) \equiv Z(Y)$. By the ultrapower theorem of Keisler and Shelah (see [6]) there is an ultrafilter D such that the respective ultrapowers $\prod_D Z(X)$ and $\prod_D Z(Y)$ are isomorphic. Our method is not good enough (perhaps rightly so) to get $\prod_D C^*(X) \cong \prod_D C^*(Y)$; however what we do get is a variant $\prod'_D C^*(X) \cong \prod'_D C^*(Y)$ where, for any Tichonov space W, $\prod'_D C^*(W)$ is obtained from $\prod_D C^*(W)$ as follows. Let d: $C^*(W) \to \prod_D C^*(W)$ be the canonical elementary embedding. First a subring $\prod^\circ_D C^*(W) \subseteq \prod_D C^*(W)$ is specified which contains the image of d. Next an ideal is defined on this subring and q: $\prod^\circ_D C^*(W) \to \prod'_D C^*(W)$ is the canonical quotient map. It is then shown that $d' = q \circ d$ is an embedding of unitary rings. (This construction, analogous to "throwing away the infinite elements and moding out the infinitesimals", is not vastly different in concept from the "nonstandard hull" construction in [9]. In fact we construct a compact Hausdorff space W_D from W so that $\prod'_D C^*(W) \cong C(W_D)$.)

So let ϕ be a positive-universal L_R-sentence true in $C^*(X)$. Then $\prod_D C^*(X) \models \phi$. Since ϕ is universal, $\prod^\circ_D C^*(X) \models \phi$. Since ϕ is positive and $\prod'_D C^*(X)$ is a homomorphic image of $\prod^\circ_D C^*(X)$, ϕ is true there too; hence $\prod'_D C^*(Y) \models \phi$. Since $C^*(Y)$ is a unitary subring of $\prod'_D C^*(Y)$, we get $C^*(Y) \models \phi$.

Now suppose X and Y are normal Tichonov and $F(X) \equiv F(Y)$. Our construction will show how to get $\prod'_D C^*(X) \cong \prod'_D C^*(Y)$ from $\prod_D F(X) \cong \prod_D F(Y)$. Then to prove 2.13(ii) we use Isbell's examples X and Y. They are both Boolean spaces, hence normal, and $F(X) \equiv F(Y)$. Thus by 2.13(i), $C(X) = C^*(X) \equiv_0 C^*(Y) = C(Y)$, in spite of the fact that $Z(X) \not\equiv Z(Y)$.

Let X be any Tichonov space. We show how to link $\prod_D Z(X)$ with $\prod'_D C^*(X)$ using a topological construction called (for reasons which will emerge later) the "ultracoproduct", which arises as a consequence of viewing the usual ultraproduct (and reduced product in general) as a direct limit of products (see [3], [4], [5] and [8]). We will define the ultraproduct here in a more set theoretical way, however, and bring in category notions only when necessary.

Let $\langle\langle X_i, \mathcal{T}_i\rangle : i \in I\rangle$ be an indexed collection of topological spaces and let D be an ultrafilter on I. The *topological ultraproduct* $\prod_D \langle X_i, \mathcal{T}_i\rangle$ ($\prod_D X_i$ for short) is the space whose points are members of the set $\prod_D X_i$ (i.e. D-equivalence classes $f_D = \{g \in \prod_{i\in I} X_i : \{i : g(i) = f(i)\} \in D\}$ for $f \in \prod_{i \in I} X_i$) and whose open sets are basically generated by "open ultraboxes" $\prod_D U_i$, where $U_i \in \mathcal{T}_i$ for each $i \in I$. (This construction is studied extensively in [1], [2] and elsewhere.) It is easy to verify that if $\mathcal{B}_i \subseteq \mathcal{T}_i$ is an open base then ultraboxes $\prod_D U_i$, where $U_i \in \mathcal{B}_i$, also generate the ultraproduct topology. Furthermore, if $\mathcal{C}_i$ is a base of closed sets then "closed ultraboxes" $\prod_D C_i$, where $C_i \in \mathcal{C}_i$, generate the closed sets in the ultraproduct.

