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Extreme points of a continuum

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A B S T R A C T

We import into continuum theory the notion of *extreme point of a convex set* from the theory of topological vector spaces. We explore how extreme points relate to other established types of "edge point" of a continuum; for example we prove that extreme points are always shore points, and that any extreme point is also non-block if the continuum is either decomposable or irreducible (in particular, metrizable). In addition we discuss some continuum-theoretic analogues of the celebrated Krein-Milman theorem.

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1. Introduction

In functional analysis, an **extreme point** of a convex set K in a real topological vector space is a point e which does not lie *strictly between* any two points of K. That is, if $e \in [a,b]_{\mathsf{L}} := \{(1-t)a+tb : 0 \le t \le 1\}$, for $a, b \in K$, then $e = a$ or $e = b$. Extreme points play a significant role in the study of convexity in topological vector spaces. One of the landmark results in the subject is the Krein-Milman theorem, which we state in two parts.

1.1 Theorem. *[12, Theorems V,7.4 and V,7.8] Let X be a locally convex topological real vector space, with K a compact convex subset. Let $\mathsf{E}(K)$ be the set of extreme points of K. Then we have the following:*

(1) *K is the closed convex hull of $\mathsf{E}(K)$.*
(2) *If K is the closed convex hull of $S \subseteq K$, then $\mathsf{E}(K) \subseteq S^-$, the topological closure of S.*

The first part of the theorem says—roughly—that the subset of extreme points of a compact convex set spans the set; the second part adds that the subset is $\le$-least in the family of all spanning sets, where $\le$ is the pre-order defined by $A \le B$ if and only if $A \subseteq B^-$.

Our intent in this paper is to define a reasonable notion of *extreme point* in the context of continua, and to relate this to more well-known versions of what it means to be "at the edge" of a continuum. Another aspiration is to give meaning to the expression, "Krein-Milman theorem for continua."

2. Preliminaries

In this paper a **compactum** is a compact Hausdorff space and a **continuum** is a connected compactum. A topological space with no points at all is vacuously a continuum. The term *continuum* for us is synonymous with *Hausdorff continuum*, used by some authors who wish to emphasize that metrizability is not assumed. A subset of a topological space is **degenerate** if it contains exactly one point, and is **nondegenerate** of it contains at least two. A subset of a space is a **subcompactum** (resp. **subcontinuum**) if it is a compactum (resp. continuum) in its subspace topology. We denote by $\mathcal{K}(X)$ the family of subcontinua of X. For $S \subseteq X$, we write $[S]_X$ for the **subcontinuum hull** of S, namely the intersection of all members of $\mathcal{K}(X)$ containing S. For $S = \{a, b\}$, $[S]_X$ is written $[a, b]_X$, and is called the **subcontinuum interval** determined by a and b. When there is no ambiguity, we write *hull* and *interval* for *subcontinuum hull* and *subcontinuum interval*, respectively. Also we drop subscripts if there is only one continuum whose intervals we are considering. Hulls are subcompacta, which are subcontinua only under certain circumstances (see Proposition 3.12 below). Saying that $c \in [a, b]$ holds is a way of asserting that c **lies between** a and b, and is the K-*betweenness relation* studied in [4–6]. If $c \in [a, b] \setminus \{a, b\}$, we say c lies **strictly between** a and b. If, in addition, we have $a \notin [c, b]$ and $b \notin [a, c]$, we say that c lies **way between** a and b.

Returning to the convexity-theoretic notion of *extreme point* above, it is easy to express the definition in continuum-theoretic terms: simply replace the segment $[a, b]_{\mathsf{L}}$ with the continuum interval $[a, b]$. However, as we shall see below, the condition that e cannot lie strictly between two points of the continuum is a bit strong, and we rather take as our definition of e being an **extreme point** of X the slightly weaker condition that e does not lie *way* between any two points of X. We denote by

$$\mathsf{E}(X) := \{e \in X : \forall a, b \in X (e \in [a, b] \Rightarrow a \in [e, b] \text{ or } b \in [a, e])\}$$

the set of extreme points of X. Whereas in the convexity context there is no difference between *strict betweenness* and *way betweenness*, the distinction is quite marked in the case of continua.

By a **point type** for continua, we mean a proper class function $\mathfrak{T}$ which assigns to each continuum X a subset $\mathfrak{T}(X)$ such that whenever $h : X \to Y$ is a homeomorphism between continua, the restriction of h to

$\mathfrak{T}(X)$ is a homeomorphism onto $\mathfrak{T}(Y)$. The proper class function E is the point type of central interest in this article.

