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CO-ELEMENTARY EQUIVALENCE FOR COMPACT HAUSDORFF SPACES AND COMPACT ABELIAN GROUPS

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Dedicated to Bernhard Banaschewski
on the occasion of his sixtieth birthday

The notions of elementary equivalence and elementary mapping in first order model theory have category-theoretic reflections in many well-known topological settings. We study the dualized notions in the categories of compact Hausdorff spaces and compact abelian groups.

0. Introduction

This report is intended as a sequel to the author's papers [4,6]; our main goals being to answer certain of the questions raised (some raised implicitly), and generally to tie up some loose ends left therein. Although we must claim full responsibility for the somewhat ungainly terminology, the idea of co-elementary equivalence has its historic roots in a paper by T. Ohkuma [14]. However, from a personal perspective, our inspiration can be traced directly to lively conversations we had with B. Banaschewski, G. Bruns, and E. Nelson while we were visiting McMaster University early in 1974. Thus the entire topic is dear to this author's heart, as well as apropos of a mid-career retrospective given in honor of Professor Banaschewski.

We use the following theorem, a main result of [4], as a focus for the present paper:

0.1 Theorem (Bankston [4]). *Let X and Y be two Tichonov spaces (resp. normal Hausdorff spaces) whose lattices $Z(X)$ and $Z(Y)$ (resp. $F(X)$ and $F(Y)$) of zero sets (resp. closed sets) are elementarily equivalent, in the sense of first order logic. Then their unital rings $C^*(X)$ and $C^*(Y)$ of bounded continuous real-valued functions satisfy the same positive-universal sentences.*

Unfortunately, this theorem is not very sharp. In our bedazzlement with the ultraproduct-ultracoproduct technique we discovered for the proof, we failed to notice that the conclusion happens to be true under almost no hypotheses at all. (We would like to thank R. Gurevič for piquing our suspicions in this direction.) While the story has a happy ending, to be related in Section 2, let us begin by reviewing briefly the five major steps of the proof.

Step 1. Assuming elementary equivalence, $Z(X) \equiv Z(Y)$, we employ the Keisler–Shelah ultrapower theorem [7] to find isomorphic ultrapowers $\prod_{\mathscr{D}} Z(X) \cong \prod_{\mathscr{D}} Z(Y)$.

Step 2. Extend (sic) the lattice isomorphism between $\prod_{\mathscr{D}} Z(X)$ and $\prod_{\mathscr{D}} Z(Y)$ to a homeomorphism between the topological ultracopowers $\sum_{\mathscr{D}} X$ and $\sum_{\mathscr{D}} Y$ ([4, Lemma 3.2], also [6, Proposition 1.10]).

Step 3. Use [4, Lemma 3.1] (also [6, Proposition 1.6]), which asserts that an ultracoproduct $\sum_{\mathscr{D}} X_i$ is (naturally) homeomorphic to the ultracoproduct $\sum_{\mathscr{D}} \beta(X_i)$ of Stone–Čech compactifications, to establish a homeomorphism $\sum_{\mathscr{D}} \beta(X) \simeq \sum_{\mathscr{D}} \beta(Y)$.

Step 4. The topological ultracoproduct is constructed as an inverse limit of coproducts in the category **KH** of compact Hausdorff spaces and continuous maps [6]. This is precisely dual to the usual construction of ultraproducts in the category of all relational structures (and atomic relation preserving maps) of a particular similarity type (or in any full subcategory which happens to be an elementary productive class). Thus, using the Gel'fand–Kolmogorov duality theorem, we conclude that $C^*(X)$ and $C^*(Y)$ have isomorphic ultrapowers in the category $C[\mathbf{KH}]$ of rings of continuous real-valued functions with compact Hausdorff domains ($C^*(X) = C(\beta(X))$). Writing $\prod_{\mathscr{D}}^{C} C^*(X) \cong \prod_{\mathscr{D}}^{C} C^*(Y)$, we show that these unital rings are obtained from the usual ultrapowers by "throwing away the infinite elements and dividing out the ideal of infinitesimals". (This is all spelled out in [4]. We should also note that this construction is better known as the "Banach ultrapower" [8,11].)

Step 5. Having established that $\prod_{\mathscr{D}}^{C} C^*(X)$ is a quotient of a subring of $\prod_{\mathscr{D}} C^*(X)$, it is an easy model-theoretic argument to show that $C^*(X)$ and $C^*(Y)$ satisfy the same positive-universal sentences.

The rest of this paper is a commentary on various aspects of the proof, and a prospectus on analogous results in the setting of topological groups. In Section 1 we consider "dualized model theory in **KH**", and answer some questions arising in [6]; in Section 2 we explore the weakness of the conclusion in Step 5, and replace it with a much stronger one; and in Section 3 we examine some of the difficulties inherent in transporting Theorem 0.1 to the setting of compact abelian groups.

To complete our introductory remarks, let us consider for a moment Step 4. Define two compact Hausdorff spaces X and Y to be *co-elementarily equivalent* (in symbols $X \equiv Y$) if they have homeomorphic ultracopowers. We will go into more detail in Section 1; as for now suffice it to say that this relation is an equivalence relation between objects in **KH** [6, Proposition 3.2.1] which respects Lebesgue covering dimension and connectedness [6, Theorem 3.2.4]. Letting X be Boolean, that is $\dim(X) = 0$, we have that Y is Boolean whenever $Y \equiv X$; and by Stone duality, Boolean spaces X and Y are co-elementarily equivalent if and only if their Boolean algebras $B(X)$ and $B(Y)$ of clopen sets are elementarily equivalent.

The operations $B(\cdot)$, $Z(\cdot)$, $F(\cdot)$, $C^*(\cdot)$, etc. are what we call "first order representations" in [4–6, 18, 19]. The fact that $C^*(\cdot)$ is a duality when restricted to **KH** makes Step 4 work. Unfortunately, the class $C[\mathbf{KH}]$ of rings is badly behaved from a model-theoretic point of view; what we would much rather have is a duality $R : \mathbf{KH} \to \mathscr{K}$, where $\mathscr{K}$ is an elementary class of relational structures, closed under arbitrary direct powers. Then ultrapowers in $\mathscr{K}$ would be the usual ones, and we could conclude, "Then $R(X) \equiv R(Y)$.", in Theorem 0.1.

In a seminar talk which we gave at McMaster University in the middle 1970's, we considered the existence of such a duality R as very unlikely; and conjectured in [3], with partial supporting results, that indeed R could not exist because ultracoproducts in **KH** would behave in a pathological way.

The conjecture was settled in the fall of 1983 by Banaschewski [1], and independently by J. Rosický [15]. ([6, Theorem 3.1.1] summarizes the situation; also there is further discussion in [5].) While both proofs are ingenious, and quite dissimilar, neither makes use of any pathology in the ultracoproduct construction in **KH**. In fact it is our growing belief, supported by the results of [5,6] and the present paper (see Section 1), that no such pathology will ever be found. Thus, while the result of Banaschewski and Rosický is a negative one, it stands as a challenge to us to try and discover why topological ultracoproducts behave so predictably.

1. Co-elementary equivalence and co-elementary maps in KH

There are several equivalent ways of representing the topological ultracoproduct (see [6, 9, 19]); the most informative is via the compactification of topological ultraproducts.

