MAPPING PROPERTIES OF CO-EXISTENTIALLY CLOSED CONTINUA

PAUL BANKSTON

ABSTRACT. A continuous surjection between compacta is called co-existential if it is the second of two maps whose composition is a standard ultracopower projection. A continuum is called co-existentially closed if it is only a co-existential image of other continua. This notion is not only an exact dual of Abraham Robinson's existentially closed structures in model theory, it also parallels the definition of other classes of continua defined by what kinds of continuous images they can be. In this paper we continue our study of co-existentially closed continua, especially how they (and related continua) behave in certain mapping situations.

1. INTRODUCTION

By a **compactum** we mean a compact Hausdorff space, a **continuum** is a connected compactum. A **subcompactum** (resp., **subcontinuum**) of a space is just a subspace that is itself a compactum (resp., continuum).

Given a compactum X and an ultrafilter $\mathcal{D}$ on an index set I (i.e., $\mathcal{D}$ is a maximal filter in the Boolean power set algebra of I), the **ultracopower of X via $\mathcal{D}$** is denoted $XI \setminus \mathcal{D}$. One easy way to describe this construction is to regard I as a discrete space, letting $p : X \times I \to X$ and $q : X \times I \to I$ be the standard projection maps. Applying the Stone-Čech compactification functor $\beta(\ )$ (see, e.g., [23, 24]), we regard $\mathcal{D}$ as a point in $\beta(I)$ and define the ultracopower to be the inverse image of $\mathcal{D}$ under q^{β}. We denote by $p_{X,\mathcal{D}}$ the restriction of p^{β} to $XI \setminus \mathcal{D}$. It is the **standard ultracopower (codiagonal) projection**, a continuous surjection to X.

The construction of ultracopowers (and, more generally, of ultracoproducts) of compacta first appeared in [1]; and in [10] R. Gurevič further exploited the connection between ultracoproducts of compacta and ultraproducts of lattices to settle some questions raised in [1]. Ultracopowers of **arcs** (i.e., homeomorphic copies of the closed unit interval) also figured prominently in the independent work of J. Mioduszewski [17] to study the Stone-Čech compactification of the half-open unit interval. (See [11, 21, 22, 25] for further work along these lines.)

A continuous surjective mapping $f : X \to Y$ between compacta is called a **map of level ≥ 0**. Given $n < \omega$, f is called a **map of level $\geq n + 1$** if there is an ultracopower $YI \setminus \mathcal{D}$ and a map $g : YI \setminus \mathcal{D} \to X$ of level $\geq n$ such that the composition $f \circ g$ equals $p_{Y,\mathcal{D}}$. This defines inductively an ordinal-indexed hierarchy of maps between compacta (the **co-elementary hierarchy**); for any limit ordinal

α, f is **of level** $\geq \alpha$ if f is of level $\geq \beta$ for all $\beta < \alpha$. By Theorem 2.10 in [5], the hierarchy ends at level $\geq \omega$, at which point we have the **co-elementary** maps (i.e., maps $f : X \to Y$ for which there exists a homeomorphism of ultracopowers $h : XI\backslash\mathcal{D} \to YJ\backslash\mathcal{E}$ such that $p_{Y,\mathcal{E}} \circ h = f \circ p_{X,\mathcal{D}}$). Because of this, we will consider only ordinal levels up to and including ω in the sequel. Terms like $\alpha \pm 1$ are defined to be α, if α is infinite, and are defined as usual otherwise.

Maps of level ≥ 1 are referred to as **co-existential**. These mappings were introduced in [5] as topological analogues (in a category dual sense, see [16]) of existential embeddings in model theory (see [12, 15, 20]); they also arise naturally *from* existential embeddings, giving us more than just an analogue. Suppose X and Y are compacta with **lattice bases** $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively. (This means they are closed-set bases that are bounded lattices under union and intersection.) If $f : \mathcal{B}_Y \to \mathcal{B}_X$ is an existential embedding (think of one field being algebraically closed relative to a larger field), and if $f^* : X \to Y$ is the natural continuous surjection induced by f, then f^* is co-existential.

Here is a summary of what is already known about co-existential maps. (See, e.g., [19] for definitions of continuum-theoretic notions.)

Theorem 1.1.   (1) (Theorems 2.4 and 2.7 in [6]) *Co-existential maps are weakly confluent; in the case of locally connected range, they are monotone.*
  (2) (Proposition 2.5 in [6] and Theorem 7.1 in [7]) *Co-existential maps preserve the topological properties of: being infinite, being disconnected, being totally disconnected, being an indecomposable continuum, being a hereditarily indecomposable continuum, and being a hereditarily decomposable continuum.*
  (3) (Theorem 2.6 in [6] and Theorem 2.5 in [7]) *Co-existential maps (resp., maps of level ≥ 2) preserve or lower (resp., preserve) covering dimension.*
  (4) (Corollaries 5.4 and 5.6 in [8]) *Co-existential maps (resp., maps of level ≥ 2) preserve or lower (resp., preserve) the multicoherence degree of continua.*
  (5) (Proposition 2.7 in [4] and Theorem 2.7 in [6]) *A function from an arc to a compactum is a co-existential (equivalently, a co-elementary) map if and only if the range is an arc and the map is a continuous monotone surjection.*

In analogy with the model-theoretic notion of a relational structure being existentially closed relative to a class of structures of which it is a member (again, see [12, 15, 20]) we define a **co-existentially closed continuum** to be a continuum that can be only a co-existential image under maps whose domains are continua. Co-existentially closed continua were first introduced in [6] (which was written before [5], despite appearing later). There are other well-known classes of continua defined in a similar fashion; most notably we have Class(C) (resp., Class(W)), the class of metrizable continua that can be only confluent (resp., weakly confluent) images under maps whose domains are metrizable continua. These two classes were first studied by A. Lelek; one of the most interesting results being that Class(C) consists precisely of the hereditarily indecomposable metrizable continua. (See [19] for details.)