This said, let us turn to the process of compactifying a Tichonov space. (For notation and terminology, as well as historical background, the reader is referred to [12].) Suppose $\langle X_i : i \in I\rangle$ is a family of Tichonov spaces. Then the families $Z(X_i)$ are "normal bases" of closed sets, in the sense of O. Frink; and it is easy to show that the zero-set ultraboxes $\prod_D Z_i$ form a normal base for the topological ultraproduct. (Of course this base is in natural one-one correspondence with the usual ultraproduct $\prod_D Z(X_i)$, and we will abuse notation accordingly from time to time. Note that a zero-set ultrabox $\prod_D Z_i$ is not in general a zero-set, unless the Z_i are also open.) With any normal base $\mathcal{N}$ on a Tichonov space Y one can form the compactification $\omega(\mathcal{N})$, whose points are $\mathcal{N}$-ultrafilters and whose closed sets are generated by sets of the form $N^\# = \{p \in \omega(\mathcal{N}): N \in p\}$. For example if $\mathcal{N} = Z(Y)$ then $\omega(\mathcal{N})$ is the Stone-Čech compactification βY. (When Y is normal, $\beta Y = \omega(F(Y))$ as well.) We now define the *ultracoproduct*, via D, of the collection $\langle X_i : i \in I\rangle$ of Tichonov spaces to be $\sum_D X_i = \omega(\prod_D Z(X_i))$.

The first relatively straightforward observation is that $\sum_D X_i$ can be viewed as a subspace of the Stone-Čech compactification of the disjoint union $\dot{\bigcup}_{i\in I} X_i$. In particular $\sum_D X_i = \{p \in \beta(\dot{\bigcup}_{i\in I} X_i)$: for all $J \in D$, $\dot{\bigcup}_{i\in J} X_i \in p\}$ (= the zero-set ultrafilters which "extend" D). A typical basic closed set is

$$\sigma_D Z_i = \sum_D X_i \cap \left(\prod_D Z_i\right)^\# = \left\{p \in \sum_D X_i : \dot{\bigcup_{i\in I}} Z_i \in p\right\}.$$

The ultra(co)product notation for spaces also extends to maps between spaces. In particular if $\theta_i: X_i \to Y_i$ is a continuous map for each $i \in I$ then so is $\prod_D \theta_i$: $\prod_D X_i \to \prod_D Y_i$, defined by $\prod_D \theta_i(f_D) = g_D$, where $g(i) = \theta_i(f(i))$. Moreover the ultracoproduct map $\sum_D \theta_i: \sum_D X_i \to \sum_D Y_i$, defined by $\sum_D \theta_i(p) = q$, where $\prod_D Z_i \in q$ iff $\prod_D \theta_i^{-1}[Z_i] \in p$, is continuous and extends $\prod_D \theta_i$.

3.1 LEMMA. *The ultracoproducts $\sum_D X_i$ and $\sum_D \beta X_i$ are naturally homeomorphic. More precisely, if $\eta_i: X_i \to \beta X_i$ is the compactification embedding for each $i \in I$, then $\sum_D \eta_i$ is a homeomorphism.*

PROOF. Let $\eta: \prod_D X_i \to \sum_D X_i$ and $\eta': \prod_D \beta X_i \to \sum_D \beta X_i$ be the compactification embeddings, and define $\phi: \prod_D \beta X_i \to \sum_D X_i$ by $\phi(f_D) = \{\prod_D Z_i \in \prod_D Z(X_i)$: $\{i: Z_i \in f(i)\} \in D\}$. It is easy to check that ϕ is a topological embedding and $\phi \circ (\prod_D \eta_i) = \eta$. That $(\sum_D \eta_i) \circ \phi = \eta'$ is also easy to verify; we show that $\sum_D \eta_i$ is a bijection. Indeed if $p \in \sum_D X_i$ with $\prod_D Z_i \in p$ then $\prod_D Z_i^\# \in (\sum_D \eta_i)(p)$, since $\prod_D Z_i$ is the inverse image of $\prod_D Z_i^\#$ under $\prod_D \eta_i$. So if $p \neq p'$ in $\sum_D X_i$ then there are zero-set ultraboxes $\prod_D Z_i \in p$ and $\prod_D Z_i' \in p'$ which are disjoint. Hence $\prod_D Z_i^\# \cap \prod_D Z_i'^\# = \varnothing$ and $(\sum_D \eta_i)(p) \neq (\sum_D \eta_i)(p')$. To see that $\sum_D \eta_i$ is onto, note that since $\sum_D \eta_i$ is one-one and $(\sum_D \eta_i) \circ \phi = \eta'$, the image of $\sum_D \eta_i$ (whose domain is compact) is dense and closed in $\sum_D \beta X_i$. □