The strongest point type we know of expressing being "in the middle" of a continuum is that of being a cut point. Recall that $c \in X$ is a **cut point** if $X \setminus \{c\}$ is disconnected; hence the weakest point type that expresses being "at the edge" is the negation of this. Point c is a **non-cut** point of X if $X \setminus \{c\}$ is connected. For continuum X, let $\mathsf{NC}(X)$ be its set of non-cut points. We say of a point type $\mathfrak{T}$ that it is an **edge point type** if $\mathfrak{T}(X) \subseteq \mathsf{NC}(X)$ for each continuum X.

2.1 Proposition. E *is an edge point type.*

Proof. Suppose c is a cut point of X; say $X \setminus \{c\} = U \cup V$, where U and V are nonempty disjoint open sets. Fix $a \in U$ and $b \in V$. Then $c \in [a,b]$. However [18, Theorem §46,II,4], $U \cup \{c\}$ and $V \cup \{c\}$ are subcontinua of X that witness the statements $b \notin [a,c]$ and $a \notin [c,b]$, respectively; hence $c \notin \mathsf{E}(X)$. □

If $[S] = X$ for some $S \subseteq X$, we say X is **irreducible about** S (or, S **spans** X). A continuum is **irreducible** if it is irreducible about a two-point subset.

Let us say that point type $\mathfrak{T}$ is **Krein-Milman (i) for class $\mathfrak{C}$ of continua** if X is irreducible about $\mathfrak{T}(X)$ for all $X \in \mathfrak{C}$. The type is **Krein-Milman (ii) for $\mathfrak{C}$** if, whenever $X \in \mathfrak{C}$ is irreducible about $S \subseteq X$, it follows that $\mathfrak{T}(X) \subseteq S^-$. ($S^-$ and S° are the closure and interior of S, respectively.) The point type NC is Krein-Milman (i) for the class of all continua, thanks to the non-cut point existence theorem (see, e.g., [24]).

2.2 Theorem. *Let X be a continuum. Then $X = [\mathsf{NC}(X)]$.*

On the other hand, NC is not Krein-Milman (ii) for all continua; witness the following.

2.3 Example. Let X be the $\sin \frac{1}{x}$*-continuum* in the euclidean plane; namely, $X = A \cup C$, where A is the segment $[\langle 0,-1\rangle, \langle 0,1\rangle]_{\mathsf{L}}$ and $C = \{\langle t, \sin \frac{1}{t}\rangle : 0 < t \leq 1\}$. Then $\mathsf{NC}(X) = A \cup \{e\}$, where $e = \langle 1, \sin 1\rangle$. If $a \in A$ is arbitrary and $S = \{a,e\}$, then $X = [S]$. But $\mathsf{NC}(X)$ is not contained in $S^- = S$.

2.4 Remark. By Theorem 2.2, every nondegenerate continuum contains at least two non-cut points; one with exactly two is called an **arc**. (These two non-cut points are commonly referred to as the *end points* of the arc.) Metrizable arcs are all homeomorphic to the closed unit interval in the real line, but non-metrizable arcs, e.g., long lines and lexicographically ordered squares, can be quite unlike what we usually think of as an "arc" in the real world. A subcontinuum interval $[a,b]$, when connected, is irreducible about $\{a,b\}$, but it may have many non-cut points other than a and b: in Example 2.3, $[a,e] = X$ whenever $a \in A$. Hence $[a,e]$ contains all points of A as non-cut points. This example also shows that the interval $[a,e]$ does not determine the set $\{a,e\}$. Intervals are arcs only under quite limited circumstances.

A space X is **continuumwise connected** if for each $a,b \in X$ there is a subcontinuum $K \subseteq X$ with $\{a,b\} \subseteq K$. A **continuum component** of X is a continuumwise connected subset which is not properly contained in another continuumwise connected subset of X. Each Hausdorff space is partitioned into its continuum components.

If c is a point of a continuum X such that $X \setminus \{c\}$ is not continuumwise connected, then c is called a **weak cut point**; otherwise c is a **strong non-cut point**. Denote by SNC the point type *strong non-cut*. The following is immediate, and asserts that SNC is the more obvious continuum-theoretic analogue of *extreme* from convexity theory.

2.5 Proposition. *Let X be a continuum. A point of X is strong non-cut if and only if it cannot lie strictly between any two points of X.*

So strong non-cut points are trivially extreme, but we shall see that extreme points may easily be weak cut as well.

