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Quotient theorems via ultracoproducts

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ABSTRACT

The following quotient theorem is well known: *Every (not necessarily metrizable) continuum X is a weakly confluent image of a continuum Y which is hereditarily indecomposable and of covering dimension one.* Here we use the topological ultracoproduct construction to modify the result to say that for any prescribed infinite cardinal α, we may arrange for Y to have at least α composants. Furthermore, if the generalized continuum hypothesis (GCH) is assumed, we may add that Y has as many composants as it has points. Another ultracoproduct argument allows us to conclude that each nondegenerate continuum X is a co-elementary image of a continuum Y with at least two nonblock points. And if we invoke GCH, we may arrange for Y to be spanned by its set of nonblock points. We also pay attention to the "size" of the continuum Y, relative to that of X (and α).

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1. Introduction

In the present context, continua are connected compact Hausdorff spaces that are not necessarily metrizable. By a *quotient theorem for continua* we mean a statement of the rough form: *Every continuum X (with restrictions possible) is a "special image" of a "special continuum" Y.*

Quotient theorems are dual (in the category-theoretic sense) to embedding theorems; here are some classical examples. (Please refer to Section 2 for specialized terminology.)

Theorem 1.1.

(1) *[19, Theorem 2] Every continuum X is a retractive image of a continuum Y which is indecomposable and has ≥ 2 composants.*
(2) *[30, Theorem 2 and Corollary 2.1] For a given infinite cardinal α, every continuum X is a retractive image of a continuum Y which is indecomposable and has $\geq \alpha$ composants.*
(3) *[20, Theorem 4.4] Every continuum X is a weakly confluent image of a continuum Y which is hereditarily indecomposable, has dimension one, and has weight equal to the weight $w(X)$ of X.*

This article presents four new quotient results; i.e., Theorems 1.2, 1.4, 1.8, and 1.9. The first two of these address the composant structure of continuum Y in item (3) above.

Theorem 1.2. *For a given infinite cardinal α, every continuum X is a weakly confluent image of a continuum Y which is hereditarily indecomposable, has dimension one, has $\geq 2^{\alpha}$ composants, and has weight $\leq 2^{(\alpha \cdot 2^{w(X)})}$.*

Remarks 1.3.

(1) A metrizable version of Theorem 1.1 (1) was first proved by Bellamy in [12]. Bellamy's pre-image Y is metrizable too–i.e., $w(Y) = w(X)$-however, the version given by Gordh in [19] does not specify $w(Y)$. A cursory analysis of his construction yields $w(Y) \leq |X|$, but Smith [33] has recently been able to tweak the argument to obtain $w(Y) = w(X)$: where a well ordering of the points of X was originally used to create a "long line" as part of the construction, it is possible to well order a dense subset of X, of cardinality $w(X)$, to the same end.
(2) In Smith's construction proving Theorem 1.1 (2), an inverse limit involving the initial ordinal α as an index set is used, and Y actually has $\geq 2^{\alpha}$ composants. Moreover, an analysis of the construction reveals that $w(Y) \leq \alpha \cdot w(X)$ (which, in ZFC, is just $\max\{\alpha, w(X)\}$).
(3) Theorems 1.1 (1, 2) may also be viewed as embedding theorems, as any retractive map entails an embedding in the opposite direction. Therefore it is not possible to place a blanket restriction on the dimension of the pre-image (*à la* Theorem 1.1 (3)) because the dimension of Y dominates that of (its closed subspace) X.
(4) The metrizable instance of Theorem 1.1 (3) was first proved by Maćkowiak and Tymchatyn in [25].
(5) The weakly confluent quotient map in Theorem 1.1 (3) cannot be upgraded to be retractive or confluent, as both kinds of mapping preserve hereditary indecomposability. If we restrict X to be hereditarily indecomposable, though, the map we get from Theorem 1.1 (3) is automatically confluent because *all* mappings from continua onto hereditarily indecomposable continua are confluent [18]. The quotient cannot be retractive, even in this case, because of item (3) above.

Our second quotient theorem presents a version of Theorem 1.1 (3) in which the number of composants of Y is large, relative to the size of Y itself. Decomposable continua have at most three composants; hence only indecomposable continua have any chance at possessing many. Since the composants of an indecomposable continuum form a partition, this suggests defining a continuum Y to be **fine** (resp., w**-fine**) if the number of its composants is the cardinality $|Y|$ of Y (resp., $\geq w(Y)$). For example, in Theorem 1.2, with $\alpha \geq 2^{w(X)}$, we see that Y is w-fine. Also, in Theorem 1.1 (2) (see Remark 1.3 (2)), with $\alpha \geq w(X)$, the continuum Y is of weight $\leq \alpha$ and has $\geq 2^{\alpha}$ composants. Thus Y is fine.

Nondegenerate metrizable continua have weight ω and cardinality $\mathfrak{c} := 2^{\omega}$. Janiszewski and Kuratowski used a Baire category argument to prove–in the premier issue of *Fundamenta Mathematicae* [21]–that nondegenerate indecomposable metrizable continua have uncountably many composants, and are *a fortiori* w-fine. A few years later, Mazurkiewicz [26] showed that the number of composants is actually $\mathfrak{c}$; hence we know that, for nondegenerate metrizable continua, the properties of being fine, being w-fine, and being indecomposable are equivalent.

On the other hand it is possible for an indecomposable nonmetrizable continuum to have just one or just two composants [13,31]. In addition, while it is unknown whether a nondegenerate hereditarily indecomposable continuum can have just one composant, it can have just two [32]. In view of this, we may safely add that Y is fine in the conclusion of Theorem 1.1 (3) in the metrizable case only.

The argument we use to prove Theorem 1.2 is inadequate for producing a continuum Y that is fine. In order to ensure this property, a very different approach–one involving the generalized continuum hypothesis (GCH)–seems to be required. And even with this powerful axiom we cannot add fineness of the pre-image to the conclusion of Theorem 1.1 (3) without a weight adjustment.

This brings us to the second of our quotient theorems.

Theorem 1.4. (GCH) *For a given infinite cardinal α, every continuum X is a weakly confluent image of a continuum Y which is hereditarily indecomposable, has dimension one, is fine, and has weight $2^{(\alpha \cdot w(X))}$.*

Remark 1.5. Fineness in a nonmetrizable continuum can be axiom-sensitive: Consider the Stone-Čech remainder $\mathbb{H}^*$ of the ray $\mathbb{H} := [0, \infty)$. This continuum is well known to be indecomposable, with weight $\mathfrak{c}$ and cardinality $2^{\mathfrak{c}}$. If we assume the continuum hypothesis (CH) $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ composants, and is therefore fine. [29]. However, under the near coherence of filters axiom (NCF, also equiconsistent with ZFC), there is but one composant [27]. In this set-theoretic universe, $\mathbb{H}^*$ is very "coarse" indeed. (See also [15, Theorem 4.1].)