We now bring in a small amount of category theoretic language. Let $\mathscr{A}$ be a category with products, let $\langle A_i : i \in I\rangle$ be an indexed family of $\mathscr{A}$-objects, and let D be a filter (not necessarily ultra-) on I. For each $J \supseteq K \in D$ let $\rho_{JK}: \prod_{i\in J}^{\mathscr{A}} A_i \to \prod_{i\in K}^{\mathscr{A}} A_i$ be the natural "restriction" morphism. This gives a directed system of morphisms (D is directed by reverse inclusion); and its direct limit, when it exists, is called the *$\mathscr{A}$-reduced product via D* and denoted $\prod_D^{\mathscr{A}} A_i$. (A typical category where this description of reduced products coincides with the usual one is a Horn class of relational structures, plus all homomorphisms (see [3], [4] and [5]).)

The main advantage of a category theoretic format here is that we can talk about "dual notions" without too much fuss. In particular if the opposite category $\mathscr{A}^{\mathrm{OP}}$ has products (i.e. if $\mathscr{A}$ has coproducts) then the reduced coproduct $\sum_D^{\mathscr{A}} A_i$ in $\mathscr{A}$ is simply the reduced product in $\mathscr{A}^{\mathrm{OP}}$.

Let us now look at the category KH of compact Hausdorff spaces and continuous maps. If $\langle X_i : i \in I\rangle$ and D are given then $\sum_D^{KH} X_i$ is the inverse limit of the coproducts $\beta(\dot{\bigcup}_{i\in J} X_i)$, $J \in D$, where the connecting maps are the Stone-Čech liftings of inclusions. Using elementary properties of inverse limits in KH, we are led to the convenient description of $\sum_D X_i$ given above, namely as a subspace of $\beta(\dot{\bigcup}_{i\in I} X_i)$. (If D is not an ultrafilter, however, we do not get $\sum_D^{KH} X_i$ as a compactification of the topological reduced product (see Remark 3.4(i)).)

At this point we bring in the celebrated duality theorem of A. O. Gel'fond and A. N. Kolmogorov (see [7]) which establishes a duality between KH and the class of unitary rings (together with unitary ring homomorphisms) $RCF = \{C(X): X \in KH\}$. Thus if $\langle X_i : i \in I\rangle$ is a family of compact Hausdorff spaces and D is any filter on I, then $C(\sum_D^{KH} X_i)$ and $\prod_D^{RCF} C(X_i)$ are (naturally) isomorphic.

3.2. LEMMA. *Suppose $\langle X_i : i \in I\rangle$ and $\langle Y_i : i \in I\rangle$ are two families of Tichonov spaces, D is an ultrafilter on I, and θ: $\prod_D Z(X_i) \to \prod_D Z(Y_i)$ is an isomorphism. Then θ gives rises to a homeomorphism $\bar{\theta}$: $\sum_D X_i \to \sum_D Y_i$ (extending the obvious homeomorphism induced by θ between the topological ultraproducts).*

PROOF. We identify $Z_D \in \prod_D Z(X_i)$ with the zero-set ultrabox $\prod_D Z(i)$. Then $\bar{\theta}(p) = \{\theta(Z_D) : Z_D \in p\}$ gives the desired homeomorphism. □

Getting back to the original problem, let X and Y be our two Tichonov spaces such that $Z(X) \equiv Z(Y)$, and let D be chosen so that the ultrapowers are isomorphic. By 3.2, the corresponding ultracopowers $\sum_D X$ and $\sum_D Y$ are homeomorphic, by 3.1 we have $\sum_D \beta X \simeq \sum_D \beta Y$, and by duality it follows that $\prod_D^{RCF} C(\beta X) \cong \prod_D^{RCF} C(\beta Y)$. Since $C^*(X) \cong C(\beta X)$ for any Tichonov space, we will be done once we show that for any compact Hausdorff space X, $\prod_D^{RCF} C(X)$ can be described as an extension of $C(X)$ which is also a quotient of a subring of $\prod_D C(X)$.