Let $\langle X_i : i \in I \rangle$ be any family of topological spaces, and let $\mathscr{D}$ be an ultrafilter on the index set I. The *topological ultraproduct* (see [2]) is the space $\prod_{\mathscr{D}} X_i$, a topological quotient of the box product, whose points are $\mathscr{D}$-equivalence classes of functions $x \in \prod_{i \in I} X_i$ (i.e., $[x] = [x]_{\mathscr{D}} = \{y \in \prod_{i \in I} X_i : \{i : y_i = x_i\} \in \mathscr{D}\}$), and whose open (closed) sets are basically generated by 'open (closed) ultraboxes' $\prod_{\mathscr{D}} M_i$, where M_i is open (closed) in X_i. It is easy to verify that if $\mathscr{B}_i$ is an open (closed) basis for X_i, then ultraboxes $\prod_{\mathscr{D}} B_i$, $B_i \in \mathscr{B}_i$, generate the ultraproduct topology in the appropriate sense.

Suppose X is Tichonov. Then $Z(X)$ is a 'normal' basis in the sense of Wallman–Frink [17]; if X is also normal, then $F(X)$ is a normal basis as well. When $\langle X_i : i \in I \rangle$ is a family of Tichonov spaces, the lattice ultraproduct $\prod_{\mathscr{D}} Z(X_i)$ is, via the obvious identification $[\langle Z_i : i \in I \rangle]_{\mathscr{D}} \mapsto \prod_{\mathscr{D}} Z_i$, a normal basis for the topological ultraproduct. Hence $\prod_{\mathscr{D}} X_i$ is also Tichonov. ($\prod_{\mathscr{D}} X_i$ can fail to be normal, even if the X_i's are compact Hausdorff [2].) We can thus form the Wallman–Frink compactification $\omega(\prod_{\mathscr{D}} Z(X_i))$: points are $\prod_{\mathscr{D}} Z(X_i)$-ultrafilters; closed sets are basically generated by sets

$$\left(\prod_{\mathscr{D}} Z_i\right)^{\#} = \left\{p \in \omega\left(\prod_{\mathscr{D}} Z(X_i)\right): \prod_{\mathscr{D}} Z_i \text{ contains a member of } p\right\}.$$

(Note that open sets are basically generated by sets $(\prod_{\mathscr{D}} C_i)^{\#}$, where $C_i \subseteq X_i$ is a cozero set.) The fundamental facts we need are:

(1) When $\langle X_i: i \in I\rangle$ is a family of compact Hausdorff spaces, $\omega(\prod_{\mathscr{D}} Z(X_i))$ is the **KH**-ultracoproduct $\sum_{\mathscr{D}} X_i$.

(2) If the X_i's are just Tichonov spaces, then $\omega(\prod_{\mathscr{D}} Z(X_i))$ is naturally homeomorphic to $\sum_{\mathscr{D}} \beta(X_i)$.

(3) If the X_i's are also normal, then $\omega(\prod_{\mathscr{D}} Z(X_i))$ is naturally homeomorphic to $\omega(\prod_{\mathscr{D}} F(X_i))$.

(4) Whenever $F_i \subseteq X_i$ is a compact subspace, $(\prod_{\mathscr{D}} F_i)^{\#}$ is naturally homeomorphic to $\sum_{\mathscr{D}} F_i$. (This is all proved in [6].)

One major goal in this section is to catalogue topological properties which are and which are not preserved by co-elementary equivalence.

Let P be a topological property. P is "preserved and reflected by ultracoproducts" if whenever $\langle X_i: i \in I\rangle$ is a family of compact Hausdorff spaces and $\mathscr{D}$ is an ultrafilter on I, then $\sum_{\mathscr{D}} X_i$ has property P if and only if $\{i: X_i$ has property $P\} \in \mathscr{D}$. If P is such a property, it is a triviality to see that P is preserved by co-elementary equivalence: If X has property P and $Y \equiv X$, then Y has property P. The converse is false however: Let P be the property of being infinite. This property is preserved by co-elementary equivalence because $\sum_{\mathscr{D}} X_i$ is infinite just in case, for each $n < \omega$, $\{i: |X_i| > n\} \in \mathscr{D}$ ($|\cdot|$ denotes cardinality). (This is [6, Proposition 1.4].)

Several preservation results are proved in [6]. The most useful for our purposes here are:

(1) Having Lebesgue covering dimension n, $n < \omega$, is preserved and reflected by ultracoproducts. ([6, Theorem 2.2.2] states $\dim(X) = \dim(\sum_{\mathscr{D}} X)$, but the proof works for the stronger assertion.)

(2) $B(\sum_{\mathscr{D}} X_i) \cong \prod_{\mathscr{D}} B(X_i)$ ([6, Proposition 1.5]; also [4, Lemma 4.6]).

(3) As a consequence of (2), connectedness is preserved and reflected by ultracoproducts.

As motivation for our new results, note that since $F(X) \equiv F(Y)$ implies $X \equiv Y$, there can be no more than c (= continuously many) $\equiv$-classes in **KH**. Also, since $X \equiv Y$ if and only if $B(X) \equiv B(Y)$ for $\dim(X) = \dim(Y) = 0$, and since the theory of Boolean algebras has only countably many complete extensions (the Tarski invariants theorem [7]), there are exactly $\aleph_0$ $\equiv$-classes of Boolean spaces. Let $n < \omega$, and let $\mathbf{KH}_n \subseteq \mathbf{KH}$ consist of all spaces of covering dimension n. [6, Theorem 3.2.5] states that there are exactly c $\equiv$-classes in **KH**, and the proof uses the preservation and reflection of n-dimensionality to construct c mutually non-co-elementarily equivalent spaces of infinite dimension. Here we improve on that result by showing that for each positive $n < \omega$, $\mathbf{KH}_n$ has c $\equiv$-classes. First we need some preservation results concerning continua.

Recall that a *continuum* is a connected compact Hausdorff space. ($\sum_{\mathscr{D}} X_i$ is a continuum if and only if $\{i: X_i \text{ is a continuum}\} \in \mathscr{D}$.) If X is a continuum and $n<\omega$, define an *n-wheel* on X to be a cover $\{K\} \cup \{L_j : j<n\}$ of X by subcontinua in such a way that:

(i) $K \setminus \bigcup_{j<n} L_j \neq \emptyset$ (K is the 'hub');

(ii) $L_j \setminus K \neq \emptyset$ for $j<n$ (L_j is a 'spoke'); and

(iii) for $j<k<n$, $L_j \cap L_k = \emptyset$.

X is *n-odic* if X has an n-wheel, but no m-wheel for $m>n$. Note that circles are 1-odic, arcs are 2-odic, and X is 0-odic if and only if X is 'indecomposable' (i.e., X is not the union of two proper subcontinua).