Before leaving the topic of Theorem 1.1 and its possible variants, there is another classical quotient theorem, related to Theorem 1.1 (2), but with the number of composants of the pre-image continuum being small rather than large. We include it here for its intrinsic interest.

Theorem 1.6. *[31, Main Theorem] Every continuum X is a retractive image of a continuum Y which is indecomposable and has exactly one composant (or exactly two composants).*

Remark 1.7. The metrizable instance of Theorem 1.6 was first proved by Bellamy in [13] (see the Corollary after Lemma 2). There is no mention of $w(Y)$ in that paper or in the general version given by Smith in [31]; however, Professor Smith [33] has since made a more careful analysis of the ω_1-indexed inverse limit constructions used in both papers to conclude that $w(Y) = \omega_1 \cdot w(X)$.

Our last two quotient theorems concern relative composants. Call a continuum X *totally unblocked* (resp., *point-unblocked*) if for each proper (resp., degenerate) subcontinuum $K \subseteq X$ there is a point $a \in X$ such that the *relative composant* $\kappa(K; a)$, namely the union of all subcontinua of X containing K and missing a, is dense in X.

Theorem 1.8. *Every nondegenerate continuum X is a co-elementary image of a continuum Y which is point-unblocked and of weight $\leq 2^{(2^{w(X)})}$.*

As our final quotient theorem, we strengthen the conclusion above by again assuming GCH.

Theorem 1.9. (GCH) *Every nondegenerate continuum X is a co-elementary image of a continuum Y which is totally unblocked and of weight $2^{w(X)}$.*

Bing showed [14, Theorem 5] that all metrizable continua are totally unblocked; however even being point-unblocked in general is consistently false [2, Theorem 4.8]. If the point $a \in X$ is such that $\kappa(K; a)$ is dense for some subcontinuum K, then a is called a *nonblock* point. Nonblock points are "at the edge" of a continuum; in particular they are special noncut points (see [1–3,11,16,24]). If X is totally unblocked, then it is clearly *spanned* by its set of nonblock points; i.e., no proper subcontinuum of X contains all these points. If X is nondegenerate and merely point-unblocked, then it has at least two nonblock points: first arbitrarily pick $x \in X$; then let $a \in X$ be such that $\kappa(\{x\}; a)$ is dense in X; finally let $b \in X$ be such that $\kappa(\{a\}; b)$ is dense. Then a and b are distinct nonblock points of X.

Thus we have some immediate corollaries of Theorems 1.8 and 1.9.

Corollary 1.10.

(1) *Every nondegenerate continuum X is a co-elementary image of a continuum Y which has at least two nonblock points.*
(2) (GCH) *Every nondegenerate continuum X is a co-elementary image of a continuum Y which is spanned by its set of nonblock points.*

Remarks 1.11.

(1) If we take X above to be locally connected, then the co-elementary map is also monotone [7, Proposition 2.1]. Co-elementary maps need not be monotone in general–or even confluent [10]–but they are always weakly confluent.
(2) As usually stated, the noncut point existence theorem asserts that each nondegenerate continuum contains at least two noncut points. However, as an immediate consequence of the standard proof [35], the set of noncut points of a continuum is always a spanning set.
(3) The results 1.8, 1.9, and 1.10 are not true for X a degenerate continuum, as co-elementary maps preserve nondegeneracy.

The main tool in producing our continua Y in the theorems above is the topological ultracoproduct construction. We make explicit use of regular ultrafilters in proving Theorems 1.2 and 1.8; and in Theorems 1.4 and 1.9, good ultrafilters are featured. Also, in Theorems 1.2 and 1.8, we use classical results from model theory; namely the Löweinheim-Skolem theorem and the Keisler-Shelah isomorphism theorem.

The method of proof of Theorem 1.1 (1) uses a clever *ad hoc* construction; that of (2) involves inverse limits via an uncountable directed index set. This is also true of the proof of Theorem 1.6, but the proof of Theorem 1.1 (3) uses a mix of general topology and model theory. Our ultracoproduct method generally makes things bigger; hence it is useless as a method for proving results like Theorem 1.6.

Questions 1.12.

(1) Is there a version of Theorem 1.1 (3) where the weight of the hereditarily indecomposable pre-image Y is (say) $\omega_1 \cdot w(X)$ and the number of composants of Y is guaranteed to be "small"?
(2) Is it possible to obtain smaller values for $w(Y)$ in Theorems 1.4, 1.8 and 1.9?
(3) Can one prove (without recourse to GCH) the existence of nonmetrizable fine continua that are *hereditarily* indecomposable?

2. Preliminaries

Our background set theory is ZFC, Zermelo-Fraenkel set theory plus the axiom of choice. In this setting it makes sense to speak of *the cardinality* $|X|$ of a set X.

Infinite ordinals are indicated using lower-case Greek letters; as usual, cardinals are initial ordinals (i.e., not equinumerous with any smaller ordinal). The smallest infinite ordinal is $\omega := \{0, 1, 2, \dots\}$, also a cardinal. Successive cardinals are denoted ω_1, ω_2, etc. If α is an infinite cardinal, 2^α denotes the cardinality of the power set $\wp(S)$ of any set S of cardinality α; on the other hand, α^+ is the cardinal successor of α.

A fundamental truth is that the inequality $\alpha^+ \leq 2^\alpha$ always holds, but that the corresponding equality may fail. We use the symbol GCH_α to express this equality. As usual, the *continuum hypothesis* (CH) is the statement GCH_ω; and the *generalized continuum hypothesis* (GCH) is the statement that asserts GCH_α for all infinite cardinals α. Each GCH_α is called an *instance of* GCH.

For a topological space X, $w(X)$ denotes its **weight**; i.e., the least infinite cardinal λ such that X has an open-set (or closed-set) base of cardinality $\leq \lambda$.

We use the term **compactum** to refer to a nonempty compact Hausdorff topological space; a compactum is a **continuum** if it is also connected. A (proper) **subcompactum/subcontinuum** of a space X is a (proper) subset which is a compactum/continuum in its subspace topology. A topological space with just one point is termed **degenerate**; it is **nondegenerate** if it has at least two points.