The following is a summary of what is already known about co-existentially closed continua.

Theorem 1.2.   (1) (Theorem 6.1 in [6]) *Every nondegenerate continuum is a continuous image of a co-existentially closed continuum of the same weight.*
   (2) (Theorem 2.7 in [7]) *A co-existential image of a co-existentially closed continuum is a co-existentially closed continuum.*
   (3) (Corollary 4.13 in [8]) *Every co-existentially closed continuum is a hereditarily indecomposable continuum of covering dimension one.*
   (4) (Theorem 4.1 in [7]) *There are at least two topologically distinct metrizable co-existentially closed continua.*

Remark 1.3. *We could just as easily have defined the notion of* co-existentially closed compactum; *in this setting, however, there is a simple charactization. By Theorem 3.1 in [5], the co-existentially closed compacta are precisely the **Boolean spaces** (i.e., totally disconnected compacta) without isolated points. So in particular, if* compactum *is substituted for* continuum *in Theorem 1.2 (and* infinite *is substituted for* nondegenerate), *then clauses 1 and 2 are true, and clauses 3 and 4 are false (no matter how one defines* hereditarily indecomposable compactum).

*A class of compacta closed under co-elementary images, as well as the taking of ultracoproducts (like the class of co-existentially closed compacta) is called a **co-elementary class**.*

We end this and succeeding sections with relevant (annotated) open questions.

Open Questions 1.4.   (1) *Is the* pseudo-arc *a co-existentially closed continuum? [See, e.g., [14] and [19] for extensive discussions on this very interesting space, characterized as the unique metrizable hereditarily indecomposable arc-like continuum. By Theorem 1.2(3), the answer would be yes if we could show the existence of a metrizable co-existentially closed continuum that is arc-like. By Theorem 1.2(4), there must be a co-existentially closed continuum that is not arc-like, hence not a pseudo-arc.]*
   (2) *Is the class of co-existentially closed continua co-elementary? [This would provide a "Nullstellensatz" for the class of continua. Since the class is already closed under co-elementary (indeed, co-existential) images, all we need to do is show it closed under the taking of ultracoproducts.]*

2. TERMINAL WEDGES OF MAPS

Define a **terminal wedge** of maps to be a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$, where X, Y and Z are compacta and $f : X \to Z$, $g : Y \to Z$ are continuous maps. An **initial wedge** is defined similarly, with the only difference being that the domains and ranges are reversed. Z is called the **base** of the wedge (whether terminal or initial). A terminal wedge $X \xrightarrow{f} Z \xleftarrow{g} Y$ and an initial wedge $U \xleftarrow{r} W \xrightarrow{s} V$ are **commutators of one another** if:
   (i) $X = U$ and $Y = V$; and
   (ii) $f \circ r = g \circ s$ (i.e., the obvious mapping square commutes).

Remark 2.1. *Every terminal wedge $X \xrightarrow{f} Z \xleftarrow{g} Y$ of maps has a commutator, $X \xleftarrow{r} W \xrightarrow{s} Y$ namely the **fiber product** (or **pullback**), where $W = \{(x,y) \in X \times Y : f(x) = g(y)\}$, and r and s are the restricted coordinate projections. If the*

maps f and g are of level ≥ 0, then so are the maps r and s. In the setting of continua this construction need not be connected, however. For instance, if X, Y, and Z are all **simple closed curves***; i.e., homeomorphs of (and represented by) the standard unit circle consisting of complex numbers of norm 1, and $f = g$ is the squaring map, then the components of W are the two sets $\{(x, y) \in X \times X : x = y\}$ and $\{(x, y) \in X \times X : x = -y\}$.*

Let $\alpha \leq \omega$. A compactum (resp., continuum) Z is a **level $\geq \alpha$ base compactum** (resp., **continuum**) if for every terminal wedge $X \xrightarrow{f} Z \xleftarrow{g} Y$ of maps of level $\geq \alpha$, where X and Y are compacta (resp., continua) there is a commutator $X \xleftarrow{r} W \xrightarrow{s} Y$, where W is a compactum (resp., continuum) and r and s are maps of level $\geq \alpha$. (This is a topological analogue of the notion of *amalgamation base* in model theory; see, e.g., [12].) By Remark 2.1, every compactum is a level ≥ 0 base compactum; by Theorem 1.1(2), if $\alpha \geq 1$, then a continuum is a level $\geq \alpha$ base continuum if and only if it is a level $\geq \alpha$ base compactum. One goal of this section is to show that every co-existentially closed continuum is a level ≥ 0 base continuum.