In order to prove a result concerning preservation of n-odicity, we will need the following lemma, due to R. Gurevič [9]:

1.1. Lemma. *Let $\langle X_i : i \in I\rangle$ be compact Hausdorff spaces, let $x_i \in X_i$, and let C_i be the component of x_i in X_i. Then $(\prod_{\mathscr{D}} C_i)^{\#}$ is the component of $(\prod_{\mathscr{D}} \{x_i\})^{\#}$ in $\sum_{\mathscr{D}} X_i$.*

Proof. By Fundamental Fact 4 above, we can write $(\prod_{\mathscr{D}} C_i)^{\#}$ and $(\prod_{\mathscr{D}} \{x_i\})^{\#}$ as $\sum_{\mathscr{D}} C_i$ and $\sum_{\mathscr{D}} x_i$ respectively. Let C be the component of $\sum_{\mathscr{D}} x_i$ in $\sum_{\mathscr{D}} X_i$. Since $\sum_{\mathscr{D}} C_i$ is connected and intersects C, we know $\sum_{\mathscr{D}} C_i \subseteq C$. Suppose $p \notin \sum_{\mathscr{D}} C_i$. Then we can find closed $F_i \subseteq X_i$ with $\prod_{\mathscr{D}} F_i \in p$ and $\{i: F_i \cap C_i = \emptyset\} \in \mathscr{D}$. Thus $p \in \sum_{\mathscr{D}} F_i$ and $\sum_{\mathscr{D}} F_i \cap \sum_{\mathscr{D}} C_i = \sum_{\mathscr{D}} F_i \cap C_i = \emptyset$. Since C_i is maximally connected in X_i and all spaces under consideration are compact, there is a clopen set $B_i \subseteq X_i$ separating K_i and C_i whenever they are disjoint. Thus $\sum_{\mathscr{D}} B_i$ is a clopen subset of $\sum_{\mathscr{D}} X_i$ separating p from $\sum_{\mathscr{D}} C_i$. This implies $p \notin C$, hence $\sum_{\mathscr{D}} C_i = C$. $\square$

1.2. Theorem. *For each $n<\omega$, the property of being an n-odic continuum is preserved and reflected by ultracoproducts.*

Proof. Suppose $J = \{i: X_i \text{ is } n\text{-odic}\} \in \mathscr{D}$. If for all $i \in J$, $\{K_i\} \cup \{L_{j,i} : j<n\}$ is an n-wheel for X_i, then it is easy to see that $\{\sum_{\mathscr{D}} K_i\} \cup \{\sum_{\mathscr{D}} L_{j,i} : j<n\}$ is an n-wheel for $\sum_{\mathscr{D}} X_i$.

Now suppose $\{K\} \cup \{L_j : j<m\}$ is an m-wheel for $\sum_{\mathscr{D}} X_i$. We need to show $m \leq n$. Since $K \setminus \bigcup_{j<m} L_j$ is a nonempty open set, there is some $\sum_{\mathscr{D}} x_i \in K \setminus \bigcup_{j<m} L_j$. Similarly, for $j<m$, we can find $\sum_{\mathscr{D}} y_{j,i} \in L_j \setminus K$.

Using a compactness argument, one can show easily that if F and G are disjoint closed subsets of $\sum_{\mathscr{D}} X_i$ then one can find open sets $U_i, V_i \subseteq X_i$, whose closures are disjoint, such that $F \subseteq \sum_{\mathscr{D}} \bar{U}_i$ and $G \subseteq \sum_{\mathscr{D}} \bar{V}_i$. These larger closed sets will, of course, be disjoint in $\sum_{\mathscr{D}} X_i$. Hence, for each $j<m$, we can find a closed set $\sum_{\mathscr{D}} F_{j,i}$ containing L_j such that $\sum_{\mathscr{D}} F_{j,i} \cap \sum_{\mathscr{D}} F_{k,i} = \emptyset$, $j<k<m$, and $\sum_{\mathscr{D}} x_i \notin \bigcup_{j<m} \sum_{\mathscr{D}} F_{j,i}$. Using Lemma 1.1, let $\sum_{\mathscr{D}} L_{j,i}$ be the component of $\sum_{\mathscr{D}} F_{j,i}$ containing $\sum_{\mathscr{D}} y_{j,i}$, $j<m$. Then L_j, being connected, must be contained in $\sum_{\mathscr{D}} L_{j,i}$ for $j<m$. Arguing similarly, but with less fuss, we can obtain an ultracoproduct sub-

continuum $\sum_{\mathscr{D}} K_i \supseteq K$ that contains no $\sum_{\mathscr{D}} y_{j,i}$. Thus $\{\sum_{\mathscr{D}} K_i\} \cup \{\sum_{\mathscr{D}} L_{j,i}: j<m\}$ is an m-wheel on $\sum_{\mathscr{D}} X_i$. But then $\{i: \{K_i\} \cup \{L_{j,i}: j<m\}$ is an m-wheel on $X_i\} \in \mathscr{D}$; whence $m \le n$. Thus the property of being an n-odic continuum is preserved by ultracoproducts. In order to show reflection, just argue as above, only in reverse order. □

1.3. Corollary. *The property of being an n-odic continuum, $n<\omega$, is preserved by co-elementary equivalence.*

1.4. Remark. In [6, Proposition 2.4.4] we proved the easy result that ultracoproducts of decomposable continua are decomposable, and asked whether decomposability is also reflected. Lemma 1.1 communicated to us by Gurevič in the fall of 1985, turned out to be the key ingredient for obtaining an affirmative answer.

We now use Theorem 1.2 to prove our advertised result concerning the number of $\equiv$-classes in $\mathbf{KH}_n$.

1.5. Theorem. *Let $n<\omega$. Then $\mathbf{KH}_n$ contains exactly $\aleph_0$ $\equiv$-classes if $n=0$ and exactly c $\equiv$-classes if $n>0$.*

Proof. The case $n=0$ is ancient history; let us prove the case for $n=1$. The case $n>1$ involves minimal extra work. It suffices to construct a sequence $\langle X_\alpha: \alpha<c\rangle$ such that $\dim(X_\alpha)=1$ and $X_\alpha \not\equiv X_\beta$ for $\alpha<\beta<c$. The proof is similar in structure to the proof of Theorem 3.2.5 in [6].

For $m=2,3,\ldots$, let H_m be m line segments emanating from a single point; i.e., $H_m=L_1\cup\cdots\cup L_m$, where $L_j=\{\langle x,y\rangle \in \mathbf{R}^2: 0\le x\le 1, y=jx\}$, then each H_m is plainly m-odic. Let S be the set of all sequences $s:\{2,3,\ldots\}\to\{0,1\}$. For each $s\in S$, let X_s be the Alexandrov one-point compactification of the disjoint union $X_{2,s(2)} \dot\cup X_{3,s(3)} \dot\cup \cdots$, where for $2\le m<\omega$,

$$X_{m,s(m)} = \begin{cases} \text{a singleton} & \text{if } s(m)=0, \\ H_m & \text{if } s(m)=1. \end{cases}$$

Clearly each X_s has dimension 1, and is in fact metrizable. It remains to show $X_s \not\equiv X_t$ for s,t distinct in S. Suppose $s\ne t$, say $s(k)=1$ and $t(k)=0$. For convenience, set $Y=X_s$, $Z=X_t$, $Y_m=X_{m,s(m)}$, $Z_m=X_{m,t(m)}$. Assuming $Y\equiv Z$, we can find an ultrafilter $\mathscr{D}$ and a homeomorphism $\delta:\sum_{\mathscr{D}} Y\to\sum_{\mathscr{D}} Z$. By Fundamental Fact 2, we have an induced isomorphism $\delta':\prod_{\mathscr{D}} B(Y)\to\prod_{\mathscr{D}} B(Z)$. Now for any $u\in S$, $B(X_u)$ is the finite-cofinite algebra on ω, and its atoms are the clopen sets $X_{m,u(m)}$. Thus the atoms of $B(\sum_{\mathscr{D}} X_u)$ correspond to the ultracoproducts of the spaces $X_{m,u(m)}$. Since δ' takes atoms to atoms, we infer that δ takes $\sum_{\mathscr{D}} Y_k$ to an ultracoproduct of the Z_m's. But $Y_k=H_k$, a k-odic continuum. Also no Z_m is k-odic, since Z_k is a singleton and no other Z_m is H_k. By Theorem 1.2, no ultracoproduct of the Z_m's can be k-odic. This is a contradiction, so we conclude $Y \not\equiv Z$.