Let $\langle X_i : i \in I\rangle$ be a family of compact Hausdorff spaces. The product in RCF of the rings $C(X_i)$ can be given as

$$\prod_{i\in I}^{RCF} C(X_i) = \left\{ f \in \prod_{i\in I} C(X_i) : \bigcup_{i\in I} (f(i))[X_i] \text{ is bounded in } \mathbf{R} \right\}.$$

(Hence $\prod_{i\in I}^{RCF} C(X_i) \cong C(\beta(\dot{\bigcup}_{i\in I} X_i))$.) Given an ultrafilter D on I, we then define $\bar{D} = \{f \in \prod_{i\in I}^{RCF} C(X_i)$: whenever $p \in \sum_D X_i$ and ε is a positive real number then there is a $\prod_D Z_i \in p$ such that $\bigcup_{i\in I}(f(i))[Z_i] \subseteq (-\varepsilon, \varepsilon)\}$. It is straightforward to show that $\bar{D}$ is a ring ideal and that $\prod'_D C(X_i) = \prod_{i\in I}^{RCF} C(X_i)/\bar{D}$ is the appropriate direct limit of the system $\langle \prod_{i\in J}^{RCF} C(X_i) : J \in D\rangle$. This can be seen most easily by showing $\prod'_D C(X_i) \cong C(\sum_D X_i)$. Indeed let ϕ: $\prod_{i\in I}^{RCF} C(X_i) \to C(\beta(\dot{\bigcup}_{i\in I} X_i))$ be the duality isomorphism, let ρ: $\prod_{i\in I}^{RCF} C(X_i) \to \prod'_D C(X_i)$ be the quotient map, and let σ: $C(\beta(\dot{\bigcup}_{i\in I} X_i)) \to C(\sum_D X_i)$ be dual to the inclusion map. Then the desired isomorphism from $\prod'_D C(X_i)$ to $C(\sum_D X_i)$ is given by $\sigma \circ \phi \circ \rho^{-1}$. For this to be verified it suffices to show that $\rho(f) = 0$ iff $\sigma(\phi(f)) = 0$ for any $f \in \prod_{i\in I}^{RCF} C(X_i)$, an easy exercise.

Now if we define $\prod_D^{\circ} C(X_i)$ to be the image of $\prod_{i\in I}^{RCF} C(X_i)$ under the natural quotient homomorphism from $\prod_{i\in I} C(X_i)$ to $\prod_D C(X_i)$, then the quotient homomorphism ρ above induces a quotient homomorphism from $\prod_D^{\circ} C(X_i)$ to $\prod'_D C(X_i)$. Finally, if $X_i = X$ for each $i \in I$, and if q: $\prod_D^{\circ} C(X) \to \prod'_D C(X)$ is the quotient homomorphism, then $d': C(X) \to \prod'_D C(X)$, given by $d' = q \circ d$ ($d: C(X) \to \prod_D C(X)$ being the diagonal elementary embedding), is a ring embedding. Thus if $Z(X) \equiv Z(Y)$, for X and Y Tichonov, then $C^*(X) \equiv_0 C^*(Y)$. Suppose now $F(X) \equiv F(Y)$, and X and Y are normal. We show $\sum_D X \simeq \sum_D Y$ as follows. Note that when Tichonov spaces X_i are normal, $\prod_D F(X_i)$ is a normal base for the topological ultraproduct which extends $\prod_D Z(X_i)$ and which has the property that whenever C_1 and C_2 are disjoint members of $\prod_D F(X_i)$ then there are $Z_1, Z_2 \in \prod_D Z(X_i)$ such that $C_1 \subseteq Z_1$, $C_2 \subseteq Z_2$, and the intersections $C_1 \cap Z_2$, $C_2 \cap Z_1$ are empty. Thus the continuous $\pi: \omega(\prod_D F(X_i)) \to \omega(\prod_D Z(X_i))$, given by $\pi(p) = p \cap \prod_D Z(X_i)$, is a homeomorphism, and 2.13(i), (ii) are proved. To get 2.13(iii) we need to find Boolean spaces X and Y such that X is basically disconnected, Y is not basically disconnected, and for any sentence ϕ of L_R which is either existential (i.e. the prenex normal form of ϕ contains only existential quantifiers) or positive-universal, $C(Y) \models \phi$ whenever $C(X) \models \phi$.