Theorem 2.2. *Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a terminal wedge of maps, where f and g are maps of levels $\geq \alpha$ and $\geq \beta$, respectively, $\alpha, \beta \leq \omega$.*

(1) *If α is positive, the wedge has a commutator $X \xleftarrow{r} W \xrightarrow{s} Y$, where r and s are maps of levels $\geq min\{\alpha - 1, \beta\}$ and $\geq \omega$, respectively. Furthermore, if Y is a continuum, then so is W.*

(2) *If both α and β are positive, the wedge has a commutator $X \xleftarrow{r} W \xrightarrow{s} Y$, where r and s are maps of levels $\geq \alpha - 1$ and $\geq \beta - 1$, respectively. Furthermore, if Z is a continuum, then so is W.*

Proof. Ad (1): Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be the given terminal wedge, where f and g are maps of levels $\geq \alpha \geq 1$ and $\geq \beta$, respectively. Then there is an ultracopower map $h : ZI\backslash\mathcal{D} \to X$, of level $\geq \alpha - 1$, such that $f \circ h = p_{Z,\mathcal{D}}$. Using the functoriality of $(\ )I\backslash\mathcal{D}$, we set $W := YI\backslash\mathcal{D}$ and $r := h \circ (gI\backslash\mathcal{D})$. By Corollary 2.4 in [5], the ultracopower map $gI\backslash\mathcal{D}$ is of level $\geq \beta$. Also we have the commutativity $p_{Z,\mathcal{D}} \circ (gI\backslash\mathcal{D}) = g \circ p_{Y,\mathcal{D}}$. By Proposition 2.5 in [5], the composition of two maps of level $\geq \lambda$ is again a map of level $\geq \lambda$; hence r is a map of level $\geq \min\{\alpha - 1, \beta\}$. Setting $s := p_{Y,\mathcal{D}}$, we have our advertised map of level $\geq \omega$, and $f \circ r = g \circ s$. By Theorem 1.1(2), W is a continuum if Y is.

Ad (2): Now assume both α and β are positive. Then there are:

(i) compacta U, V; maps $h : U \to X$, $j : V \to Y$, of levels $\geq \alpha - 1$ and $\geq \beta - 1$, respectively; and

(ii) maps $p : U \to Z$, $q : V \to Z$, both of level $\geq \omega$, where $f \circ h = p$ and $g \circ j = q$.

By the argument in the last paragraph, there is a compactum W and maps $m : W \to U$, $n : W \to V$, both of level $\geq \omega$, such that $p \circ m = q \circ n$. Set $r := h \circ m$ and $s := j \circ n$. Then, again by Proposition 2.5 in [5], r and s are maps of levels $\geq \alpha - 1$ and $\geq \beta - 1$, respectively, and $f \circ r = g \circ s$. Finally, if Z is a continuum, then so are U and V; hence so is W (again by Theorem 1.1(2)). $\qquad\square$

The following is now an immediate consequence of Theorem 2.2.

Corollary 2.3.   (1) *Every co-existentially closed continuum is a level ≥ 0 base continuum.*

    (2) *Every compactum (resp.,continuum) is a level $\geq \omega$ base compactum (resp.,continuum).*

The next result summarizes what we know about level $\geq \alpha$ base compacta, for $\alpha \leq \omega$.

Corollary 2.4. *All compacta are level $\geq \alpha$ base compacta for $\alpha \in \{0, \omega\}$; all Boolean spaces without isolated points are level $\geq \alpha$ base compacta for $\alpha \geq 2$.*

Proof. The first clause of the assertion follows from Remark 2.1 and Corollary 2.3(2). From Remark 1.3, the class of co-existentially closed compacta is closed under the taking of ultracoproducts. It is not hard to show, then, that continuous surjections between co-existentially closed compacta are maps of level $\geq \omega$. (This actually follows from closure under ultracopowers.) Now the property of being a co-existentially closed compactum is preserved by inverse images of maps of level ≥ 2. Consequently if $X \xrightarrow{f} Z \xleftarrow{g} Y$ is any terminal wedge of maps of level $\geq \alpha$, for $\alpha \geq 2$, and if Z is a co-existentially closed compactum, then so are X and Y; and f and g are maps of level $\geq \omega$. We finish by applying Theorem 2.2. $\square$

We turn now to strengthening the notion of *level $\geq \alpha$ base compactum/continuum.* Among several possibilities, the most obvious is to allow more than just two maps in the definition. A **generalized terminal wedge** of maps is an indexed diagram $\langle X_i \xrightarrow{f_i} Z : i \in I \rangle$, where each f_i is a continuous map from the compactum X_i to the compactum Z (the **base** of the generalized wedge). A **commutator** for the generalized terminal wedge is just a generalized *initial* wedge $\langle X_i \xleftarrow{r_i} W : i \in I \rangle$, where W is a compactum and each $r_i : W \to X_i$ is a map such that whenever i and j are in I, we have $f_i \circ r_i = f_j \circ r_j$.

Let $\alpha \leq \omega$. We define a compactum (resp., continuum) Z to be a **generalized level $\geq \alpha$ base compactum** (resp., **continuum**) if for every generalized terminal wedge $\langle X_i \xrightarrow{f_i} Z : i \in I \rangle$ of maps of level $\geq \alpha$, where each X_i is a compactum (resp., continuum), there is a commutator $\langle X_i \xleftarrow{r_i} W : i \in I \rangle$, where W is a compactum (resp., continuum) and each r_i is a map of level $\geq \alpha$. In [13] J. Krasinkiewicz proved the (surprisingly difficult) result that any arc is a generalized level ≥ 0 base continuum. It turns out that the use of the adjective *generalized*, as it applies to level $\geq \alpha$ base compacta/continua, is redundant.

Theorem 2.5. *Let $\alpha \leq \omega$. Then every level $\geq \alpha$ base compactum (resp., continuum) is a generalized level $\geq \alpha$ base compactum (resp., continuum).*

Proof. Fix $\alpha \leq \omega$ and assume Z is a level $\geq \alpha$ base continuum. Let I be any set (which we may as well assume to have cardinality ≥ 2), and fix a generalized terminal wedge $\langle X_i \xrightarrow{f_i} Z : i \in I \rangle$, with base Z, where each f_i is a map of level $\geq \alpha$. In order to obtain the desired commutator $\langle X_i \xleftarrow{r_i} W : i \in I \rangle$ such that each r_i is also a map of level $\geq \alpha$, we first assign a well ordering $<$ to I, letting i_0 denote the $<$-first element.