Now let $n > 1$, and let $[0,1]^n$ be the n-cube. Let $Y_s = [0,1]^n \dot{\cup} X_s$, where X_s is as above, $s \in S$. For particular $s, t \in S$ where $s(k) = 1$, $t(k) = 0$, repeat the above argument. Here it is convenient to use the fact that covering dimension is preserved and reflected by ultracoproducts. □

For any space X, let $w(X)$ denote the 'weight' of X, the smallest cardinality of a basis for X. It is well known that for infinite $X \in \mathbf{KH}_0$, $|B(X)| = w(X)$. (The analogous statement goes through for compact abelian groups and their discrete character groups, by Pontryagin–van Kampen duality.) Moreover, it is proved in [5] that if $R : \mathbf{KH}_0 \to \mathscr{K}$ is any duality onto an elementary productive class in which equalizers are embeddings and co-equalizers are surjections, then $|R(X)| = w(X)$ for any infinite X. Thus, a case can be made that the weight for compact Hausdorff spaces is the correct 'dual' to cardinality from the standpoint of model theory.

In [6] we asked the question (Question 3.2.7) whether every $X \in \mathbf{KH}$ is co-elementarily equivalent to a second countable (= metrizable) $Y \in \mathbf{KH}$. This is a 'Löwenheim–Skolem' type of question, and in a letter, Gurevič suggested that the methods of his paper [10] could be used to give an affirmative answer.

While this can indeed be done, we prove instead a stronger result involving co-elementary maps corresponding to the well-known Löwenheim–Skolem downward theorem.

The idea of co-elementary map is dual to that of elementary map. If A and B are two relational structures, a map $\varepsilon : A \to B$ is *elementary* (in symbols $\varepsilon : A \prec B$) if whenever $\phi(v_1, \ldots, v_n)$ is a first order formula with free variables among $\{v_1, \ldots, v_n\}$, and $a_1, \ldots, a_n \in A$, then the sentence $\phi[a_1, \ldots, a_n]$ (where a_i is 'plugged in' for v_i) is true in A just in case the sentence $\phi[\varepsilon(a_1), \ldots, \varepsilon(a_n)]$ is true in B. This is equivalent to saying that the expanded structures, with constants naming each element of A, are elementarily equivalent. Let $\Delta = \Delta_{A,\mathscr{D}} : A \to \prod_{\mathscr{D}} A$ be the canonical diagonal map. Then the ultrapower theorem gives us the following, also stated in [6]:

1.6. Proposition. *$\varepsilon : A \prec B$ if and only if there is an ultrafilter $\mathscr{D}$ and an isomorphism $\delta : \prod_{\mathscr{D}} A \to \prod_{\mathscr{D}} B$ such that $\delta \circ \Delta = \Delta \circ \varepsilon$.*

If X and Y are compact Hausdorff and $\gamma : X \to Y$ is any function, continuous or not, call γ a *co-elementary map* (in symbols $\gamma : X \succ Y$) if there is an ultrafilter $\mathscr{D}$ and a homeomorphism $\delta : \sum_{\mathscr{D}} X \to \sum_{\mathscr{D}} Y$ such that $\gamma \circ \nabla = \nabla \circ \delta$. ($\nabla = \nabla_{X,\mathscr{D}} : \sum_{\mathscr{D}} X \to X$ is the canonical co-diagonal map, always a continuous surjection (see [6]).)

The fundamental facts about co-elementary maps are:

(1) They are continuous surjections which preserve properties which are preserved by co-elementary equivalence.

(2) When restricted to $\mathbf{KH}_0$ they correspond, under Stone duality, to elementary maps between Boolean algebras.

(3) Whenever $\gamma: X \to Y$ and $\delta: Y \to Z$ are functions, the co-elementarity of $\delta \circ \gamma$ (resp. δ) follows from the co-elementarity of γ and δ (resp. γ and $\delta \circ \gamma$). (This is [6, Theorem 3.3.2].)

The Löwenheim–Skolem downward theorem states that if $\mathscr{L}$ is a countable lexicon of relation and function symbols, A is an $\mathscr{L}$-structure, and $S \subseteq A$ is any subset, then there is an elementary substructure B of A which contains S and whose cardinality is $\leq |S| \cdot \aleph_0$. An equivalent version, more suitable to dualization, is that if $\eta: C \to A$ is an embedding of $\mathscr{L}$-structures, then there is an $\mathscr{L}$-structure B and maps $\theta: C \to B$, $\varepsilon: B \to A$ such that θ is an embedding, ε is an elementary map, $\eta = \varepsilon \circ \theta$, and $|B| \leq |C| \cdot \aleph_0$. The dualized version of this in **KH** is the following:

1.7. Theorem. *Let X and Z be compact Hausdorff spaces, with $\eta: X \to Z$ a continuous surjection. Then there is a compact Hausdorff Y and maps $\theta: Y \to Z$, $\gamma: X \to Y$ such that θ is a continuous surjection, γ is a co-elementary map, $\eta = \theta \circ \gamma$, and $w(Y) \leq w(Z) \cdot \aleph_0$.*

Proof. Let $\eta: X \to Z$ be given, and assume B is a basis of closed subsets of Z, of minimal cardinality, such that B is closed under finite intersections and finite unions. Then $|B| \leq w(Z) \cdot \aleph_0$, and $\eta^{\leftarrow}: B \to F(X)$ is a lattice embedding. By the usual Löwenheim–Skolem downward theorem, there is a lattice L and maps $\delta: B \to L$, $\varepsilon: L \to F(X)$ such that δ is an embedding, ε is an elementary map, $\eta^{\leftarrow} = \varepsilon \circ \delta$, and $|L| \leq |B| \cdot \aleph_0 \leq w(Z) \cdot \aleph_0$. Since $L \equiv F(X)$ and X is a normal space, L is an atomic lattice which can naturally be viewed as a normal basis of closed sets over its set of atoms. Let Y be the Wallman–Frink compactification $\omega(L)$. Then $w(Y) \leq |L| \leq w(Z) \cdot \aleph_0$. Let $p \in \omega(L)$ be an L-ultrafilter, and define $\theta(p)$ to be the unique $z \in Z$ such that $\delta(b) \in p$ for each $b \in B$ with $z \in b$. For $x \in X$, let $\gamma(x) = \{l \in L: x \in \varepsilon(l)\}$. Then $\gamma(x) \in \omega(L)$, and it is easy to check that both θ and γ are continuous surjections, with $\eta = \theta \circ \gamma$.