The strongest edge point type we consider here is that of the co-locally connected points. The point $c \in X$ is **co-locally connected** (or, a point of **co-local connectedness**) if c has a base of open neighborhoods whose complements in X are connected. Let CC denote this point type. It is obvious that every co-locally connected point is strongly non-cut, and a simple metrizable example of the failure of the converse may be found in [10].

Summarizing what we know so far, we have the implications

$$\mathsf{CC} \Rightarrow \mathsf{SNC} \Rightarrow \mathsf{E} \Rightarrow \mathsf{NC}.$$

As shown in Example 2.10 below, extreme points may be weak cut and non-cut points can fail to be extreme. So all these implications are irreversible.

The Krein-Milman (i) property tends to favor less restrictive edge point types, while Krein-Milman (ii) favors the more restrictive ones, especially CC.

2.6 Proposition. *Let X be a continuum. If $S \subseteq X$ spans X, then $\mathsf{CC}(X) \subseteq S^-$.*

Proof. Suppose A is any closed subset of X, with $c \in \mathsf{CC}(X) \setminus A$. Then there is an open neighborhood U of c with U disjoint from A and $X \setminus U$ connected. Thus $X \setminus U$ is a proper subcontinuum containing A; hence A does not span X. $\square$

A continuum X is **aposyndetic** (see, e.g., [15]) if whenever $a, b \in X$ are distinct, there is an open set U and a subcontinuum K such that $a \in U \subseteq K \subseteq X \setminus \{b\}$. The following is a folklore result, which was probably known to G.T. Whyburn. It states that for aposyndetic continua, there is no distinction between the edge point types we consider. We include a straightforward "two-step" proof.

2.7 Theorem. *Let X be an aposyndetic continuum. Then $\mathsf{CC}(X) = \mathsf{NC}(X)$.*

Proof. Suppose $c \in X$ is a non-cut point of aposyndetic continuum X. We first show that c is strong non-cut.

For each $x \in X \setminus \{c\}$, use aposyndesis to obtain K_x, a subcontinuum of $X \setminus \{c\}$ such that $x \in K_x^\circ$. Then $\{K_x^\circ : x \in X \setminus \{c\}\}$ is an open cover of the connected set $X \setminus \{c\}$, so if $a, b \in X \setminus \{c\}$ are arbitrary, there is an $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and a subfamily $\{K(i) : 1 \le i \le n\} \subseteq \{K_x : x \in X \setminus \{c\}\}$, such that: (1) $a \in K(1)^\circ$; (2) $b \in K(n)^\circ$; and (3) $K(i)^\circ \cap K(i+1)^\circ \neq \emptyset$ for $1 \le i < n$. Thus $\bigcup_{1 \le i \le n} K(i)$ is a subcontinuum of $X \setminus \{c\}$ containing $\{a, b\}$, and we infer that c is a strong non-cut point.

In order to show c is co-locally connected, let U be a fixed open neighborhood of c. Using aposyndesis again, we pick, for each $x \in X \setminus U$, a subcontinuum M_x of $X \setminus \{c\}$ such that $x \in M_x^\circ$. Since $X \setminus U$ is compact, there is a finite subfamily $\{M(1), \dots, M(n)\} \subseteq \{M_x : x \in X \setminus U\}$ such that $X \setminus U \subseteq \bigcup_{1 \le i \le n} M(i)^\circ$. For each $1 \le i, j \le n$, use the fact that $c \in \mathsf{SNC}(X)$ to find a subcontinuum $L(i, j) \subseteq X \setminus \{c\}$ which intersects both $M(i)$ and $M(j)$. Let $V = X \setminus \left(\bigcup_{1 \le i \le n} M(i) \cup \bigcup_{1 \le i,j \le n} L(i, j)\right)$. Then $c \in V \subseteq U$, and $X \setminus V$ is connected. $\square$

The following is immediate from the results above.

2.8 Corollary. *All edge point types interpolating between CC and NC are both Krein-Milman (i) and Krein-Milman (ii) for the class of aposyndetic continua.*

A key difference between $[\cdot,\cdot]$ and $[\cdot,\cdot]_{\mathsf{L}}$ is that the latter betweenness notion satisfies the **antisymmetry** axiom: if $[a,b]_{\mathsf{L}} = [c,b]_{\mathsf{L}}$, then $a = c$. One look at the continuum $X = A \cup C$ in Example 2.3 reveals that this need not hold for $[\cdot,\cdot]$: If a, b are any two points in A, then $[a,e] = [b,e] = X$.