Next we construct inductively an inverse system $\langle U_i, g_{ij} : i \leq j \in I \rangle$ of compacta and bonding maps of level $\geq \alpha$ (i.e., each $g_{ij} : U_j \to U_i$, $i \leq j \in I$, is a map of level $\geq \alpha$, each g_{ii} is the identity map on U_i, and, for $i \leq j \leq k \in I$, $g_{ik} = g_{ij} \circ g_{jk}$), as well as maps $h_i : U_i \to X_i$, $i \in I$, also of level $\geq \alpha$, in such a way that:

   (i) h_{i_0} is the identity map witnessing that $U_{i_0} = X_{i_0}$; and
   (ii) for each $i \leq j \in I$, we have $f_j \circ h_j = f_i \circ (h_i \circ g_{ij})$.

For the moment let us assume such a construction possible. Then we set the compactum W to be the limit of the inverse system $\langle U_i, g_{ij} : i \leq j \in I \rangle$, with projection maps $g_i : W \to U_i$, $i \in I$. Then, by Theorem 3.4 in [5] (the α-*chains theorem*), each g_i is a map of level $\geq \alpha$. Set $r_i := h_i \circ g_i : W \to X_i$, $i \in I$. Then each r_i is a map of level $\geq \alpha$ (Proposition 2.5 in [5]). To show commutativity, if $i, j \in I$, say $i \leq j$, we have $f_i \circ r_i = f_i \circ (h_i \circ g_i) = f_i \circ (h_i \circ (g_{ij} \circ g_j)) = f_j \circ (h_j \circ g_j) = f_j \circ r_j$.

The actual construction is easy: at successor levels use the fact that Z is a level $\geq \alpha$ base compactum; at limit levels use inverse limits and argue as in the last paragraph.

Finally the argument above works equally well with *compactum* replaced with *continuum*.   □

We do not yet know whether the arc is a level $\geq \alpha$ base continuum for $\alpha \geq 1$, but there is a small amount we can say on the subject nonetheless. First we define a generalized terminal wedge to be **jointly injective** if for any point in the base, at most one inverse image of that point has more than one element.

Theorem 2.6. *Let $\mathcal{V} := \langle X_i \xrightarrow{f_i} Z : i \in I \rangle$ be a generalized terminal wedge consisting of arcs and maps of level ≥ 1, with $\langle X_i \xleftarrow{r_i} W : i \in I \rangle$ the associated fiber product.*

   (1) *If $\mathcal{V}$ is not jointly injective, then W is a continuum that is not an arc and no projection r_i is a map of level ≥ 2.*
   (2) *If $\mathcal{V}$ is jointly injective and the index set I is at most countable, then W is an arc and each projection r_i is a map of level $\geq \omega$.*

Proof. Ad (1): By Theorem 1.1(1), all the maps f_i are monotone surjections. So let $i_0 \in I$ and $x \in X_{i_0}$ be fixed. Then $r_{i_0}^{-1}[\{x\}]$ is the product $\prod_{i \in I} F_i$, where F_i is $\{x\}$ if $i = i_0$, and is $f_i^{-1}[\{f_{i_0}(x)\}]$ otherwise. Since each f_i is monotone, each r_i must be monotone too, and W is therefore a continuum.

Now suppose $\mathcal{V}$ is not jointly injective. Then there is some $z \in Z$ and two distinct $i_0, i_1 \in I$ such that $f_{i_0}^{-1}[\{z\}]$ and $f_{i_1}^{-1}[\{z\}]$ are subarcs of X_{i_0} and X_{i_1}, respectively. So $\prod_{i \in I} f_i^{-1}[\{z\}]$ is a subcontinuum of W that contains a homeomorphic copy of the closed unit square; hence, by elementary dimension theory, the covering dimension of W is at least 2. Thus W is a continuum that is not an arc. Furthermore, by Theorem 1.1(3), no projection r_i can be a map of level ≥ 2.

Ad (2): Next suppose $\mathcal{V}$ is jointly injective and that I is at most countable. We lose no generality in assuming $I = \omega$, and we write $\mathcal{V} = \langle X_n \xrightarrow{f_n} Z : n < \omega \rangle$. By Theorem 1.1(5), since the projections r_n are monotone surjective maps, all we need to show is that W is an arc. And for this, since our index set is countable and therefore W is a metrizable continuum, it suffices to show (by a classic result of R. L. Moore, see Theorem 6.17 in [19]) that W has just two noncut points (the minimum number allowed for any nondegenerate continuum, see Theorem 6.6 in [19]).

To set things up, assume each arc X_n is the standard unit interval $[0, 1]$. For each $n < \omega$ we may order X_n via the usual ordering or its reverse, depending upon whether or not f_n is $\leq$-preserving with respect to the usual ordering. For this

reason, we are safe in the assumption that each f_n is $\leq$-preserving. Denote points in W as sequences $x = \langle x_0, x_1, \ldots \rangle$, and define $x < y$ in W to hold just in case $x \neq y$, and if n is the first index where $x_n \neq y_n$, then $x_n < y_n$. (This is just the lexicographic ordering restricted to W.) Since each $f_n(0)$ is 0 and each $f_n(1)$ is 1, the points $p := \langle 0, 0, \ldots \rangle$ and $q := \langle 1, 1, \ldots \rangle$ are respectively the minimal and the maximal elements in this linear ordering on W. We show that every other point of W is a cut point.