Now $\varepsilon: L \to F(X)$ is an elementary embedding; hence there is an ultrafilter $\mathscr{D}$ and an isomorphism $\lambda: \prod_{\mathscr{D}} L \to \prod_{\mathscr{D}} F(X)$ such that $\Delta \circ \varepsilon = \lambda \circ \Delta$. Let Y_0 be the set of atoms of L, and think of elements of $\prod_{\mathscr{D}} L$ as ultraproducts $\prod_{\mathscr{D}} A_i$, where $A_i \in L$, $A_i \subseteq Y_0$. A typical member of $\omega(L)$ is an L-ultrafilter; a typical basic closed set is of the form $A^{\#} = \{p: A \in p\}$ for $A \in L$. A typical member of $\sum_{\mathscr{D}} \omega(L)$ is an ultrafilter of sets $\prod_{\mathscr{D}} C_i$, where C_i is closed in $\omega(L)$. A typical member of $\sum_{\mathscr{D}} X$ is an ultrafilter of sets $\prod_{\mathscr{D}} K_i$, where $K_i \subseteq X$ is closed. Define, for $p \in \sum_{\mathscr{D}} \omega(L)$, $\bar{\lambda}(p) = \{\prod_{\mathscr{D}} K_i \in \sum_{\mathscr{D}} X: \prod_{\mathscr{D}} A_i^{\#} \in p\}$, where $\prod_{\mathscr{D}} K_i = \lambda(\prod_{\mathscr{D}} A_i)$. The straightforward verification that $\bar{\lambda}: \sum_{\mathscr{D}} \omega(L) \to \sum_{\mathscr{D}} X$ is a continuous bijection, hence a homeomorphism; and that the appropriate diagram commutes, is left to the reader. Thus γ is a co-elementary map. $\square$

1.8. Corollary. *Every compact Hausdorff space is co-elementarily equivalent to a compact metrizable space.*

1.9. Remark. If $R(\cdot)$ is a first order representation, it is natural to ask whether there is a Löwenheim–Skolem result for $R(\cdot)$; namely whether, given $X \in \mathbf{KH}$, one can always find Y compact metrizable such that $R(Y) \equiv R(X)$. Clearly this is true for $B(\cdot)$, but it is false for $F(\cdot)$, $Z(\cdot)$, and $C(\cdot)$: Pick $X \in \mathbf{KH}_0$ extremally disconnected and infinite. If $Y \in \mathbf{KH}$ and $F(Y) \equiv F(X)$, then Y is also extremally disconnected and infinite. If $Z(Y) \equiv Z(X)$, then Y is basically disconnected and infinite. Hence $B(Y)$ is an infinite countably complete Boolean algebra, hence uncountable. Consequently, $w(Y) = |B(Y)|$ is uncountable and Y fails to be metrizable. If $C(Y) \equiv C(X)$, then, by a result of A. Macintyre (see [4]), $Z(Y) \equiv Z(X)$. An interesting problem area would be to determine, given $R(\cdot)$, the smallest cardinal number λ_R such that whenever $X \in \mathbf{KH}$, there is $Y \in \mathbf{KH}$ with $R(Y) \equiv R(X)$ and $w(Y) \leq \lambda_R$.

1.10. Corollary. *Any topological property which holds for all separable metrizable spaces but not for all compact Hausdorff spaces (e.g. hereditary normality, the equality of covering dimension and the inductive dimensions) fails to be preserved by co-elementary equivalence.*

In answer to [6, Question 2.2.4(i)], we have the following:

1.11. Corollary. *Co-elementary equivalence does not preserve large, or small, inductive dimension.*

Proof. Let X be the classic example, due to A.L. Lunc (see [13]), of a compact Hausdorff space such that $\dim(X) = 1$ and $\operatorname{ind}(X) = \operatorname{Ind}(X) = 2$. Let $Y \equiv X$ be compact and metrizable. Then $\dim(Y) = 1$ because covering dimension is preserved by co-elementary equivalence. But now Y is separable metrizable, and basic dimension theory dictates that $\operatorname{ind}(Y) = \operatorname{Ind}(Y) = \dim(Y) = 1$. □

1.12. Remark. In [6] we catalogued several topological properties which are not preserved by co-elementary equivalence, but which also do not obtain for all separable metrizable spaces. Some of these properties are: point-homogeneity, being an F-space, basic disconnectedness, extremal disconnectedness, and path connectedness.

2. Step 5 revisited: the happy ending

Returning to the topic of Theorem 0.1, let us first examine why the conclusion is so weak. Let $\mathscr{L}$ be a lexicon of relation and function symbols. A *positive-universal* formula is one built up from the atomic formulas of $\mathscr{L}$ using conjunctions, disjunctions, and universal quantification.

2.1. Proposition. *Let E be a fixed $\mathscr{L}$-structure, and let $\mathscr{K}$ be a class of $\mathscr{L}$-structures satisfying*:

(1) *each $A \in \mathscr{K}$ embeds in a power E^I of E in such a way that the image of A under the embedding contains all constant maps from I to E*; *and*

(2) *for each $A \in \mathscr{K}$ and finite subset $F \subseteq E$, there is some $f \in A$ (viewing A as a substructure of E^I) with $f[I] \supseteq F$. Then every two members of $\mathscr{K}$ satisfy the same positive-universal sentences from $\mathscr{L}$.*

Proof. A typical positive-universal sentence σ can be written in the form $\forall v_1 \cdots v_n \bigwedge_{1 \le i \le k} \bigvee_{1 \le j \le l} \theta_{ij}$, where each θ_{ij} is atomic with at most the variables $\mathbf{v} = v_1, \ldots, v_n$ free.

Case 1: σ is $\forall \mathbf{v}\theta$, θ atomic. If $E \models \sigma$, then $E^I \models \sigma$, so $A \models \sigma$ whenever $A \subseteq E^I$. Conversely, if $A \models \sigma$, then $E \models \sigma$ since E is a homomorphic image of A. Thus the proposition holds in this case.

Case 2: σ is $\forall \mathbf{v} \bigvee_{1 \le j \le l} \theta_j$, where each θ_j is atomic. Let $A \in \mathscr{K}$, and assume for each $1 \le j \le l$ that $A \not\models \forall \mathbf{v}\theta_j$. We show $A \not\models \sigma$. Indeed, if $A \models \exists \mathbf{v} \neg \theta_j$, then $E \models \exists \mathbf{v} \neg \theta_j$ by Case 1. So plug in $a_{k,j} \in E$ for the variable v_k, $1 \le k \le n$. Then $E \models \neg\theta_j[a_{1,j}, \ldots, a_{n,j}]$ for some such choice; and we have an $l \times n$ array of $a_{i,j}$'s. For $1 \le i \le n$, let $f_i \in A$ contain $\{a_{i,1}, \ldots, a_{i,l}\}$ in its range. Then $A \models \neg\theta_j[f_1, \ldots, f_n]$; whence $A \models \exists \mathbf{v} \bigwedge_{1 \le j \le l} \neg\theta_j$ and $A \not\models \sigma$. So if $A, B \in \mathscr{K}$ and $A \models \sigma$, then $A \models \forall \mathbf{v}\theta_j$ for some $1 \le j \le l$. Thus $B \models \forall \mathbf{v}\theta_j$ by Case 1. Hence $B \models \sigma$.

Case 3: σ is $\forall \mathbf{v} \bigwedge_{1 \le i \le k} \bigvee_{1 \le j \le l} \theta_{ij}$, where each θ_{ij} is atomic. Then σ is $\bigwedge_{1 \le i \le k} \sigma_i$, where σ_i is $\forall \mathbf{v} \bigvee_{1 \le j \le l} \theta_{ij}$. Let $A, B \in \mathscr{K}$. If $A \models \sigma$, then $A \models \sigma_i$ for each $1 \le i \le k$. By Case 2, $B \models \sigma_i$ for each $1 \le i \le k$, so $B \models \sigma$. □

An immediate application of Proposition 2.1 is that we can take E to be any interval of real numbers equipped with all continuous operations and any relations we like. Let $\mathscr{K}$ be all relevant structures of continuous bounded E-valued functions with infinite normal topological spaces for domains. Then Tietze's extension theorem trivially ensures that condition (2) is satisfied in the hypothesis of Proposition 2.1.