A subset S of a continuum X **spans** X (or, X is **irreducible about** S) if no proper subcontinuum of X contains S. A continuum is **irreducible** if it has a two-point spanning set.

A continuous map $f : Y \to X$ between compacta is **retractive** if there is a continuous $g : X \to Y$ such that $f \circ g$ is the identity map on X. The map f is **weakly confluent** if each subcontinuum of X is the image under f of a subcontinuum of Y. Clearly retractive maps are weakly confluent, but the converse is easily shown to be false. As for co-elementary maps, these are "dual" to elementary embeddings in model theory and will be defined in the next section. Suffice it to say that co-elementary maps are weakly confluent maps (but not retractive in general) that preserve a host of properties of compacta.

Throughout this paper, *dimension* always refers to covering dimension, defined as follows. Given $n \in \omega$, a compactum X is of **dimension** $\leq n$ if for each open cover $\mathcal{U}$ of X, there is a finite open cover $\mathcal{V}$ that refines $\mathcal{U}$, and is such that each point of X lies in $\leq n + 1$ sets in $\mathcal{V}$. If compactum X is of dimension $\leq n$ for some $n \in \omega$, then the least such n is the **dimension** of X. Having dimension zero is equivalent to being a Boolean space; for a nondegenerate continuum X, having dimension ≤ 1 is equivalent to being of dimension one.

A continuum is **decomposable** if it is the union of two of its proper subcontinua; **indecomposable** otherwise. The continuum is **hereditarily indecomposable** if each of its subcontinua is indecomposable; equivalently if any two of its subcontinua are either disjoint or setwise comparable.

Given a proper subcontinuum A of continuum X and $S \subseteq X \setminus A$, the **composant at** A **relative to** S is the union

$$\kappa(A; S) := \bigcup \{K : K \text{ is a proper subcontinuum of } X \text{ and } A \subseteq K \subseteq X \setminus S\}.$$

The **composant** at a proper subcontinuum A is $\kappa(A) := \kappa(A; \emptyset)$. As a consequence of boundary bumping [23, Theorem §47,III,4], the composant at any proper subcontinuum of X is dense in X. A **composant** of X is a composant at a point of X. Note that a continuum is irreducible if and only if it has at least two composants.

Clearly, if $A \subseteq B$ are subcontinua, with B proper, then $\kappa(A; S) \supseteq \kappa(B; S)$ for any $S \subseteq X \setminus B$. Hence the composant at a proper subcontinuum is contained in the composant at any point of the subcontinuum.

If X is an indecomposable continuum and A is a proper subcontinuum, then $\kappa(A) = \kappa(a)$ for any $a \in A$. (We drop curly brackets when either A or S is a singleton.) In this case the composants partition the continuum into dense subsets; the number of these subsets in nondegenerate metrizable continua is the cardinality $\mathfrak{c} := 2^{\omega}$ of the reals [26, Théorème 1]. In indecomposable continua of weight as "little" as ω_1, however, the number of composants can be one or two (see [13,31,32]).

When $\kappa(A; a)$ fails to be dense in X, we say a **blocks** A. If a blocks every subcontinuum of $X \setminus \{a\}$, then a is a **block point** of X; otherwise a is a **nonblock point**.

If each $a \in X \setminus A$ blocks A, we say that A is **blocked**; otherwise A is said to be **unblocked**.[2] A continuum X is **totally (un)blocked** if each of its proper subcontinua is (un)blocked. The continuum is **point-unblocked** if each of its degenerate subcontinua is unblocked.

An indecomposable continuum with at least two composants is totally unblocked, as any proper subcontinuum must lie in one of the composants and no point from a different composant can block it. As mentioned in the paragraph following Theorem 1.9 above, all metrizable continua are totally unblocked; however this condition is consistently false [2, Theorem 4.8] in general: under NCF, the Stone-Čech remainder $\mathbb{H}^*$, while being indecomposable, has but one composant and is also totally blocked. (On the other hand, a theorem of Rudin says that, under CH, $\mathbb{H}^*$ has $2^{\mathfrak{c}}$ composants–see [29], also [34, Theorem 9.23])–and is *a fortiori* totally unblocked.)

3. The ultraproduct and ultracoproduct constructions

We employ the notation and terminology of [11] and related papers. Given a space X, we denote by $F(X)$ its bounded lattice of closed subsets, and by $B(X)$ the sublattice of clopen subsets. A bounded sublattice $\mathcal{A}$ of $F(X)$ is a **lattice base** for X if each member of $F(X)$ is an intersection of sets in $\mathcal{A}$. So a compactum X is a Boolean space precisely when $B(X)$ is a lattice base, and is a continuum precisely when $B(X) = \{\emptyset, X\}$.

The topological ultraproduct/ultracoproduct constructions give us an important source of nonmetrizable topological spaces; they also provide an avenue for bringing model-theoretic methods into topology.

Definition 3.1.

(1) (Ultraproducts of Sets): Start with an infinite index set I (always assumed to be topologically discrete) and let $\vec{X} = \langle X_i : i \in I\rangle$ be an I-sequence of sets, with $\mathcal{D}$ an ultrafilter on I. A typical element of the Cartesian product of this I-sequence is itself an I-sequence of the form $\vec{x} := \langle x_i : i \in I\rangle$, where $x_i \in X_i$ for each i. If $\vec{x}, \vec{y}$ are two such I-sequences, we write $\vec{x} \sim_{\mathcal{D}} \vec{y}$ to mean that $\{i \in I : x_i = y_i\} \in \mathcal{D}$. The $\sim_{\mathcal{D}}$-equivalence class containing $\langle x_i : i \in I\rangle$ is denoted $\vec{x}^{\mathcal{D}}$ (or $\prod_{\mathcal{D}} x_i$); the **$\mathcal{D}$-ultraproduct**, denoted $\vec{X}^{\mathcal{D}}$ (or $\prod_{\mathcal{D}} X_i$), comprises these equivalence classes. If $A_i \subseteq X_i$ for each $i \in I$, then $\vec{A}^{\mathcal{D}} = \prod_{\mathcal{D}} A_i$ has its obvious meaning, and is commonly referred to as an **ultrabox** in $\vec{X}^{\mathcal{D}}$. We thus identify $\vec{x}^{\mathcal{D}}$ with $\prod_{\mathcal{D}}\{x_i\}$.