Fix $a \in W \setminus \{p, q\}$. We endeavor to show that the "half-open" intervals $[p, a)$ and $(a, q]$ (necessarily forming a cover of $W \setminus \{a\}$ by nonempty disjoint sets) are open sets in W. For each $m < \omega$, set $U_m := W \cap (\prod_{n<\omega} B_n)$ and $V_m := W \cap (\prod_{n<\omega} C_n)$, where, for $n < \omega$, B_n (resp., C_n) is $[0, a_n)$ (resp., $(a_n, 1]$) if $n = m$, and is $[0, 1]$ otherwise. Then U_m and V_m are disjoint open subsets of W, each contained in $W \setminus \{a\}$; a point $x \in W$ is in U_m (resp., V_m) if and only if $x_m < a_m$ (resp., $x_m > a_m$). We are done once we show that, for some $m < \omega$, $[p, a) = U_m$ and $(a, q] = V_m$. Indeed, equality will follow once we show $[p, a) \subseteq U_m$ and $(a, q] \subseteq V_m$.

We start by noticing that, since the maps f_n are $\leq$-preserving, there must be some entry, say a_m, of a that is neither 0 nor 1. Since all the compositions $f_n \circ r_n$ are equal, we may fix a single $b \in Z$ that is equal to $f_n(a_n)$ for each n. If it happens that $b = 0$, then, by joint injectivity, it must be the case that $a_n = 0$ for all $n \neq m$. Suppose $x < a$. Then $x_n < a_n$ for some $n < \omega$; hence $x_m < a_m$. Thus $[p, a) \subseteq U_m$. If now $x > a$ and it is not the case that $x_m > a_m$, then (because each f_n is $\leq$-preserving) $f_m(x_m) \leq f_m(a_m) = 0$. By joint injectivity we know that $x_n = 0$ for $n \neq m$. But $x_n > a_n$ for some $n < \omega$, and no such n can equal m. This contradiction tells us that $(a, q] \subseteq V_m$.

In the event $b = 1$ we argue much as we did above, so suppose $0 < b < 1$. Then $0 < a_n < 1$ for every entry a_n of a. Pick (the unique) m such that $f_m^{-1}[\{b\}]$ is nondegenerate, if there is one; otherwise let m be arbitrary. The argument is now similar to that in the last paragraph: If $x < a$, but $x_m \geq a_m$, then for some $n \neq m$ we have $x_n < a_n$. By joint injectivity, $f_n(a_n) < b$; hence $f_m(x_m) < b$. Since $x_m \geq a_m$ and f_m is $\leq$-preserving, we have a contradiction. Thus $[p, a) \subseteq U_m$; similarly we conclude $(a, q] \subseteq V_m$, and the proof is complete. $\square$

Remark 2.7. *Define a continuum to be a **generalized arc** if it has exactly two noncut points (like an arc, but not necessarily metrizable). From the proof of Theorem 2.6(2) it is easy to show that the fiber product of a terminal wedge of generalized arcs and monotone continuous surjections is again a generalized arc, no matter what the size of the index set. (Of course a suitable lexicographic order on the fiber product depends on a well ordering of that index set.)*

Open Questions 2.8. (1) *Is the pseudo-arc a level ≥ 0 base continuum? [Clearly yes if Open Question 1.4(1) has an affirmative answer, because of Corollary 2.3(1).]*

(2) *Which compacta are level ≥ 1 base compacta?*

(3) *Give nondegenerate examples of continua that are level $\geq \alpha$ base continua, $1 \leq \alpha < \omega$. For example, is the arc a level ≥ 1 base continuum? [If Open Question 1.4(2) had an affirmative answer; indeed if the class of co-existentially closed continua were closed under the formation of ultra-copowers, we could conclude, as in Corollary 2.4, that the co-existentially closed continua are level $\geq \alpha$ base continua for $\alpha \geq 2$.]*

(4) *Find examples (if such exist) of continua that are* not *level $\geq \alpha$ base continua, for $\alpha < \omega$.*

3. α-EQUIVALENCE

In this section we show that the co-existentially closed compacta (resp., continua) share much in common with one another. Let $1 \leq \alpha \leq \omega$. For compacta X and Y, we say X is α-**dominated by** Y (and write $X \leq_\alpha Y$) whenever there is an initial wedge $X \overset{f}{\leftarrow} Z \overset{g}{\to} Y$, where f and g are maps of level $\geq \alpha - 1$ and $\geq \omega$, respectively.

Remark 3.1.   (1) *The model-theoretic analogue of $X \leq_\alpha Y$ may be written the same way, $A \leq_\alpha B$, and interpreted to mean that every Π^0_α sentence true in B is also true in A. This is equivalent to there being an embedding of level $\geq \alpha - 1$ from A into an ultrapower of B (see, e.g., [9]).*
(2) *Clearly saying $X \leq_\alpha Y$ is weaker than having X be an image of Y under a map of level $\geq \alpha - 1$.*
(3) *If $X \leq_1 Y$ and Y is connected (resp., discrete with $\leq n$ points), then so is X.*
(4) *In the definition of $X \leq_\alpha Y$ above, the map g may be taken to be of level $\geq \alpha$. For in that case, we may construct an ultracopower triangle over g and use the fact that the class of maps of any fixed level $\geq \beta$ is closed under composition.*