Define a continuum to be **antisymmetric** [6] if its subcontinuum betweenness relation satisfies the antisymmetry axiom. This is easily seen to be equivalent to saying that if $\langle a,b,c\rangle$ is any triple of points with $b \neq c$, then there is a subcontinuum which contains a and exactly one of b, c.

Aposyndetic continua are antisymmetric [6, Theorem 3.2]; the *harmonic fan* (the cone over the ordered space $\omega + 1$, where $\omega := \mathbb{N} \cup \{0\}$) is antisymmetric without being aposyndetic (see Example 3.7 below).

2.9 Proposition. *If X is an antisymmetric continuum, then* $\mathsf{E}(X) = \mathsf{SNC}(X)$.

Proof. Suppose X is antisymmetric and $c \notin \mathsf{SNC}(X)$. Then we have $a, b \in X$ with $c \in [a,b] \setminus \{a,b\}$. By antisymmetry, there is a subcontinuum K such that either $a, c \in K \subseteq X \setminus \{b\}$ or $a, b \in K \subseteq X \setminus \{c\}$. Since $c \in [a,b]$, the first alternative must obtain; hence $b \notin [a,c]$. Similarly we show $a \notin [c,b]$; hence $c \notin \mathsf{E}(X)$. □

2.10 Example. The $\sin \frac{1}{x}$-continuum $X = A \cup C$ from Example 2.3 fails to be antisymmetric because $\mathsf{E}(X) \neq \mathsf{SNC}(X)$: the end point $e \in C$ is the only strong non-cut point (it is even co-locally connected), but the end points of A are also extreme. (The other points of A are all non-cut points which fail to be extreme.) In summary, this example shows $\mathsf{E} \not\Rightarrow \mathsf{SNC}$, as well as $\mathsf{NC} \not\Rightarrow \mathsf{E}$.

In the next section we show that extreme points are much more than merely non-cut.

3. Relating E to other edge point types

Two important and well-studied point types to which we want to compare E are the following. A point c in continuum X is a **non-block point** if there is at least one continuum component of $X \setminus \{c\}$ that is dense in X. (So a point is a **block point** if it is not a non-block point.) A point $c \in X$ is a **shore point** if "X has arbitrarily large subcontinua missing c;" that is, whenever $\mathcal{U}$ is a finite family of nonempty open subsets of X, there is a subcontinuum of $X \setminus \{c\}$ which intersects each $U \in \mathcal{U}$. (In the metric setting this is equivalent to saying that there are subcontinua of $X \setminus \{c\}$ that are arbitrarily close to X, relative to the Hausdorff metric.) Let NB and S be the point types *non-block* and *shore*, respectively.

3.1 Proposition. *For any continuum X, we have* $\mathsf{SNC}(X) \subseteq \mathsf{NB}(X) \subseteq \mathsf{S}(X) \subseteq \mathsf{NC}(X)$.

Proof. If c is a block point of X, then $X \setminus \{c\}$ must have at least two continuum components. This makes c a weak cut point of X, and gives us the first inclusion.

Suppose $c \in \mathsf{NB}(X)$. Then there is a continuum component A of $X \setminus \{c\}$ that is dense in X. Fix $b \in A$. If $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a finite family of nonempty open subsets of X, we may pick, for each $1 \leq i \leq n$, a point $a_i \in A \cap U_i$. For each such i, there is a subcontinuum $K_i \subseteq X \setminus \{c\}$ containing both a_i and b. Thus $\bigcup_{i=1}^n K_i$ is a subcontinuum of $X \setminus \{c\}$ intersecting each set in $\mathcal{U}$, and the second inclusion is established.

Finally if $c \in X$ is a cut point, let U and V be two nonempty disjoint open subsets whose union equals $X \setminus \{c\}$. Then no subcontinuum of $X \setminus \{c\}$ can intersect both U and V; hence the family $\{U, V\}$ witnesses that c is not a shore point of X. □

The fact that the implications $\mathsf{CC} \Rightarrow \mathsf{SNC} \Rightarrow \mathsf{NB} \Rightarrow \mathsf{S} \Rightarrow \mathsf{NC}$ are all proper for metrizable continua is demonstrated in [10]. One aim of this paper is to fit E into the hierarchy of edge point types.

