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CLOPEN SETS IN HYPERSPACES

PAUL BANKSTON

Abstract. Let $X$ be a space and let $H(X)$ denote its hyperspace ($= \text{all nonempty closed subsets of } X \text{ topologized via the Vietoris topology}$). Then $X$ is Boolean ($= \text{totally disconnected compact Hausdorff}$) iff $H(X)$ is Boolean; and if $B$ denotes the characteristic algebra of clopen sets in $X$ then the corresponding algebra for $H(X)$ is the free algebra generated by $B$ modulo the ideal which "remembers" the upper semilattice structure of $B$.

0. Introduction. This note concerns the algebraic topology of the hyperspace $H(X)$ of a compact Hausdorff space $X$. Two antithetical situations immediately arise, namely when $X$ is connected (i.e., a continuum) and when $X$ is totally disconnected (i.e., a Boolean space). In the first situation one can study the homotopy of $H(X)$. Indeed, in 1931 Borsuk and Mazurkiewicz [2] showed that $H(X)$ is path connected if $X$ is metric; and in an unpublished paper, Banaschewski was able to remove the metrizability condition. Although the calculation of homotopy for $H(X)$ is still an open problem when $X$ is a general continuum (even a metric continuum), the question has long been settled for $X$ a Peano space ($= \text{locally connected metric continuum}$). In this case there is a beautiful succession of increasingly stronger results: $H(X)$ is Peano (Vietoris, 1923); $H(X)$ is contractible (Woydyslawski, 1938); $H(X)$ is an AR (Woydyslawski, 1939); and $H(X)$ is the Hilbert cube (Curtis and Schori, 1974). The historical details up to 1939 are in [2]; the last result is in [1]. Needless to say $H(X)$ has uninteresting homotopy when $X$ is Peano.

Our interest here lies in the second of the above situations, that is where $X$ is Boolean; and the algebraic object we study is the characteristic Boolean algebra $\chi(X)$ of clopen subsets of $X$. We then have a computational result which takes the following form: Let $B = \chi(X)$. Then $\chi(H(X))$ is the quotient of the free algebra generated by $B$ divided by the ideal which "remembers" the upper semilattice structure of $B$. Although the theorem makes sense without requiring that $H(X)$ be Boolean, the proof we present requires this property; and in fact it is easy to show (modulo classical results) that $X$ is Boolean iff $H(X)$ is Boolean as well.
1. Preliminaries on hyperspaces

1.1 Definition. Let $X$ be a space, $S \subseteq X$. Define $S^1 = \{C : C \subseteq X \text{ closed nonempty}, C \subseteq S\}$, and $S^2 = \{C : C \subseteq X \text{ closed nonempty}, C \cap S \neq \emptyset\}$. The hyperspace $H(X)$ of $X$ consists of all closed nonempty subsets of $X$ topologized by taking subbasic sets of the form $U_1, U_2$ for $U \subseteq X$ open. The resulting topology is called the Vietoris topology.

An interesting and helpful fact is the following:

1.2 Lemma. Let $r = \{U_1, \ldots, U_m\}$ be a finite set of open subsets of $X$ and denote by $r^\#_r$ the set $\{C \in H(X) : C \subseteq U_1 \cup \cdots \cup U_m, \text{ and for } 1 \leq k \leq m, C \cap U_k \neq \emptyset\}$. Then the collection of all such sets $r^\#_r$ forms a basis for the Vietoris topology.

Proof. Let $r = \{U_1, \ldots, U_m\}, s = \{V_1, \ldots, V_n\}$, with $U = \cup r, V = \cup s$; and let $t$ be the set $\{U \cap V, U_1 \cap V, \ldots, U_m \cap V, U \cap V_1, \ldots, U \cap V_n\}$. Then $t^\# = r^\# \cap s^\#$, so the collection forms a basis for some topology $\tau$. Now if $U \subseteq X$ is open then $U^1 = \{U\}^\#$, $U^2 = \{U,X\}^\#$, whence $\tau$ contains the Vietoris topology. On the other hand if $r = \{U_1, \ldots, U_m\}$ then $r^\# = \{U_1 \cup \cdots \cup U_m\}^1 \cap U_1^2 \cap \cdots \cap U_m^2$, so $\tau$ is in fact the Vietoris topology itself. \qed

Remarks. (i) For metrizable $X$, say with metric $d$, $H(X)$ is metrizable as well via the well-known Hausdorff metric over $d$; and the Vietoris topology is precisely the derived metric topology.

(ii) Let $f : X \to Y$ be a continuous closed map and define $H(f) : H(X) \to H(Y)$ in the obvious way, i.e. $H(f)(C) = f''C = \{f(x) : x \in C\}$. Then $H(f)$ is continuous, indeed

$$H(f)^{-1}(\{U_1, \ldots, U_m\}^\#) = \{f^{-1}(U_1), \ldots, f^{-1}(U_m)\}^\#.$$

$H(f)$ need not be a closed map, however.