Proposition 3.2.   (1) *The relations $\leq_\alpha$, $1 \leq \alpha \leq \omega$, are transitive.*
(2) *If $2 \leq \alpha \leq \omega$ and $X \leq_\alpha Y$, then $Y \leq_{\alpha-1} X$.*

Proof. *Ad* (1): Suppose $X \leq_\alpha Y \leq_\alpha Z$; say we have initial wedges $X \overset{f}{\leftarrow} U \overset{g}{\to} Y$ and $Y \overset{h}{\leftarrow} V \overset{j}{\to} Z$, where f and h are maps of level $\geq \alpha - 1$, and g and j are maps of level $\geq \omega$. Using Theorem 2.2(1), we have a commutator $U \overset{r}{\leftarrow} W \overset{s}{\to} V$ for the terminal wedge $U \overset{g}{\to} Y \overset{h}{\leftarrow} V$, where r and s are maps of levels $\geq \alpha - 1$ and $\geq \omega$, respectively. Because mapping composition preserves level (Proposition 2.5 in [5]), $f \circ r$ and $j \circ s$ now witness that $X \leq_\alpha Z$ holds.

Ad (2): Suppose $X \leq_\alpha Y$; say we have the initial wedge $X \overset{f}{\leftarrow} U \overset{g}{\to} Y$, where f and g are maps of levels $\geq \alpha - 1$ and $\geq \omega$, respectively. Since $\alpha - 1 \geq 1$, we have an initial wedge $X \overset{k}{\leftarrow} Z \overset{h}{\to} U$, where k is a map of level $\geq \omega$, h is a map of level $\geq \alpha - 2$, and $k = f \circ h$. Then the initial wedge $X \overset{k}{\leftarrow} Z \overset{g \circ h}{\to} Y$ witnesses the fact that $Y \leq_{\alpha-1} X$. $\square$

We now define two compacta X and Z to be α-**equivalent**, $1 \leq \alpha \leq \omega$ (in symbols, $X \equiv_\alpha Y$), if each is α-dominated by the another. By Proposition 3.2(1), $\equiv_\alpha$ is a genuine equivalence relation; by Proposition 3.2(2), ω-dominance and ω-equivalence are the same relation.

Remark 3.3. *In* [1] *the notion of co-elementary equivalence was introduced: two compacta X and Y are* **co-elementarily equivalent** *if an ultracopower of one is homeomorphic to an ultracopower of the other. This is the topological version of elementary equivalence in model theory, thanks to the ultrapower theorem of Keisler and Shelah (see* [9]*). Indeed, two Boolean spaces are co-elementarily equivalent if and only if their Boolean lattices of closed-open sets are elementarily equivalent;*

moreover, two compacta are co-elementarily equivalent if some lattice base of one is elementarily equivalent to some lattice base of the other. It is not hard to show that co-elementary equivalence and ω-equivalence are the same relation. [Given a witness $X \xleftarrow{f} Z \xrightarrow{g} Y$ for the ω-equivalence of X and Y; i.e., both f and g are co-elementary maps, use the definition, plus the fact that compositions of co-elementary maps are co-elementary, to justify the assertion that we may take g to be a standard ultracopower projection map. Now apply the definition of co-elementary map *to f, and use the fact (see [1]) that an ultracopower of an ultracopower of a compactum is itself an ultracopower of that compactum.]*

Proposition 3.4. (1) *Any continuum is 1-dominated by any co-existentially closed continuum.*

(2) *Any two co-existentially closed continua are 2-equivalent.*

(3) *Any continuous map from one co-existentially closed continuum onto another is a map of level ≥ 2.*

(4) *Any two co-existentially closed continua that are continuous images of each other are 3-equivalent.*

Proof. Ad (1): Given continua X and Y, where Y is co-existentially closed, we let $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ be the standard projection maps from the topological product, also a continuum. Thus q is a map of level ≥ 1. By Remark 3.1 (4), this suffices to conclude that $X \leq_1 Y$.

Ad (2): Given co-existentially closed continua X and Y, we consider again the initial wedge $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. We infer $X \leq_2 Y$ using the facts that q is a map of level ≥ 1 and X is a co-existentially closed continuum. By the symmetry of the situation, $Y \leq_2 X$.

Ad (3): Let $f : X \to Y$ be a continuous surjection, where both X and Y are co-existentially closed continua. Then, because Y is co-existentially closed, we have a continuum Z, a map $g : Z \to Y$, of level $\geq \omega$, and a map $h : Z \to X$, of level ≥ 0, such that $f \circ h = g$. Because X is co-existentially closed, however, h is actually of level ≥ 1; hence f is a map of level ≥ 2.

Ad (4): This follows quickly from 3.4 (3) above. If $f : Y \to X$ is a continuous surjection between co-existentially closed continua, then f is a map of level ≥ 2; hence $X \leq_3 Y$. □

Remark 3.5. *Two compacta that are 2-equivalent either both share or both fail to share any topological property of compacta that is preserved by both ultracopowers and images of maps of level ≥ 1. Indeed, if $X \leq_2 Y$ and Y is a Boolean space (resp., an indecomposable continuum, a hereditarily indecomposable continuum, a compactum of covering dimension $\leq n$, a continuum of multicoherence degree $\leq n$), then so is X (see [5, 6, 7, 8]). It is possible for X to be an indecomposable continuum when Y is a decomposable one; in fact, maps of level ≥ 1 do not preserve decomposability in continua. However, if $X \equiv_2 Y$, then one is a decomposable continuum if and only if the other is.*

If we are willing to restrict our domain of discourse to **Peano compacta** (i.e., compacta that are both metrizable and locally connected), then we can also add

being an arc (or a simple closed curve) to the list of properties mentioned in Remark 3.5. The following strengthens the main result (Theorem 0.6) of [2].