2.2. Corollary. *Let $X, Y \in$ **KH** be any two infinite spaces. Then $C(X)$ and $C(Y)$ satisfy the same positive-universal sentences.*

In order to rectify the situation, let us, for the remainder of this section, view $C(X)$ as a Banach space. Specifically, the relevant lexicon includes the vector space operations, a unary operation of scalar multiplication for each rational scalar, and two additional unary relations P and Q: Px (resp. Qx) is to mean that the norm of x is to be ≤ 1 (resp. ≥ 1). A formula is *positive-bounded* if it is built up from the atomic formulas using the finitary logical operations of conjunction and disjunction, and 'bounded quantification': $\forall x(Px \to \cdots)$ and $\exists x(Px \wedge \cdots)$.

The above notions are the invention of Henson [11] who went on to define a natural, but technical, notion of 'approximate satisfaction' between Banach spaces and sentences. A main result of [11] is that two Banach spaces approximately satisfy the same positive-bounded sentences if and only if they have isometrically isomorphic Banach ultrapowers.

For completeness, we reproduce Henson's definition of approximate satisfaction here. Let σ be an arbitrary positive-bounded formula, $1 \leq m < \omega$. The 'approximation' σ_m is obtained from σ using induction on complexity: If σ is atomic, we replace $x = y$ by $Pm \cdot (x - y)$; Px by $P(1 - 1/m) \cdot x$; and Qx by $Q(1 + 1/m) \cdot x$. For more complex formulas, use the identities $(\sigma \wedge \tau)_m = \sigma_m \wedge \tau_m$, $(\sigma \vee \tau)_m = \sigma_m \wedge \tau_m$, $(\forall x(Px \to \sigma))_m = \forall x(Px \to \sigma_m)$, and $(\exists x(Px \wedge \sigma))_m = \exists x(Px \wedge \sigma_m)$.

If A is any structure appropriate to our lexicon (e.g. a Banach space) and σ is a positive-bounded sentence (possibly with constants from A), we say that A *approximately satisfies* σ if $A \models \sigma_m$ for each $1 \leq m < \omega$.

What all this means to us is that, at Step 4 of the proof of Theorem 0.1, we can infer easily that the Banach ultrapowers $\prod_{\mathscr{D}}^{C} C^*(X)$ and $\prod_{\mathscr{D}}^{C} C^*(Y)$ are isometrically isomorphic as Banach spaces. A new Step 5, using Henson's theorem, allows us to infer that $C^*(X)$ and $C^*(Y)$ approximately satisfy the same positive-bounded sentences. Hence we have the following substitute for Theorem 0.1:

2.3. Theorem. *Let X and Y be two Tichonov spaces (resp. normal spaces) such that $Z(X) \equiv Z(Y)$ (resp. $F(X) \equiv F(Y)$). Then their Banach spaces $C^*(X)$ and $C^*(Y)$ of bounded continuous real-valued functions approximately satisfy the same positive-bounded sentences.*

2.4. Remark. By Henson's theorem [11] and our Step 4, the conclusion of Theorem 2.3 is equivalent to saying $\beta(X) \equiv \beta(Y)$. Thus, by various results concerning the sharpness of co-elementary equivalence (including Theorem 1.5), we see that this replacement for Theorem 0.1 is much more substantial.

3. Toward an analogue to Theorem 0.1 for compact abelian groups

In view of the classical Pontryagin–van Kampen duality between compact Hausdorff abelian groups and (discrete) abelian groups (see [12]), the temptation is overwhelming to try to effect an analogue to Theorem 0.1. Let **KAb** be the category of compact Hausdorff abelian groups and continuous homomorphisms, and let **Ab** be the category of abelian groups and homomorphisms. Let $T \in$ **KAb** be the circle group, that is, the multiplicative group of complex numbers of unit norm. For $G \in$ **KAb**, let $D(G)$ be the group of continuous homomorphisms (characters) $\chi : G \to T$. For $A \in$ **Ab** let $D^{-1}(A)$ be the compact abelian group of all homomorphisms (characters again) $\chi : A \to T$. The compact topology on $D^{-1}(A)$ is inherited from the topological power T^A. The famed result of Pontryagin–van Kampen is that $D: \mathbf{KAb} \to \mathbf{Ab}$ is a category duality. One of the beauties of this duality is that the right-hand side, as with Stone duality, is a class of structures in which the ultraproduct construction is the usual one.

Our analogue to Theorem 0.1 should conclude that $D(G) \equiv D(H)$ whenever $R(G) \equiv R(H)$ for suitably chosen first order representation $R(\cdot)$ on **KAb**.

Two first order representations which leap to mind are $F(G)$ = the closed set lattice of G, and $U(G)$ = the underlying group of G. Of course both of these are 'forgetful', and one should not expect either to replace $R(\cdot)$ in our analogue. The candidate for $R(\cdot)$ which we would like to champion is the obvious 'composite' of $F(\cdot)$ and $U(\cdot)$; namely define $M(G)$ to be $F(G)$ with the group structure of $U(G)$ on the set of atoms. Our analogue can now be stated, but only as speculation.

3.1. Conjecture. *Let G and $H \in$* **KAb**, *and assume $M(G) \equiv M(H)$. Then $D(G) \equiv D(H)$.*

An attempted proof might go as follows: Step 1 is no problem; Step 2 looks reasonable (it was easy in the compact Hausdorff case); there is no need for Step 3; Step 4 is simply an application of the duality theorem, and we conclude $\prod_{\mathscr{D}} D(G) \cong \prod_{\mathscr{D}} D(H)$; and Step 5 is the easy direction of the ultrapower theorem. As we shall see, Step 2 is the stumbling block.

3.2. Remark. There are twelve versions of Conjecture 3.1 when we allow the various first order representations above to be substituted (so as not to obtain a tautology). Some are trivially true, while others can fairly easily be shown false. Still others, we have no firm answers for. We believe that all are false, save the trivial ones and Conjecture 3.1. As an example, let us show the converse is false. We will actually do more and find $G, H \in \mathbf{KAb}$ such that $D(G) \equiv D(H)$, but $F(G) \not\equiv F(H)$. Let $\mathbf{Z}_2$ be the two-element group, let $G = \mathbf{Z}_2^{\omega}$, and let $H = \mathbf{Z}_2^{\omega_1}$. By duality (see [12]), $D(G)$ and $D(H)$ are respectively the direct copowers $\mathbf{Z}_2^{(\omega)}$ and $\mathbf{Z}_2^{(\omega_1)}$. These groups have equal Szmielew invariants (see [16]) and are hence elementarily equivalent. Now G has a metrizable topology, and is hence hereditarily normal. In particular, the complement in G of any point (= atom of the lattice $F(G)$) is normal. It is easy to write this statement down as a first order sentence in the language of lattices. However, the removal of a point from an uncountable product of discrete spaces ruins normality (since one can embed an uncountable power ω^{ω_1} of the integers as a closed subset, and ω^{ω_1} is not normal, by a theorem of Stone [17]). An alternate proof uses ultrapowers, and is more in the spirit of this paper. Let G be totally disconnected. Then (see [12]) $D(G)$ is a torsion group. Choose G so that $D(G)$ has elements of arbitrarily high order (say $G = \prod_{n=2}^{\infty} \mathbf{Z}_n$), and let $\mathscr{D}$ be a free ultrafilter on ω. Then $\prod_{\mathscr{D}} D(G)$ is not a torsion group; hence $D^{-1}(\prod_{\mathscr{D}} D(G))$ is not totally disconnected. Thus, $F(G) \not\equiv F(D^{-1}(\prod_{\mathscr{D}} D(G)))$. However, $D(G) \equiv D(D^{-1}(\prod_{\mathscr{D}} D(G)))$.