Assume all spaces henceforth to be $T_1$. Then for each $X$ there is a natural injection $i : X \to H(X)$ taking $x \in X$ to its singleton $\{x\}$. $i$ is evidently a topological embedding (and in the metric case an isometry). Moreover, if $X$ is Hausdorff then $i$ is closed as well. For suppose $C \in H(X) - i''X$, with $a, b \in C$ distinct. Let $U = U(a), V = V(b)$ be a disjoint pair of open sets. Then $\{U, V, X\}$ is a nbd of $C$ missing $i''X$.

Now a standard result of Vietoris (see [4]) is that $X$ is compact Hausdorff iff $H(X)$ is compact Hausdorff. By extending this theorem we can easily prove that $X$ is Boolean iff $H(X)$ is Boolean. To see this, assume $H(X)$ is Boolean. Then, since $X$ embeds via $i$ as a closed subset of $H(X)$, $X$ is Boolean as well. Conversely if $X$ is Boolean then $X$ has a basis of clopen sets so that if $C_1, C_2 \in H(X)$ are distinct with, say, $a \in C_1 - C_2$, then there is a clopen $U$ containing $a$ and missing $C_2$. Thus $\{U, X\}^\#$ is a clopen nbd of $C_1$ missing $C_2$, proving that $H(X)$ is “ultra-Hausdorff” hence (in view of compactness) Boolean.

An alternative description of $H(X)$ for $X$ Boolean goes as follows (the straightforward details being left to the reader): Let $B$ be the algebra of clopen sets in $X$ and let $\sigma^*(B)$ be the set of proper filters in $B$. For $s = \{b_1, \ldots, b_n\} \subseteq B$ let $s^\# = \{p \in \sigma^*(B) : b_1 \lor \cdots \lor b_n \in p\}$ and for $1 \leq k \leq n, b \in p,$
Then the sets $s^\#$ form a topological basis; and the resulting space is precisely $H(X)$. The proof of this fact hinges upon the Stone duality between the closed nonempty subsets of a Boolean space and the proper filters of the corresponding Boolean algebra.

2. The main theorem. Let $\text{BTop}$ denote the category of Boolean spaces and continuous maps with $\text{Boo}$ the category of Boolean algebras and homomorphisms. Let

$$\text{BTop} \xrightarrow{x} \text{Boo} \xleftarrow{\sigma} \text{BTop}$$

be the statement of Stone duality; where $x(X)$ is the characteristic algebra of clopen subsets of $X$, and $\sigma(B)$ is the Stone space of ultrafilters of $B$ topologized by taking as basis sets all sets of the form $b^0 = \{ u \in \sigma(B) : b \subseteq u \}$ as $b$ ranges over $B$. The Stone Duality Theorem says that $x$ and $\sigma$ are contravariant natural equivalences. Now if $X$ is compact, $Y$ is Hausdorff and $f: X \rightarrow Y$ continuous then $f$ is automatically a closed map. Thus in particular the hyperspace operator $H: \text{BTop} \rightarrow \text{BTop}$ is functorial. A corollary of our theorem will be that there is an (algebraically defined) endofunctor $H': \text{Boo} \rightarrow \text{Boo}$ which makes the category-theoretic diagram

$$\begin{array}{ccc}
\text{BTop} & \xrightarrow{H} & \text{BTop} \\
\chi \downarrow \uparrow \sigma & & \chi \downarrow \uparrow \sigma \\
\text{Boo} & \xrightarrow{H'} & \text{Boo}
\end{array}$$

commutative.

We now define $H'$. Let $B$ be a Boolean algebra. For ease of notation we will assume $B$ to be a field of sets and so use the usual set-theoretic notation for the Boolean algebraic operations. Now let $F(B)$ denote the free Boolean algebra generated by the elements of $B$. In this context we will use the connectives of elementary logic to denote the operations in $F(B)$ and use square brackets to distinguish the set $U$ in $B$ from its “name” $[U]$ in $F(B)$. So if $B = x(X)$ for some $X \in \text{BTop}$ then $\{ \cup, \cap, X - (\cdot), \emptyset, X \}$ denote the Boolean operations in $B$, whereas $\{ \vee, \wedge, \neg, 0, 1 \}$ denote the corresponding “formal” operations in $F(B)$. Typical elements of $F(B)$ include words of the form $[X], [\emptyset], [U], [U \cap (X - V)] \vee \neg[W]$, etc.

Given $B = x(X)$ we define the ideals $I_1, I_2$ in $F(B)$ as follows:

$I_1$ is generated by the words $\{ [U] \land \neg[V] : U \subseteq V \text{ in } B \}$ and $\{ [U] \land [V] \land \neg[U \land V] : U, V \in B \}$.

$I_2$ is generated by the words $\{ [U] \land \neg[V] : U \subseteq V \text{ in } B \}$ and $\{ [U \lor V] \land \neg([U] \lor [V]) : U, V \in B \}$.

Intuitively $I_1$ (resp. $I_2$) “remembers” the lower (resp. upper) semilattice structure of $B$ so that $F(B)/I_1$, say, “believes” that $[U] \subseteq [V]$ whenever $U \subseteq V$, and that $[U] \land [V] = [U \cap V]$.