(2) (Ultraproducts of Spaces): If each X_i is a topological space and $\vec{A} = \langle A_i : i \in I\rangle$ is an I-sequence of sets where each A_i is open/closed in X_i, then $\vec{A}^{\mathcal{D}}$ is called an *open/closed ultrabox*. The **topological $\mathcal{D}$-ultraproduct**–also denoted $\vec{X}^{\mathcal{D}}$–has $\prod_{\mathcal{D}} X_i$ for underlying set and all open ultraboxes for an open-set base; equivalently, all closed ultraboxes for a closed-set base. The assignment $\vec{x} \mapsto \vec{x}^{\mathcal{D}}$ is a continuous open map from the Cartesian product–endowed with the box topology–onto the topological $\mathcal{D}$-ultraproduct.

(3) (Ultracoproducts of Compacta): Let each X_i now be a compactum, with $\mathcal{A}_i$ a lattice base for X_i. Being a lattice base for a compactum is a first-order property of lattices (see [8]), and hence the Łoś ultraproduct theorem [17] tells us that the ultraproduct lattice $\prod_{\mathcal{D}} \mathcal{A}_i$ is a lattice base for a compactum X. Following the well-known Wallman construction, points of X are maximal filters of the lattice $\prod_{\mathcal{D}} \mathcal{A}_i$. For each ultrabox $\prod_{\mathcal{D}} S_i$ in $\prod_{\mathcal{D}} X_i$, let $(\prod_{\mathcal{D}} S_i)^{\sharp}$ be the set of $\mu \in X$ such that some member of μ is contained in $\prod_{\mathcal{D}} S_i$. Then a typical basic closed set for X is of the form $(\prod_{\mathcal{D}} A_i)^{\sharp}$, where $\prod_{\mathcal{D}} A_i \in \prod_{\mathcal{D}} \mathcal{A}_i$. ($\prod_{\mathcal{D}} \mathcal{A}_i$ is naturally identified with ultraboxes taken from the respective lattice bases.) Each inclusion

[2] *Unblocked* is also called *coastal* in [1,2,11].

map $\mathcal{A}_i \subseteq F(X_i)$ induces a homeomorphism between the Wallman constructions based on $\langle \mathcal{A}_i : i \in I\rangle$ and $\langle F(X_i) : i \in I\rangle$, respectively. Hence the construction is independent of choice of constituent lattice bases, and we now refer to it as the **topological $\mathcal{D}$-ultracoproduct** $\vec{X}_\mathcal{D} = \sum_\mathcal{D} X_i$ of the I-sequence $\vec{X}$. For any $\vec{x}^\mathcal{D} \in \vec{X}^\mathcal{D}$, $(\vec{x}^\mathcal{D})^\sharp$ consists of exactly one point, which we denote $\vec{x}_\mathcal{D} = \sum_\mathcal{D} x_i$. The assignment $\vec{x}^\mathcal{D} \mapsto \vec{x}_\mathcal{D}$ defines a topological embedding from $\vec{X}^\mathcal{D}$ onto a dense subset of $\vec{X}_\mathcal{D}$. In this way we may view $\vec{X}^\mathcal{D}$ as a dense subspace of $\vec{X}_\mathcal{D}$ (or $\vec{X}_\mathcal{D}$ as a compactification of $\vec{X}^\mathcal{D}$). Finally, when each closed $F_i \subseteq X_i$ is regarded as a compactum, the closed set $(\prod_\mathcal{D} F_i)^\sharp$, as a subspace, is naturally homeomorphic to the ultracoproduct $\sum_\mathcal{D} F_i$.

When each X_i is the same compactum X, then $\prod_\mathcal{D} X_i$ and $\sum_\mathcal{D} X_i$ are respectively known as the **$\mathcal{D}$-ultrapower** $X^\mathcal{D}$ and the **$\mathcal{D}$-ultracopower** $X_\mathcal{D}$ of X. In this case there is a continuous surjection $p_\mathcal{D} : X_\mathcal{D} \to X$, known as the **codiagonal map**: for $\mu \in X_\mathcal{D}$, $x = p_\mathcal{D}(\mu)$ if and only if for each open neighborhood U of x in X, we have $\mu \in (U^\mathcal{D})^\sharp$. There is also a **diagonal map** $d_\mathcal{D} : X \to X^\mathcal{D}$, defined by $x \mapsto x^\mathcal{D}$, the single element of $\{x\}^\mathcal{D}$. This function, rarely continuous, is a right-inverse for $p_\mathcal{D}$; i.e., $p_\mathcal{D}(d_\mathcal{D}(x)) = x$, for all $x \in X$. In the context of nonstandard analysis, the restriction of $p_\mathcal{D}$ to $X^\mathcal{D}$ is known as the *standard part* map from $X^\mathcal{D}$ onto X.

A map $f : X \to Y$ is **co-elementary** if there are ultrafilters $\mathcal{D}$ and $\mathcal{E}$, and a homeomorphism $h : X_\mathcal{D} \to Y_\mathcal{E}$ such that $f \circ p_\mathcal{D} = p_\mathcal{E} \circ h$. Prototypical examples of co-elementary maps are the codiagonal maps. Co-elementary maps preserve dimension, (hereditary) decomposability, and (hereditary) indecomposability. With the exception of hereditary decomposability, all these properties are also reflected (see [9]).

Remark 3.2. As the terminology suggests, ultracoproducts and ultraproducts are dual notions from the viewpoint of category theory (see [7,8]): while ultraproducts may be described as "direct limits of products," ultracoproducts may be described dually as "inverse limits of coproducts."

The following is a basic fact about ultracoproducts of compacta.

Lemma 3.3. *[6, Lemma 4.6] Let $\langle X_i : i \in I\rangle$ be an I-sequence of compacta, with $\mathcal{D}$ an ultrafilter on I. Then: $B(\sum_\mathcal{D} X_i)$ is isomorphic to $\prod_\mathcal{D} B(X_i)$. Consequently:*

(1) *$\sum_\mathcal{D} X_i$ is a Boolean space if and only if*

$$\{i \in I : X_i \text{ is a Boolean space}\} \in \mathcal{D}.$$

(2) *$\sum_\mathcal{D} X_i$ is a continuum if and only if*

$$\{i \in I : X_i \text{ is a continuum}\} \in \mathcal{D}.$$

While many properties of compacta/continua fail to carry over to ultracoproducts–e.g., metrizability, number of composants–there are some key properties that do.