In [19] it is proved that every nondegenerate metrizable continuum has at least two shore points; in [10] it is recognized that almost the same argument shows that every nondegenerate metrizable continuum has at least two non-block points. In both papers the following result of R. H. Bing is crucial.

3.2 Theorem. *[8, Theorem 5] Let X be a metrizable continuum, with S a nonempty proper subset of X. Then there is a point $c \in X$ such that the union of all subcontinua intersecting S and excluding c is dense in X.*

An almost immediate consequence of this is the fact that both NB and S are Krein-Milman (i) for metrizable continua.

3.3 Corollary. *Let X be a nondegenerate metrizable continuum. Then $[\mathsf{NB}(X)] = [\mathsf{S}(X)] = X$.*

The proof of Theorem 3.2 relies on the Baire category theorem and the second countability of X, but the first author showed [1, Corollary 5.11] that Bing's proof can be modified to allow the weaker assumption that X is separable.

3.4 Corollary. *Let X be a nondegenerate separable continuum. Then $[\mathsf{NB}(X)] = [\mathsf{S}(X)] = X$.*

When separability is dropped, however, the existence of shore and non-block points is not assured. Under the near coherence of filters axiom NCF (consistent with the axioms of ZFC, see [9]), the Stone-Čech remainder $\mathbb{H}^* := \beta(\mathbb{H}) \setminus \mathbb{H}$ of the half-open ray $\mathbb{H} := [0, \infty)$ consists entirely of block points [2, Theorem 4.8]. Furthermore, while all points of $\mathbb{H}^*$—or any indecomposable continuum—are shore points [3, Theorem 3], NCF implies that when two disjoint copies of $\mathbb{H}^*$, are *spot-welded* at a single point of each, the result is devoid of shore points [3, proof of Theorem 1].

We do not know whether all extreme points are non-block. (They are, under certain quite broad circumstances, see Corollary 3.10 below.) In the mean time, we show that extreme points are at least shore points. The proof is facilitated by the introduction of a new point type, which acts as an interpolant: $c \in X$ is a **nested point** if the family of closures of the continuum components of $X \setminus \{c\}$ is *nested*; that is, totally ordered by inclusion. Let $\mathsf{N}(X)$ be the set of nested points of X.

3.5 Lemma. *Let X be a continuum. For $c \in X$, let A be a continuum component of $X \setminus \{c\}$. Then $c \in A^-$.*

Proof. Suppose $c \notin A^-$. Then we may use boundary bumping [18, Theorem §47,III,4] to obtain a subcontinuum $M \subsetneq X \setminus \{c\}$ properly containing the subcontinuum A^-. This contradicts the fact that A is a continuum component of $X \setminus \{c\}$; hence we infer that $c \in A^-$. □

3.6 Theorem. *Let X be a continuum. Then $\mathsf{E}(X) \subseteq \mathsf{N}(X) \subseteq \mathsf{S}(X)$.*

Proof. For $e \in X$, let $\mathcal{A}$ be the family of continuum components of $X \setminus \{e\}$. Assume $e \in \mathsf{E}(X)$ and suppose for the moment that there are $A, B \in \mathcal{A}$ with A^- and B^- incomparable (i.e., neither is contained in the other). Let $a \in A \setminus B^-$ and $b \in B \setminus A^-$. Then we have $e \in [a, b]$, since otherwise a and b would be contained in some subcontinuum of $X \setminus \{e\}$, contradicting the assumption that $A \neq B$. However, by Lemma 3.5, the subcontinuum B^- (resp. A^-) witnesses that $a \notin [e, b]$ (resp. $b \notin [a, e]$), contradicting our assumption that e is an extreme point. This establishes that $\mathsf{E}(X) \subseteq \mathsf{N}(X)$.

Assume $e \in \mathsf{E}(X)$, with $\mathcal{U} = \{U_1, \ldots, U_n\}$ a finite family of nonempty open subsets of X. Our job is to find a subcontinuum of $X \setminus \{e\}$ that intersects each $U \in \mathcal{U}$. For $1 \leq i \leq n$, we may choose $a_i \in U_i \setminus \{e\}$, with $A_i \in \mathcal{A}$ the continuum component containing a_i. Since $\mathcal{A}^-$ is nested, there is some $A \in \mathcal{A}$ with $A^- \supseteq \bigcup_{i=1}^n A_i^-$. In particular we have $\{a_1, \ldots, a_n\} \subseteq A^-$. So A^- intersects each open set U_i; hence we may fix $b_i \in A \cap U_i$, $1 \leq i \leq n$. Fix $a \in A$. Then, for each i, we have a subcontinuum K_i, with $\{a, b_i\} \subseteq K_i \subseteq A$. Thus $K = \bigcup_{i=1}^n K_i$ is a subcontinuum of $X \setminus \{e\}$ intersecting each $U \in \mathcal{U}$, and the proof is complete. □

By way of addressing the inevitable question of whether E, N, and S are distinct edge point types, the following examples show the answer to be *yes*.