3.3. LEMMA. *Let $\langle X_i : i \in I \rangle$ be compact Hausdorff spaces and let D be an ultrafilter. Then $\sum_D X_i$ is Boolean iff $\{i : X_i$ is Boolean$\} \in D$. Moreover, if D is countably incomplete and if $\{i : X_i$ is infinite$\} \in D$ then $\sum_D X_i$ is not basically disconnected.*

PROOF. One can show easily that the reduced coproduct construction in KH, when restricted to the full subcategory BS of Boolean spaces, is precisely the reduced coproduct construction in BS. Thus $\sum_D X_i$ is Boolean whenever $\{i: X_i$ is Boolean$\} \in D$. Suppose $\{i: X_i$ is Boolean$\} \notin D$. Since D is an ultrafilter, we lose no generality by assuming that for each $i \in I$ there is an infinite compact connected Y_i and an embedding $\theta_i: Y_i \to X_i$. Now Lemma 4.6 of [3] shows that $B(\sum_D W_i) \cong \prod_D B(W_i)$ for any family $\langle W_i : i \in I \rangle$ of compact Hausdorff spaces; hence $\sum_D W_i$ is connected iff $\{i: W_i$ is connected$\} \in D$. Thus $\sum_D Y_i$ above is connected. Also $\sum_D \theta_i$ extends $\prod_D \theta_i$. Since each θ_i is a topological embedding of an infinite space, so too is the ultraproduct map. Thus $\sum_D \theta_i$ is not a constant, and $\sum_D X_i$ therefore fails to be Boolean.

To verify the second part of the lemma, assume that $\sum_D X_i$ is basically disconnected. Then $B(\sum_D X_i) \cong \prod_D B(X_i)$ is a countably complete Boolean algebra (see [12]). So if $\{i: X_i$ is infinite$\} \in D$ then $\prod_D X_i$ is infinite, and hence $\sum_D X_i$ is also infinite. By the above, each X_i is a Boolean space, so $B(X_i)$ is an infinite Boolean algebra. Therefore $\prod_D B(X_i)$ is a countably complete infinite Boolean algebra. But if D is countably incomplete then $\prod_D B(X_i)$ is also ω_1-saturated (see [6]); hence infinite increasing chains in order type ω, of which there are plenty, fail to have suprema; and we have a contradiction. □

To finish the proof of 2.13(iii), let X be any infinite basically disconnected compact Hausdorff space, let D be a countably incomplete ultrafilter, and let $Y = \sum_D X$. Then Y is a Boolean space which is not basically disconnected by 3.3. Now let ϕ be any L_R-sentence such that, for any Boolean space W, W is basically disconnected iff $C(W) \models \phi$. (We know there is one by 2.2 and 2.4.) Then ϕ cannot be equivalent to an existential or a positive-universal sentence, for in either case $\prod'_D C(X) \cong C(\sum_D X) = C(Y) \models \phi$, implying that Y is basically disconnected. □

3.4. REMARKS. (i) We chose the definition of $\sum_D X_i$ as a particular compactification of the topological ultraproduct for reasons of exposition. Actually one should present the reduced coproduct using the inverse limit recipe and then prove that it is a compactification of $\prod_D X_i$ in the case D is an ultrafilter. If D is nonmaximal, $\sum_D X_i$ is not a compactification of $\prod_D X_i$; for suppose each X_i is a singleton. Then $\prod_D X_i$ must also be a singleton; however $\sum_D X_i$ is in natural one-one correspondence with the set of ultrafilters on I which extend D, and there are lots of those when D is not an ultrafilter.

(ii) Any duality theorem where one of the categories is a Horn class of relational structures inspires a reduced coproduct construction in the dual category. For example, if $\langle X_i : i \in I \rangle$ is a family of compact abelian groups then one can define the reduced coproduct using Pontryagin duality. Although we know very little of this construction, we can show quite easily that $\sum_D X_i$ is almost never a compactification of the reduced product $\prod_D X_i$ (naturally a topological abelian group), even when D is an ultrafilter. To see this, let D be countably incomplete. Then $\prod_D X_i$ has a P-space topology (see [1, Theorem 4.1]). If $\prod_D X_i$ were to embed as a dense subgroup of $\sum_D X_i$ then each point of $\prod_D X_i$ would be a P-point of $\sum_D X_i$. But the group structure forces $\sum_D X_i$ to be point-homogeneous. Thus $\sum_D X_i$, a compact space, would have to be a P-space, hence finite.

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