Lemma 3.4. *Let $\langle X_i : i \in I\rangle$ be an I-sequence of compacta, with $\mathcal{D}$ an ultrafilter on I and $n \in \omega$. Then:*

(1) *[9, Proposition 3.2] $\sum_\mathcal{D} X_i$ has dimension $\leq n$ if and only if*

$$\{i \in I : X_i \text{ has dimension } \leq n\} \in \mathcal{D}.$$

(2) *[9, Theorems 4.5, 4.9] $\sum_\mathcal{D} X_i$ is an (hereditarily) indecomposable continuum if and only if*

$$\{i \in I : X_i \text{ is an (hereditarily) indecomposable continuum }\} \in \mathcal{D}.$$

4. Proof of Theorem 1.2

Lemma 4.1. *Let Z be a continuum. Then there is an ultrafilter $\mathcal{E}$, on an index set of cardinality $2^{w(Z)}$, and a metrizable continuum M such that $X_{\mathcal{E}}$ and $M_{\mathcal{E}}$ are homeomorphic.*

Proof. We refer the reader to the proof of [11, Theorem 4.7]. First one takes a lattice base $\mathcal{A}$ for Z, of cardinality $w(Z)$, and then–by the Löweinheim-Skolem theorem [17, Theorem 3.1.6]–lets $\mathcal{B}$ be a countable elementary sublattice of $\mathcal{A}$. Since being a lattice base for a compactum is a first-order property of lattices, there is a metrizable continuum M with a lattice base isomorphic to $\mathcal{B}$. Both Z and M have elementarily equivalent lattice bases; hence, by the Keisler-Shelah isomorphism theorem (see [17, Theorem 6.1.15 and Exercise 6.1.11]), there is an ultrafilter $\mathcal{E}$, on an index set of cardinality $2^{w(Z)}$, such that $Z_{\mathcal{E}}$ and $M_{\mathcal{E}}$ are homeomorphic. □

Start with the nondegenerate continuum X, and apply Theorem 1.1 (3) to obtain $g : Z \to X$, where Z is hereditarily indecomposable, of dimension one, of weight $w(X)$, and g is a weakly confluent map.

Lemma 4.2. *There is an ultrafilter $\mathcal{E}$ such that $Z_{\mathcal{E}}$ is an hereditarily indecomposable continuum, of dimension one, with infinitely many composants and weight $\leq 2^{(2^{w(X)})}$.*

Proof. Using Lemma 4.1, find an ultrafilter $\mathcal{E}$, on an index set of cardinality $2^{w(X)}$, and a metrizable continuum M such that $Z_{\mathcal{E}}$ and $M_{\mathcal{E}}$ are homeomorphic. Clearly we have $w(Z_{\mathcal{E}}) \leq 2^{(2^{w(X)})}$.

By Lemma 3.4, M is an hereditarily indecomposable continuum of dimension one. Since M is also metrizable, it has infinitely many composants, so let T be a an infinite set which is a *partial transversal*; i.e., no two points of T lie in the same composant of M. (T may miss some composants.)

Consider the set $T^{\mathcal{E}} \subseteq M^{\mathcal{E}} \subseteq M_{\mathcal{E}}$. This set is infinite, and we are done once we prove it is a partial transversal for the continuum $M_{\mathcal{E}}$ (also hereditarily indecomposable, of dimension one).

The continuum M is irreducible about any two points of T. Thus [11, Proposition 3.11] $M_{\mathcal{E}}$ is irreducible about any two points of $T^{\mathcal{E}}$, and we conclude that $Z_{\mathcal{E}}$ has infinitely many composants. □

An ultrafilter $\mathcal{D}$ on an infinite index set I is **free** if $I \setminus \{i\} \in \mathcal{D}$ for each $i \in I$; equivalently, if $\mathcal{D}$ consists of infinite sets only. The ultrafilter is **countably incomplete** if it is not closed under countable intersections; equivalently, if there is a nested sequence $I_0 \supseteq I_1 \supseteq \dots$ of sets in $\mathcal{D}$ such that $\bigcap_{n=0}^{\infty} I_n = \emptyset$. The ultrafilter $\mathcal{D}$ is **regular** if there exists a subfamily $\mathcal{S} \subseteq \mathcal{D}$ such that $|\mathcal{S}| = |I|$ and $\{J \in \mathcal{S} : i \in J\}$ is finite for each $i \in I$. Regular ultrafilters are known to exist in abundance; each regular ultrafilter is countably incomplete, and each countably incomplete ultrafilter is free. If the index set is countable, all three notions coincide (see [17]).

Lemma 4.3. *Suppose $\mathcal{D}$ is a regular ultrafilter on an index set of infinite cardinality λ.*

(1) *[17, Proposition 43.7] If S is an infinite set, then $|S^{\mathcal{D}}| = |S|^{\lambda}$.*
(2) *[7, Theorem 2.3.3] If X is an infinite compactum, then $w(X_{\mathcal{D}}) = w(X)^{\lambda}$.*

To finish the proof, fix an infinite cardinal α, and let $Z_{\mathcal{E}}$ be as in Lemma 4.2. The codiagonal map $p_{\mathcal{E}} : Z_{\mathcal{E}} \to Z$ is weakly confluent–indeed $K = p_{\mathcal{E}}[K_{\mathcal{E}}]$ for any subcontinuum $K \subseteq Y$–and so the composition $g \circ p_{\mathcal{E}} : Z_{\mathcal{E}} \to X$ is also weakly confluent.

Let $\mathcal{D}$ now be a regular ultrafilter on an index set of cardinality α, with $T \subseteq Z_{\mathcal{E}}$ a countably infinite partial transversal. Let Y be the *iterated ultracopower* $(Z_{\mathcal{E}})_{\mathcal{D}}$, with $f = g \circ p_{\mathcal{E}} \circ p_{\mathcal{D}} : Y \to X$. Then, by remarks above, Y is hereditarily indecomposable, of dimension one, and f is weakly confluent. Furthermore, $T^{\mathcal{D}}$ is a partial transversal for Y, and its cardinality is–by Lemma 4.3 (1)–2^α. Thus Y has at least 2^α composants. Moreover, $w(Y) \leq w(Z_{\mathcal{E}})^\alpha \leq 2^{(\alpha \cdot 2^{w(X)})}$, completing the proof of Theorem 1.2. □

Remarks 4.4.