Now define the homomorphisms $h_1, h_2$ from $F(x(X))$ to $x(H(X))$ as follows: Let $U \in x(X)$. Then we set $f_1(U) = U^1$, $f_2(U) = U^2$. (Note. $U^1, U^2$ are clopen in $H(X)$ since $H(X) - U^1 = (X - U)^2$, etc.) Since $F(x(X))$ is free, $f_1, f_2$ extend uniquely to homomorphisms $h_1, h_2$. We can now state our main theorem thusly:
2.1 Theorem. Let $X \in \text{BTop}$ with $h_1, h_2 : F(\chi(X)) \to \chi(H(X))$ given as above. Then:

(i) Both $h_1$, $h_2$ are epimorphisms.

(ii) Ker $h_1 = I_1$, Ker $h_2 = I_2$ whence

$$\chi(H(X)) \cong F(\chi(X))/I_1 \cong F(\chi(X))/I_2.$$ 

Thus if $B$ is any Boolean algebra, if we let $H'(B)$ be either of the above quotients, and if $g : B_1 \to B_2$ is a homomorphism, let $H'(g)$ be the obvious quotient homomorphism. Then

(iii) The diagram

$$\begin{array}{ccc}
\text{BTop} & \overset{H}{\rightarrow} & \text{BTop} \\
\chi \downarrow \uparrow \sigma & & \chi \downarrow \uparrow \sigma \\
\text{Boo} & \overset{H'}{\rightarrow} & \text{Boo}
\end{array}$$

commutes (up to natural equivalences).

(iv) Let $j : F(\chi(X)) \to \chi(X)$ be the natural projection. Then Ker $j = I_1 \vee I_2$ = the ideal generated by $I_1 \cup I_2$.

Proof. (i) We show that the algebra $A$ generated by the sets $U_1, U \in \chi(X)$, is all of $\chi(H(X))$. Indeed by compactness the sets $s^\#$ form a basis for $H(X)$ as $s$ ranges over the finite subsets of $\chi(X)$. Also since

$$s^\# = \{U_1, \ldots, U_m\}^\# = (U_1 \cup \cdots \cup U_m)^1 \cap (U_1^2 \cap \cdots \cap U_m^2)$$

we have that the $U_1$'s generate a basis for the Vietoris topology. Since every clopen set in $H(X)$ is compact and is a union of elements from $A$, it is a union of finitely many elements from $A$ and is thus itself in $A$. Thus $A = \chi(H(X))$.

(ii) Since $\emptyset^1 = \emptyset, X^1 = X, U^1 \subseteq V^1$ for $U \subseteq V$, and $(U \cap V)^1 = U^1 \cap V^1$ for all $U, V \in \chi(X)$, we have Ker $h_1 \supseteq I_1$. Similarly Ker $h_2 \supseteq I_2$. Let $w \in F(\chi(X))$ and assume $w$ is represented as a disjunction of conjunctions of generators and complements of generators. Such a conjunction we refer to as a minterm. If $w = w_1 \lor \cdots \lor w_n$ is a disjunction of minterms, and if the minterms of Ker $h_1$ are in $I_1$ then Ker $h_1 \subseteq I_1$. For $w \in \text{Ker } h_1 \Rightarrow w_k \in \text{Ker } h_1, \text{each } 1 \leq k \leq n$. Thus $w \in I_1$. So it suffices to prove the inclusion for minterms, of which there are three kinds: positive (only unnegated generators occur), negative, and mixed.

Positive. $w = [U_1] \land \cdots \land [U_m] \in \text{Ker } h_1$. Then $U_1 \cap \cdots \cap U_m = (U_1 \land \cdots \land U_m)^1 = \emptyset$ iff $U_1 \cap \cdots \cap U_m = \emptyset$. Now

$$([U_1] \land \cdots \land [U_m]) \land \neg[U_1 \land \cdots \land U_m]$$

$$= [U_1] \land \cdots \land [U_m] \land \neg[\emptyset] \in I_1.$$ 

But $[\emptyset] \in I_1$, so $[U_1] \land \cdots \land [U_m] \in I_1$ as well.

Negative. $w = \neg[V_1] \land \cdots \land \neg[V_n] \in \text{Ker } h_1$. Then $(X - V_1)^2 \cap \cdots \cap (X - V_n)^2 = \emptyset$ iff some $V_k = X$. But $\neg[X] \in I_1$, so $w \in I_1$ too.

Mixed. $w = [U_1] \land \cdots \land [U_m] \land \neg[V_1] \land \cdots \land \neg[V_n] \subseteq \text{Ker } h_1$. Then

$$(U_1 \cap \cdots \cap U_m)^1 \cap (X - V_1)^2 \cap \cdots \cap (X - V_n)^2 = \emptyset.$$
whence $U_1 \cap \cdots \cap U_m \subseteq V_l$ for some $1 \leq l \leq n$. Thus $[U_1 \cap \cdots \cap U_m] \land \neg[V_l] \in I_1$ so $([U_1] \land \cdots \land [U_m]) \land \neg[V_l] \in I_1$ and therefore $w \in I_1$.

The proof that $\ker h_2 = I_2$ is similar.

(iii) This follows straightforwardly from (ii).

(iv) This is proved in the same way as (ii). □

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