Given a family $\langle G_i: i \in I \rangle$ in **KAb** and an ultrafilter $\mathscr{D}$ on I, define the **KAb**-ultracoproduct to be $D^{-1}(\prod_{\mathscr{D}} D(G_i))$, and denote it by $\sum_{\mathscr{D}}^0 G_i$. Clearly two compact abelian groups are co-elementarily equivalent if and only if their character groups are elementarily equivalent. One obvious way in which $\sum_{\mathscr{D}}^0 G_i$ and the topological ultracoproduct $\sum_{\mathscr{D}} G_i$ differ is that $\sum_{\mathscr{D}} G_i$ 'almost never' supports a topological group structure (i.e., when $\mathscr{D}$ is countably incomplete and $\{i: |G_i| \geq n\} \in \mathscr{D}$ for all $n > \omega$). Another difference is in the preservation of dimension: as we saw in Remark 3.2, G can have dimension zero and be co-elementarily equivalent to a group of non-zero dimension.

The problem with Step 2 in this situation is that we do not know whether an isomorphism between ultraproducts $\prod_{\mathscr{D}} M(G_i)$ and $\prod_{\mathscr{D}} M(H_i)$ leads to a topological isomorphism between $\sum_{\mathscr{D}}^0 G_i$ and $\sum_{\mathscr{D}}^0 H_i$. As we have seen, topological ultracoproducts are compactifications of topological ultraproducts. This is no longer true in the setting of topological groups.

Denote $\prod_{\mathscr{D}} M(G_i)$ simply by $\prod_{\mathscr{D}} G_i$. This is the usual topological ultraproduct, with extra group structure. We will show how to define a continuous monomorphism $\eta: \prod_{\mathscr{D}} G_i \to \sum_{\mathscr{D}}^0 G_i$; however it is 'almost never' the case, in the same sense as above, that $\prod_{\mathscr{D}} G_i$ topologically embeds in $\sum_{\mathscr{D}}^0 G_i$. This is the main obstacle to our analogue to Step 2.

Let $G \in$ **KAb**, and let $\mathscr{D}$ be an ultrafilter. The diagonal map $\Delta: G \to \prod_{\mathscr{D}} G$ is an elementary embedding of topological groups, in the sense of $U(\cdot)$. It fails to be continuous whenever G is infinite and $\mathscr{D}$ is countably incomplete (see [2]). For each $[g] \in \prod_{\mathscr{D}} G$, there is a unique $x \in G$ such that for all open neighborhoods U of x, $[g] \in \prod_{\mathscr{D}} U$ (see [2]; use compactness of G). Denote this x by $\Lambda([g])$. Λ is the '$\mathscr{D}$-limit', or 'standard part' map.

3.3. Lemma. *$\Lambda: \prod_{\mathscr{D}} G \to G$ is a continuous homomorphism onto G, and is a left-inverse for Δ.*

Proof. Clearly $\Lambda \circ \Delta$ is the identity map on G; hence Λ is surjective. If $U \subseteq G$ is open, then $\Lambda^{\leftarrow}[U] = \bigcup \{\prod_{\mathscr{D}} V: V \subseteq G \text{ is open and } \bar{V} \subseteq U\}$ (see [2]). To prove Λ is a homomorphism, let $[g], [h] \in \prod_{\mathscr{D}} G_i$, let $x = \Lambda([g])$, $y = \Lambda([h])$, and $z = \Lambda([g] \cdot [h]^{-1}) = \Lambda([g \cdot h^{-1}])$. We show $z = x \cdot y^{-1}$. So let U be any open neighborhood of $x \cdot y^{-1}$. We must show $[g \cdot h^{-1}] \in \prod_{\mathscr{D}} U$; i.e., $\{i: g_i \cdot h_i^{-1} \in U\} \in \mathscr{D}$. By continuity of the group operations, there are open neighborhoods V of x and W of y such that if $x' \in V$ and $y' \in W$, then $x' \cdot (y')^{-1} \in U$. Now $\{i: g_i \in V\} \in \mathscr{D}$ and $\{i: h_i \in W\} \in \mathscr{D}$; hence $\{i: g_i \cdot h_i^{-1} \in U\} \supseteq \{i: g_i \in V \text{ and } h_i \in W\} \in \mathscr{D}$, as desired. □

Now let $\langle G_i: i \in I \rangle$ be any family of compact abelian groups. We define the *evaluation map* $\eta: \prod_{\mathscr{D}} G_i \to \sum_{\mathscr{D}}^0 G_i$ as follows. Regarding $\sum_{\mathscr{D}}^0 G_i$ as a closed subgroup of the power $T^{\prod_{\mathscr{D}} D(G_i)}$, we let $\eta([g])([a]) = \Lambda([r])$, where $r_i = a_i(g_i)$. (N.B.: each a_i is a continuous homomorphism from G_i to T; each g_i is an element of G_i; so each r_i is an element of T.)

3.4. Lemma. *η is well defined, with values in $\sum_{\mathscr{D}}^{0} G_i$.*

Proof. Suppose $[g]=[h]$ in $\prod_{\mathscr{D}} G_i$. Then $\{i: g_i=h_i\} \in \mathscr{D}$. Let $[a] \in \prod_{\mathscr{D}} D(G_i)$. Then $\{i: a_i(g_i)=a_i(h_i)\} \in \mathscr{D}$, so $\eta([g])=\eta([h])$; hence η is a well defined function from $\prod_{\mathscr{D}} G_i$ to $T^{\prod_{\mathscr{D}} D(G_i)}$. We need to show that $\eta([g])$ is a homomorphism. Let $[a],[b] \in \prod_{\mathscr{D}} D(G_i)$. Then $\eta([g])([a]\cdot[b]^{-1})=\eta([g])([a\cdot b^{-1}])=\Lambda([r])$, where $r_i=(a_i\cdot b_i^{-1})(g_i)$. Let $s_i=a_i(g_i)$, $t_i=b_i(g_i)$. Then $r_i=s_i\cdot t_i^{-1}$; so $\Lambda([r])=\Lambda([s]\cdot[t]^{-1})=\Lambda([s])\cdot(\Lambda([t]))^{-1}$, by Lemma 3.3. This last expression is just $\eta([g])([a])\cdot(\eta([g])([b]))^{-1}$. □

3.5. Lemma. *η is continuous.*

Proof. Let $U \subseteq T$ be open, and let $a \in \prod_{i\in I} D(G_i)$. The subbasic open set determined by U and a is denoted $[a,U]=\{\chi \in \sum_{\mathscr{D}}^{0} G_i: \chi([a]) \in U\}$. We claim that $\eta^{\leftarrow}[[a,U]]=\bigcup\{\prod_{\mathscr{D}} a_i^{\leftarrow}[V]: V\subseteq T \text{ is open and } \bar{V}\subseteq U\}$. For let $[g]\in\eta^{\leftarrow}[[a,U]]$. Then $\eta([g])([a])\in U$, so $\Lambda([r])\in U$, where $r_i=a_i(g_i)$. Thus, by results of [2], $[r]\in\prod_{\mathscr{D}} V$ for some open $V\subseteq T$ with $\bar{V}\subseteq U$. This says that $[g]\in\prod_{\mathscr{D}} a_i^{\leftarrow}[V]$. The reverse inclusion is similar. □