(1) The iterated ultracopower in the proof above is actually an ultracopower: if $\mathcal{D}$ and $\mathcal{E}$ are ultrafilters on index sets I and J respectively, the *Fubini product* $\mathcal{F} = \mathcal{D} \cdot \mathcal{E}$ is an ultrafilter on $I \times J$, and there is a homeomorphism $h : Z_{\mathcal{F}} \to (Z_{\mathcal{E}})_{\mathcal{D}}$, such that $p_{\mathcal{F}} = p_{\mathcal{E}} \circ p_{\mathcal{D}} \circ h$ (see [7, Theorem 2.1.1]). This is an analogue of a well-known result about finite iterations of ultrapowers in model theory.
(2) In the last line of the proof above, $w(Y)$ actually equals $w(Z_{\mathcal{E}})^\alpha$ because of Lemma 4.3 (2). But this is moot because the ultrafilter in the (GCH-free) Shelah proof of the isomorphism theorem is not apparently regular. Hence we cannot claim $w(Z_{\mathcal{E}}) = 2^{(2^{w(X)})}$ in Lemma 4.2.
(3) In the argument above for Theorem 1.2, we cannot get the number of composants of Y to exceed its weight. Hence there seems to be no–even consistent–way to get Y to be fine without an entirely new argument.

5. Proof of Theorem 1.4

In order to obtain a version of Theorem 1.1 (3) where the pre-image is fine, we mimic the Baire category approach from Janiszewski-Kuratowski [21]. But first we will need to examine some cardinal-indexed properties of topological ultra(co)products.

If α is an infinite cardinal and X is a topological space, $a \in X$ is a P_α**-point** if for any collection $\mathcal{U}$ of at most α open neighborhoods of a, we have $a \in (\bigcap \mathcal{U})^\circ$. (The interior and closure of a set $A \subseteq X$ are denoted A° and A^-, respectively.) A space is a P_α**-space** if each of its points is a P_α-point; equivalently if whenever $\mathcal{U}$ is a family of $\leq \alpha$ open subsets, $\bigcap \mathcal{U}$ is open as well. When $\alpha = \omega$, P_α-points (P_α-spaces) are referred to as *P-points* (*P-spaces*).

Lemma 5.1. *Suppose $\mathcal{D}$ is a regular ultrafilter on an index set I of cardinality α. Then the following statements hold.*

(1) *Every $\mathcal{D}$-ultraproduct of topological spaces is a P_α-space.*
(2) *Every $\mathcal{D}$-ultracoproduct of compacta has a dense set of P_α-points.*

Proof. The first statement follows immediately from [4, Theorem 4.1]. As for the second, suppose $\vec{X}_{\mathcal{D}}$ is an ultracoproduct of compacta, where $\mathcal{D}$ is regular. By Definition 3.1 (3), $\vec{X}^{\mathcal{D}}$ is a dense subspace of $X_{\mathcal{D}}$. By (1) above, $\vec{X}^{\mathcal{D}}$ is also a P_α-space. We need to show that each $\vec{x}_{\mathcal{D}} \in \vec{X}_{\mathcal{D}}$ is a P_α-point in the ultracoproduct.

Let $\mathcal{U}$ be a family of $\leq \alpha$ open neighborhoods of $\vec{x}_{\mathcal{D}}$ in $\vec{X}_{\mathcal{D}}$. Without loss of generality, we may take each member of $\mathcal{U}$ to be basic open. Thus there is a family $\mathcal{V}$ of $\leq \alpha$ open ultraboxes in $\vec{X}^{\mathcal{D}}$ such that $\mathcal{U}$ consists of sets $(\prod_{\mathcal{D}} U_i)^\sharp$, where $\prod_{\mathcal{D}} U_i \in \mathcal{V}$.

Note that we may regard $\prod_{\mathcal{D}} U_i$ as $(\prod_{\mathcal{D}} U_i)^\sharp \cap \vec{X}^{\mathcal{D}}$. Because $\vec{X}^{\mathcal{D}}$ is a P_α-space, there is an open ultrabox $\prod_{\mathcal{D}} V_i$ with $\vec{x}^{\mathcal{D}} \in \prod_{\mathcal{D}} V_i \subseteq \bigcap \mathcal{V}$. Hence we have $\vec{x}_{\mathcal{D}} \in (\prod_{\mathcal{D}} V_i)^\sharp \subseteq \bigcap \mathcal{U}$. □

Corollary 5.2. *Let $\mathcal{D}$ be a free ultrafilter on a countable index set. Then each topological $\mathcal{D}$-ultraproduct is a P-space, and each topological $\mathcal{D}$-ultracoproduct possesses a dense set of P-points.*

One more important condition on ultrafilters, stronger than regularity, is goodness. Let $\wp_\omega(I) := \{s \in \wp(I) : s \text{ is finite}\}$. The ultrafilter $\mathcal{D}$ on I is **good** if: (1) $\mathcal{D}$ is countably incomplete; and (2) whenever $f : \wp_\omega(I) \to \mathcal{D}$ is such that $s \subseteq t$ implies $f(s) \supseteq f(t)$, there exists $g : \wp_\omega(I) \to \mathcal{D}$ such that $g(s \cup t) = g(s) \cap g(t)$ for all $s, t \in \wp_\omega(I)$ and also such that $g(s) \subseteq f(s)$ for $s \in \wp_\omega(I)$.

Good[3] ultrafilters on an infinite index set I are regular; when I is countable, all free ultrafilters are good. Keisler originally used a GCH argument to prove the existence of good ultrafilters on any infinite index set, and Kunen [22] later provided a GCH-free proof. (Indeed, there are $2^{(2^\alpha)}$ good ultrafilters on an index set of cardinality α; details may also be found in [17].) The main use of good ultrafilters was to create a maximum amount of saturatedness in ultraproducts of relational structures; a purely combinatorial–but topologically important–version of this is the following. First recall that a family of sets satisfies the **finite intersection property** if each finite subfamily has nonempty intersection.

Lemma 5.3. *[5, Proposition 1.5] Suppose I is an index set of infinite cardinality α, with $\vec{X}$ an I-sequence of sets and $\mathcal{D}$ a good ultrafilter on I. Let $\mathcal{S}$ be a family of $\leq \alpha$ ultraboxes in $\vec{X}^{\mathcal{D}}$, and assume $\mathcal{S}$ satisfies the finite intersection property. Then $\bigcap \mathcal{S} \neq \emptyset$.*

Remark 5.4. Lemma 5.3 is a stripped-down version of Theorem 6.1.8 in [17]. In the parlance of model theory, $\mathcal{S}$ is a finitely satisfiable 1-type.