Theorem 3.6. *Let X be a Peano compactum.*
  (1) *If X is 2-dominated by an arc, then X is an arc.*
  (2) *If X is 2-dominated by a simple closed curve, then X is a simple closed curve.*

Proof. Ad (1): Suppose $X \leq_2 Y$, where Y is an arc. Then there is an ultrafilter $\mathcal{D}$ and a co-existential map $f : YI\backslash\mathcal{D} \to X$ witnessing this. We assume, for the sake of obtaining a *reductio ad absurdum*, that X is a Peano compactum that is not an arc.

We begin by noting that, since $YI\backslash\mathcal{D}$ is a continuum, so is X. Next we cite Theorem 1.1(4) twice: first to infer that $YI\backslash\mathcal{D}$ is **unicoherent** (i.e., incapable of decomposition into a union of two subcontinua with disconnected intersection); second to infer that X is unicoherent as well.

Now we cite another classic theorem of R. L. Moore (see [18], also Exercise 8.40 in [19]), used extensively in [2], to the effect that *any Peano continuum that is neither an arc nor a simple closed curve must contain a **simple triod*** (i.e., a homeomorphic copy of the letter T, the cone over a three-point discrete space). Since X is a unicoherent continuum, it cannot be a simple closed curve; hence, by the above-cited theorem of Moore, it contains a simple triod.

Using the local connectedness of X, we argue as in Lemma 1.3 in [2] to construct subcontinua K, L_1, L_2, L_3 of X such that:
  (i) K intersects each L_j, $1 \leq j \leq 3$;
  (ii) all of $K \setminus (L_1 \cup L_2 \cup L_3)$, $L_j \setminus K$, $1 \leq j \leq 3$, have nonempty interiors; and
  (ii) the subcontinua L_1, L_2, L_3 are pairwise disjoint.

In the terminology of [2], the collection $\{K, L_1, L_2, L_3\}$ constitutes a **fat 3-wheel** in X (the word *fat* referring to the various nonempty interiors). Since (by Theorem 1.1(1)) f is monotone, the inverse images of these subcontinua constitute a fat 3-wheel in $YI\backslash\mathcal{D}$.

The point of making sure sets have nonempty interiors (rather than being merely nonempty) is to enable the construction (see the proof of Lemma 1.3 in [2]) of a 3-wheel $\{\sum_\mathcal{D} K_i, \sum_\mathcal{D} L_{1,i}, \sum_\mathcal{D} L_{2,i}, \sum_\mathcal{D} L_{3,i}\}$, consisting of ultracoproduct subcontinua, where $\sum_\mathcal{D} K_i \supseteq f^{-1}[K]$, etc. Then, for almost every index i (modulo $\mathcal{D}$), $\{K_i, L_{1,i}, L_{2,i}, L_{3,i}\}$ constitutes a 3-wheel in the arc Y. This is impossible; hence X must indeed be an arc.

Ad (2): Retaining the notation of the argument above, but taking Y now to be a simple closed curve, assume X is a Peano compactum 2-dominated by Y. Then X is a Peano continuum. And, because simple closed curves cannot contain 3-wheels, X is either an arc or a simple closed curve. Now suppose X is an arc. Then X contains a fat 2-wheel $\{K, L_1, L_2\}$ (like a fat 3-wheel, but with one less "spoke") such that $X = K \cup L_1 \cup L_2$. Arguing as above, we may cover $YI\backslash\mathcal{D}$ with a 2-wheel $\{\sum_\mathcal{D} K_i, \sum_\mathcal{D} L_{1,i}, \sum_\mathcal{D} L_{2,i}\}$. This gives rise to the existence of a 2-wheel cover of the simple closed curve Y, an impossibility.   □

Remark 3.7. *The condition of local connectedness cannot be removed from Theorem 3.6. This follows from Theorem 2.10 in [3]: Every infinite compactum is ω-equivalent to a compactum of the same weight, which is not locally connected.*

With the aid of Theorem 3.6(1), we can immediately obtain a variation on Theorem 2.6 by weakening the hypothesis that all spaces involved are arcs and strengthening the hypothesis that all maps involved are of level ≥ 1.

Corollary 3.8. *Let* $\mathcal{V} := \langle X_i \xrightarrow{f_i} Z : i \in I \rangle$ *be a generalized terminal wedge consisting of Peano compacta, where Z is an arc and the f_i are maps of level ≥ 2. Let $\langle X_i \xleftarrow{r_i} W : i \in I \rangle$ be the associated fiber product.*

(1) *If $\mathcal{V}$ is not jointly injective, then W a continuum that is not an arc and no projection r_i is a map of level ≥ 2.*
(2) *If $\mathcal{V}$ is jointly injective and the index set I is at most countable, then W is an arc and each projection r_i is a map of level $\geq \omega$.*

Proof. In light of Theorem 2.6, all we need to do is show that the spaces X_i are arcs. But if the arc Z is a level ≥ 2 image of the Peano compactum X_i, then $Z \leq_3 X_i$; hence, by Proposition 3.2(2), $X_i \leq_2 Z$. Now apply Theorem 3.6(1). $\square$

There is also an analogue of Theorem 3.6 for the pseudo-arc. Recall that a compactum X is **arc-like** if for any open cover $\mathcal{U}$ of X, there exists a continuous map from X onto an arc such that each point-inverse under that map is contained in a member of $\mathcal{U}$. Of course every arc-like compactum is connected; moreover it has covering dimension one.