3.6. Lemma. *η is a homomorphism.*

Proof. Let $[g],[h]\in\prod_{\mathscr{D}} G_i$, let $[a]\in\prod_{\mathscr{D}} D(G_i)$. We need to show

$$\eta([g]\cdot[h]^{-1})([a]) = \eta([g])([a])\cdot(\eta([h]([b]))^{-1}.$$

This is straightforward application of Lemma 3.3. □

3.7. Lemma. *η is an injection.*

Proof. This uses the key idea in the proof of the duality theorem, that if $G\in\mathbf{KAb}$ and $g\neq 1$ in G, then for some χ in $D(G)$, $\chi(g)\neq 1$. Represent $z\in T$ as $e^{i\theta}$ for unique $-\pi<\theta\leq\pi$. Write $\theta=\alpha(z)$. For each $n\in\mathbf{Z}$, the map $e^{i\theta}\mapsto e^{in\theta}$ is a character on T; so for each $z\in T$, $z\neq 1$, there is a character $\chi\in D(T)$ such that $|\alpha(z)|\geq\pi/4$.

Now suppose $[g]\in\prod_{\mathscr{D}} G_i$ is not 1, say $\{i: g_i\neq 1\}\in\mathscr{D}$. For each such i, choose $a_i\in D(G_i)$ so that $|\alpha(a_i(g_i))|\geq\pi/4$. Then $\eta([g])([a])\neq 1$, hence $[g]$ is not in the kernel of η. □

Lemmas 3.4 to 3.7 prove the following:

3.8. Theorem. *$\eta:\prod_{\mathscr{D}} G_i\to\sum_{\mathscr{D}}^{0} G_i$ is a continuous monomorphism of groups.*

Our assertion that $\prod_{\mathscr{D}} G_i$ 'almost never' topologically embeds as a subgroup of $\sum_{\mathscr{D}}^{0} G_i$ can be made precise as follows:

3.9. Proposition. *Let $\phi:\prod_{\mathscr{D}} G_i\to\sum_{\mathscr{D}}^{0} G_i$ be any continuous homomorphism. If*

$\prod_{\mathscr{D}} G_i$ is infinite, i.e., if $\{i: |G_i| \geq n\} \in \mathscr{D}$ for all $n < \omega$, and if $\mathscr{D}$ is countably incomplete, then ϕ is not a topological embedding.

Proof. If $\phi : \prod_{\mathscr{D}} G_i \to \sum^0_{\mathscr{D}} G_i$ is a topological embedding as well as a group homomorphism, let G be the closure in $\sum^0_{\mathscr{D}} G_i$ of the image of ϕ. Then G is a compact subgroup; hence its topology is point-homogeneous. If $\prod_{\mathscr{D}} G_i$ is infinite, then so is G. If, furthermore, $\mathscr{D}$ is countably incomplete, then $\prod_{\mathscr{D}} G_i$ is a P-space. (See [2]: every point x is a P-point, i.e., whenever U_n is an open neighborhood of x for each $n < \omega$, there is an open U containing x and contained in each U_n.) Hence G is an infinite compact group with a dense subspace which is a P-space. This says that each point of the dense subspace must be a P-point of G. By point-homogeneity, G must itself be a P-space. But compact P-spaces are finite. Thus, no continuous homomorphism from $\prod_{\mathscr{D}} G_i$ to $\sum^0_{\mathscr{D}} G_i$ can be a topological embedding. □

3.10. Remarks. (i) All we know about η has now been expressed. We do not know, for example, whether the image $\eta[\prod_{\mathscr{D}} G_i]$ is generally dense in $\sum^0_{\mathscr{D}} G_i$; and we do not know whether η can be used to achieve a continuous isomorphism between $\sum^0_{\mathscr{D}} G$ and $\sum^0_{\mathscr{D}} H$ from an isomorphism between $\prod_{\mathscr{D}} M(G)$ and $\prod_{\mathscr{D}} M(H)$. If, on the other hand, Conjecture 3.1 turns out to be false, then we will know that η plays a much weaker role than its counterpart in **KH**.

(ii) We have mentioned little about co-elementary equivalence $\equiv$ in **KAb**. Although there are interesting questions as to which properties P of compact abelian groups are preserved by $\equiv$, the problem boils down to an analysis of duality and of elementary equivalence of abelian groups. Thus, P is preserved by co-elementary equivalence if and only if $D(P)$ is preserved by elementary equivalence. For example, given $G \in$ **KAb**, (the underlying space of) G is connected if and only if $U(G)$ is divisible, if and only if $D(G)$ is torsion-free [12]. Thus, the property of connectedness (or of divisibility) is preserved by co-elementary equivalence. On the other hand, G is zero-dimensional if and only if $D(G)$ is a torsion group. This implies that zero-dimensionality is not preserved.

(iii) As to the number of $\equiv$-classes in **KAb**, the answer is immediately c: use duality and count Szmielew invariants [16].

(iv) Since duality converts weight of compact groups to cardinality of their discrete character groups, the analogue of Theorem 1.7 goes through without difficulty.

References

[1] B. Banaschewski, More on compact Hausdorff spaces and finitary duality, Canad. J. Math. 36 (1984) 1113–1118.

[2] P. Bankston, Ultraproducts on topology, Gen. Top. Appl. 7 (1977) 283–308.

[3] P. Bankston, Some obstacles to duality in topological algebra, Canad. J. Math. 34 (1982) 80–90.

[4] P. Bankston, Expressive power in first order topology, J. Symbolic Logic 49 (1984) 478–487.
[5] P. Bankston, First order representations of compact Hausdorff spaces, in: Categorical Topology (Heldermann, Berlin, 1984) 23–28.
[6] P. Bankston, Reduced coproducts of compact Hausdorff spaces, J. Symbolic Logic 52 (1987) 404–424.
[7] C.C. Chang and H.J. Keisler, Model Theory (North-Holland, Amsterdam, 1973).
[8] D. Dacunha-Castelle and J. Krivine, Applications des ultraproducts à l'étude des espaces et des algèbres de Banach, Studia Math. 41 (1972) 315–334.
[9] R. Gurevič, On ultracoproducts of compact Hausdorff spaces, J. Symbolic Logic 53 (1988) 294–300.
[10] R. Gurevič, Topological model theory and factorization theorems, Preprint.
[11] C.W. Henson, Nonstandard hulls of Banach spaces, Israel J. Math. 25 (1976) 108–144.
[12] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. I (Springer, Berlin, 1963).
[13] J. Nagata, Modern Dimension Theory (Wiley, New York, 1965).
[14] T. Ohkuma, Ultrapowers in categories, Yokahama Math. J. 14 (1966) 17–37.
[15] J. Rosický, Categories of models, Fernuniversität Hagen, Sem. Math. Inf. 19 (1984) 377–413.
[16] W. Szmielew, Elementary properties of abelian groups, Fund. Math. 41 (1955) 203–271.
[17] R.C. Walker, The Stone–Čech Compactification (Springer, Berlin, 1974).
[18] P. Bankston, Model-theoretic characterizations of arcs and simple closed curves, Proc. A.M.S. 104 (1988) 898–904.
[19] P. Bankston, Taxonomies of model-theoretically defined topological properties, J. Symbolic Logic, to appear.