Good ultrafilters confer additional important cardinal-indexed properties on topological ultra(co)products. If α is an infinite cardinal and X is a topological space, X is an **almost P_α-space** if whenever $\mathcal{U}$ is a family of $\leq \alpha$ open subsets of X and $\bigcap \mathcal{U} \neq \emptyset$, then $(\bigcap \mathcal{U})^\circ \neq \emptyset$ as well. Clearly each P_α-space is an almost P_α-space. And while a compactum cannot be a P-space without being finite, infinite compacta can be almost P_α-spaces for any cardinal α.

Lemma 5.5. *Suppose $\mathcal{D}$ is a good ultrafilter on an index set I of cardinality α. Then every $\mathcal{D}$-ultracoproduct of compacta is an almost P_α-space.*

Proof. Let $\vec{X}_{\mathcal{D}}$ be an ultracoproduct of compacta, where $\mathcal{D}$ is a good ultrafilter on I. Let $\alpha = |I|$, and suppose $\mathcal{U}$ is a family of $\leq \alpha$ open subsets of $\vec{X}_{\mathcal{D}}$, with $\bigcap \mathcal{U} \neq \emptyset$. It is easy to see that we lose no generality in assuming each member of $\mathcal{U}$ is basic open.

So, as in the proof of Lemma 5.1, we have a family $\mathcal{V}$ of $\leq \alpha$ open ultraboxes in $\vec{X}^{\mathcal{D}}$, such that $\mathcal{U}$ consists of sets $(\prod_{\mathcal{D}} U_i)^\sharp$, where $\prod_{\mathcal{D}} U_i \in \mathcal{V}$. We generally cannot infer that $\bigcap \mathcal{V}$ is nonempty just because $\bigcap \mathcal{U}$ is, but the fact that $\bigcap \mathcal{U}$ is nonempty immediately gives us that $\mathcal{U}$ satisfies the finite intersection property. By straightforward facts about ultraproducts, we can then infer that $\mathcal{V}$ also satisfies the finite intersection property. Hence, by Lemma 5.3, there is a point $\vec{x}^{\mathcal{D}}$ in $\bigcap \mathcal{V}$. But now we have $\vec{x}_{\mathcal{D}} \in \bigcap \mathcal{U}$. Good ultrafilters are regular; so, by Lemma 5.1, $\vec{x}_{\mathcal{D}}$ is a P_α-point of $X_{\mathcal{D}}$. From this we infer that $(\bigcap \mathcal{U})^\circ \neq \emptyset$. □

Corollary 5.6. *Let $\mathcal{D}$ be a free ultrafilter on a countable index set. Then all topological $\mathcal{D}$-ultracoproducts are almost P-spaces.*

Remarks 5.7.

(1) Lemma 5.5 is stated, with the weaker hypothesis that the ultrafilter is merely regular, as Theorem 2.3.7 in [7]. The proof is incorrect, however, as it infers a space to be an almost P_α-space from the fact that it has a dense set of P_α-points. For a simple counterexample to this assertion, consider the ordinal space $X = \omega + 1$: Each point of the dense subset $\omega = \{0, 1, \dots\}$ is isolated in X, hence a P-point. On the other hand, let $U_n = \{n, n+1, \dots, \omega\}$, $n \in \omega$. Then $\mathcal{U} = \{U_n : n \in \omega\}$ is a countable family of open sets, $\bigcap \mathcal{U} = \{\omega\}$ is nonempty, but $(\bigcap \mathcal{U})^\circ = \emptyset$. Hence X is not an almost P-space.

[3] The definition of *good* often does not include countable incompleteness (see [17]), but we have no present use for this weaker notion.

(2) While it is true that a continuum is an almost P_α-space if it has an α-saturated lattice base, it seems likely that a proof of this would require an argument similar to that of Lemma 5.5 above.

The last cardinal-indexed property we will need is the following: for an infinite cardinal α, a topological space X is **α-Baire** if whenever $\mathcal{U}$ is a family of $\leq \alpha$ dense open subsets of X, we have that $\bigcap \mathcal{U}$ is dense in X. Compacta are always ω-Baire, by the Baire category theorem. In [5] Theorem 2.2 says that if $\mathcal{D}$ is a good ultrafilter on an index set of cardinality α, then all topological $\mathcal{D}$-ultraproducts are α-Baire.[4] In the case of ultracoproducts of compacta, we can get a higher degree of Baireness. The proof follows along the lines of usual Baire category arguments, with a couple of modifications.

Lemma 5.8. *Let α be an infinite cardinal. If a compactum is an almost P_α-space, then it is α^+-Baire.*

Proof. Let the compactum X be an almost P_α-space, with $\mathcal{U}$ a family of $\leq \alpha^+$ dense open subsets of X. Let $V \subseteq X$ be nonempty open; we need to show $V \cap (\bigcap \mathcal{U}) \neq \emptyset$.

First let $\langle U_\xi : \xi < \alpha^+ \rangle$ be a well-ordering of $\mathcal{U}$. Using transfinite induction, we construct a sequence $\langle B_\xi : \xi < \alpha^+ \rangle$ of nonempty open sets as follows: Define $B_0 = V$; and assume, for fixed $0 < \delta < \alpha^+$, that $\langle B_\xi : \xi < \delta \rangle$ has been defined so that $(\bigcap_{\xi<\delta} B_\xi) \cap (\bigcap_{\xi<\delta} U_\xi) \neq \emptyset$. Let $\mathcal{F} = \{B_\xi : \xi < \delta\} \cup \{U_\xi : \xi < \delta\}$. Then $|\mathcal{F}| \leq \alpha$ and $\bigcap \mathcal{F} \neq \emptyset$. Since X is an almost P_α-space, $\bigcap \mathcal{F}$ has nonempty interior; so let B_δ be any nonempty open set such that $B_\delta^- \subseteq \bigcap \mathcal{F}$. To finish the induction step, note that $(\bigcap_{\xi<\delta+1} B_\xi) \cap (\bigcap_{\xi<\delta+1} U_\xi) = B_\delta \cap \bigcap \mathcal{F} \cap U_\delta = B_\delta \cap U_\delta \neq \emptyset$, since U_δ is dense in X. This completes the construction of $\langle B_\xi : \xi < \alpha^+ \rangle$.

Clearly $\langle B_\xi^- : \xi < \alpha^+ \rangle$ is a nested sequence of nonempty closed subsets of the compactum X; hence there is a point x in its intersection. Thus $x \in V$. Also, for fixed $\delta < \alpha^+$, we have $x \in B_{\delta+1} \subseteq U_\delta$; so $x \in V \cap \bigcap \mathcal{U}$. $\square$

The following is a slight generalization of a result [28, Proposition 11.14] that is well known in the context of metrizable continua.