Theorem 3.9. *Let X be a metrizable arc-like compactum that is 2-dominated by a pseudo-arc. Then X is a pseudo-arc.*

Proof. Suppose $X \leq_2 Y$, where Y is a pseudo-arc. Then there is an ultrafilter $\mathcal{D}$ and a co-existential map $f : YI\backslash\mathcal{D} \to X$, as per the definition. By Corollary 4.10 in [8], $YI\backslash\mathcal{D}$ is a hereditarily indecomposable continuum because Y is. By Theorem 1.1(2) above, then, X is now a hereditarily indecomposable metrizable continuum that is also arc-like. This (see [14]) characterizes X as a pseudo-arc. $\square$

Open Questions 3.10. (1) *Can we remove the assumption of being arc-like from Theorem 3.9? [The answer is no if the pseudo-arc is a co-existentially closed continuum. For, by Theorem 1.2(4), we may choose X to be a co-existentially closed continuum that is not arc-like. If Y is a pseudo-arc, assumed to be co-existentially closed, then Proposition 3.4(2) tells us that $X \leq_2 Y$.]*

(2) *Is the image of a pseudo-arc under a map of level ≥ 1 again a pseudo-arc? [The nondegenerate image of a pseudo-arc under a continuous map is a pseudo-arc if the map is either open or monotone; it is still an open question whether the map may be taken to be merely confluent (see Theorem 4.15 and Question 4.17 in [14]).]*

(3) *How large a family of pairwise non–3-equivalent co-existentially closed continua is it possible to have? [This number could conceivably be the power of the continuum. On the other hand, if the class of co-existentially closed continua is co-elementary (even closed under ultracopowers), then all of its members are ω-equivalent to one another (as in the case with the class of co-existentially closed compacta).]*

4. INVERSE LIMITS

Theorem 1.2(2) above states that the class of co-existentially closed continua is
closed under images of maps of level ≥ 1; here we show the class to be closed
also under limits of inverse systems in which the bonding maps are continuous
surjections. We first recall an earlier result.

Lemma 4.1 (Lemma 3.1 in [5]). *Let $\alpha \leq \omega$, let $h : Y \to X$ be a function between
compacta, and let $\mathcal{A}$ be a lattice base for X. Suppose that for each finite $\delta \subseteq \mathcal{A}$
there is a map $g_\delta : X \to Z_\delta$, of level $\geq \alpha$, such that:*

  (i) *$g_\delta \circ h$ is a map of level $\geq \alpha$; and*
  (ii) *for each $A \in \delta$, $g_\delta^{-1}[g_\delta[A]] = A$.*

Then h is a map of level $\geq \alpha$.

Theorem 4.2. *Let $\alpha \leq \omega$, and let $\langle I, \leq \rangle$ be a directed set, with $\langle X_i, f_{ij} : i \leq j \in I \rangle$
an inverse system of compacta and continuous bonding maps. Suppose further that
there is a compactum Y and, for each $i \in I$, a map $h_i : Y \to X_i$ of level $\geq \alpha$ such
that $f_{ij} \circ h_j = h_i$, for $i \leq j$ in I. If X is the limit of the system, with projection
maps $g_i : X \to X_i$, $i \in I$, and if $h : Y \to X$ is the limit of the maps h_i (i.e.,
$g_i \circ h = h_i$, $i \in I$), then h is a map of level $\geq \alpha$.*

Proof. For any topological space Z, define $F(Z)$ to be the bounded lattice of closed
subsets of Z. $F(\ )$ is a contravariant functor, converting continuous surjections to
lattice embeddings. So, applying $F(\ )$ to the inverse system in question, we obtain
a directed system $\langle F(X_i), f_{ij}^F : i \leq j \in I \rangle$ of closed-set lattices and lattice homo-
morphisms. Each h_i is a continuous surjection; hence so is each f_{ij}. Consequently
the functions f_{ij}^F are lattice embeddings. Let A be the limit of this directed system
of lattices. Then (see, e.g., [5]) A is isomorphic to a lattice base $\mathcal{A}$ for the inverse
limit X above, so we may view the two lattices as the same . For each $i \in I$, let
$r_i : F(X_i) \to \mathcal{A}$ be the limit embedding. for each finite $\delta \subseteq \mathcal{A}$, fix $i_\delta \in I$ so that
r_{i_δ} includes δ in its image. Then the maps $g_{i_\delta} : X \to X_{i_\delta}$ satisfy condition (ii) of
Lemma 4.1; they also satisfy condition (i) because $g_{i_\delta} \circ h = h_{i_\delta}$. Thus h is a map
of level $\geq \alpha$. $\square$

Corollary 4.3. *The class of co-existentially closed continua is closed under limits
of inverse systems with surjective bonding maps.*

Proof. Let $\langle I, \leq \rangle$ be a directed set, with $\langle X_i, f_{ij} : i \leq j \in I \rangle$ an inverse system of
co-existentially closed continua and continuous surjective bonding maps. If X is
the limit of the system, with projection maps $g_i : X \to X_i$, $i \in I$, and if $h : Y \to X$
is a surjective map between continua, let $h_i := g_i \circ h$, $i \in I$. Then each h_i is a map
of level ≥ 1 because each X_i is co-existentially closed. By Theorem 4.2, h is of level
≥ 1 as well; hence X is a co-existentially closed continuum. $\square$

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DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, MARQUETTE UNIVERSITY, MILWAUKEE, WI 53201-1881

E-mail address: `paulb@mscs.mu.edu`