Lemma 5.9. *Let X be a continuum, with $a \in X$. Then $\kappa(a)$ is a union of $\leq w(X)$ proper subcontinua, each containing a.*

Proof. Let $\mathcal{U}$ be an open-set base for $X \setminus \{a\}$, consisting of $w(X)$ nonempty sets. For each $U \in \mathcal{U}$, let C_U be the component of $X \setminus U$ containing a. Then C_U is a proper subcontinum of X containing a; hence $\bigcup\{C_U : U \in \mathcal{U}\} \subseteq \kappa(a)$. On the other hand, if $x \in \kappa(a)$, then there is a proper subcontinuum $K \subseteq X$ with $a, x \in K$. Since $\mathcal{U}$ is an open base, there is some $U \in \mathcal{U}$ such that $U \subseteq X \setminus K$. Both K and C_U are subcontinua of $X \setminus U$ containing a; hence $K \subseteq C_U$. This shows that

$$\kappa(a) = \bigcup\{C_U : U \in \mathcal{U}\},$$

a union of $\leq w(X)$ proper subcontinua, each containing a. $\square$

We are now ready to finish the proof of Theorem 1.4. Let X be a continuum, and α an infinite cardinal. By Theorem 1.1 (3), there is a weakly confluent $g : Z \to X$, where Z is an hereditarily indecomposable continuum of dimension one, and $w(Z) = w(X)$.

Now, fix a good ultrafilter $\mathcal{D}$ on an index set of cardinality $\lambda = \alpha \cdot w(Z)$. Let $Y = Z_{\mathcal{D}}$, with $f = g \circ p_{\mathcal{D}}$. Then, as argued in the last section, we know that: (1) Y is hereditarily indecomposable, of dimension one; and (2) $f : Y \to X$ is weakly confluent.

[4] In [5] "α-Baire" means intersections of $< \alpha$ (not $\leq \alpha$) dense open sets are dense.

Furthermore we know: (3) $w(Y) = w(X)^\lambda = 2^\lambda$, by Lemma 4.3 (2); and (4) Y is an almost P_λ-space, by Lemma 5.5. Thus Y is λ^+-Baire, by Lemma 5.8. So assume GCH_λ. Then, by Lemma 5.9, and because proper subcontinua of indecomposable continua are nowhere dense, we know that any union of $\leq \lambda^+$ composants of Y has empty interior. Hence Y has at least λ^{++} composants. Now, $|Y| \leq 2^{(\lambda^+)}$. Hence, if GCH_{λ^+} also holds, then Y is fine. This finishes the proof of Theorem 1.4. □

6. Proof of Theorem 1.8

Lemma 6.1. *[11, Theorem 5.5] Let $\langle X_i : i \in I\rangle$ be an I-sequence of totally unblocked continua, with $\mathcal{D}$ an ultrafilter on I. Then $\vec{X}_{\mathcal{D}}$ is point-unblocked.*

Remark 6.2. Theorem 5.5 in [11] uses total unblockedness in the factor continua to show that each proper subcontinuum of $\vec{X}_{\mathcal{D}}$ is unblocked if it intersects $\vec{X}^{\mathcal{D}}$. That fact is then used to infer that the ultracoproduct is point-unblocked.

So if we start with a nondegenerate continuum X, use Lemma 4.1 to obtain a metrizable continuum M and homeomorphic ultracopowers $X_{\mathcal{D}}$ and $M_{\mathcal{D}}$, where $\mathcal{D}$ is an ultrafilter on an index set of cardinality $2^{w(X)}$. Since M is metrizable, Bing's [14, Theorem 5] tells us M is totally unblocked. But then Lemma 6.1 says that $X_{\mathcal{D}}$ is point-unblocked; moreover, it has weight $\leq 2^{(2^{w(X)})}$. So let $Y = X_{\mathcal{D}}$, with $f = p_{\mathcal{D}}$. This finishes the proof of Theorem 1.8. □

7. Proof of Theorem 1.9

Given the intermediate results of Section 5, this is now an almost trivial consequence of Bing's theorem [14, Theorem 5].

A topological space X is a ***wb*-space** if it is $w(X)$-Baire. The proof of Bing's theorem, showing each metrizable continuum to be totally unblocked, hinges on the fact that every metrizable continuum is a *wb*-space, and may be easily modified to show the following.

Lemma 7.1. *Every wb-continuum is totally unblocked.*

Remarks 7.2.

(1) Recall that the **density** $d(X)$ of a space X is the least infinite cardinal α such that X has a dense subset of cardinality $\leq \alpha$; X is a ***db*-space** if it is $d(X)$-Baire. Clearly every *wb*-space is a *db*-space, and D. Anderson has improved significantly on the lemma above by showing [1, Corollary 5.10] that every *db*-continuum is totally unblocked. (We do not need this extra strength for our argument, however.)

(2) A point a in continuum X is called a **shore point** if for any finite family $\mathcal{U}$ of nonempty open subsets of X, there is a subcontinuum of $X \setminus \{a\}$ which meets every set in $\mathcal{U}$. Bing's theorem was used in [24] to prove that all nondegenerate metrizable continua have at least two shore points. Shore points are obviously noncut, and nonblock points are easily shown to be shore [16]. Hence the fact that each metrizable continuum is spanned by its set of shore points is immediate.

So we fix our nondegenerate continuum X. Setting $\lambda = w(X)$, we choose a good ultrafilter $\mathcal{D}$ on an index set of cardinality λ. Set $Y = X_{\mathcal{D}}$. Then $w(Y) = 2^\lambda$, by Lemma 4.3 (2); and, by Lemmas 5.5 and 5.8, Y is λ^+-Baire. So if GCH_λ holds, then Y is a *wb*-space. We set $f = p_{\mathcal{D}} : Y \to X$. Then Y is totally unblocked, by Lemma 7.1, and f is co-elementary. This finishes the proof of Theorem 1.9. □

Parting Question 7.3. We were intrigued by the fact that the proof of Theorem 1.4–unlike that of Theorem 1.9–apparently requires two instances of GCH (after initial conditions are set). In particular, can one prove–in ZFC + CH–that each metrizable continuum is a weakly confluent image of a continuum which is hereditarily decomposable, of dimension one, fine, and of weight $\mathfrak{c}$?

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