3.7 Examples.

(1) Let X be the *harmonic fan* in the plane, given as follows: Let $a = \langle 0, 0\rangle$, $b = \langle 1, 0\rangle$, with $b_n = \langle 1, \frac{1}{n}\rangle$, $n \in \mathbb{N}$. Then $X = H \cup \left(\bigcup_{n=1}^{\infty} D_n\right)$, where H is $[a, b]_{\mathsf{L}}$ and each D_n is $[a, b_n]_{\mathsf{L}}$. As mentioned earlier, X is antisymmetric; so the extreme points, namely $b, b_1, b_2, \ldots$, are also strong non-cut. The rest of the other non-cut points constitute the set $H \setminus \{a, b\}$. All of these points are weak cut and non-block (hence shore). They are also nested: the complement of each of these points has just two continuum components, one of which is dense. Hence $\mathsf{N} \not\Rightarrow \mathsf{E}$.

(2) Let $X = A \cup C$ be the $\sin \frac{1}{x}$-continuum from Example 2.10. The end point of C is the only strong non-cut point; the other two extreme points are the end points of A. If $c \in A$ is any point other than an end point, then c is a shore point of X which is not nested: $X \setminus \{c\}$ has three continuum components, one of which is dense, the other two having incomparable closures. Hence $\mathsf{S} \not\Rightarrow \mathsf{N}$.

Thus far we have established the following diagram of proper implications.

$$\begin{array}{ccccccccccc} \mathsf{CC} & \Rightarrow & \mathsf{SNC} & \Rightarrow & \mathsf{E} & \Rightarrow & \mathsf{N} & \Rightarrow & \mathsf{S} & \Rightarrow & \mathsf{NC} \\ & & & \searrow & & & & \nearrow & & & \\ & & & & & \mathsf{NB} & & & & & \end{array}$$

In the sequel we will show that the implication $\mathsf{E} \Rightarrow \mathsf{NB}$ holding in general would impose some rather demanding limitations on the types of continua that are allowed to exist. These limitations involve the decomposability/indecomposability divide.

Recall that a continuum is **decomposable** if it is the union of two proper subcontinua; it is **unicoherent** if it is not the union of two subcontinua whose intersection is disconnected. A well-known characterization result [18, Theorem §48,V,2] says that a continuum is indecomposable if and only if each of its proper subcontinua is nowhere dense.

If c is a point in continuum X, define the **composant at** c **in** X to be the set $\kappa(c) := \bigcup\{K \in \mathcal{K}(X) \setminus \{X\} : x \in K\}$. Composants of a continuum are clearly continuumwise connected; they are also (by boundary bumping) dense in the continuum. A continuum X is irreducible if and only if it has more than one composant; the composants of an indecomposable continuum form a partition of the continuum. It is well known [20, Théorème 1] that the cardinality of the set of composants of a nondegenerate metrizable indecomposable continuum is that of the real line, but [7] non-metrizable indecomposable continua with just one composant are known to exist. These are what we call **Bellamy continua**, and can have weight as low as $\aleph_1$. Bellamy continua play an important role in the question of when extreme points are non-block.

3.8 Proposition. *Let X be an irreducible indecomposable continuum. Then each point of X is a weak cut point which is also non-block.*

Proof. Suppose $c \in X$ is fixed. Then the composant $\kappa(c)$ is dense in X. Since X is irreducible, we know that $\kappa(c)$ is a nondegenerate proper subset of X. So let $a \in \kappa(c) \setminus \{c\}$, with $b \in X \setminus \kappa(c)$. Then $[a, b] = X$; hence $c \in [a, b] \setminus \{a, b\}$. This shows c to be a weak cut point.

Again because of irreducibility, there is at least one composant of X disjoint from $\kappa(c)$. Each such composant is a continuum component of $X \setminus \{c\}$ and is dense in X. This shows that c is a non-block point. $\square$

3.9 Theorem. *Let X be a nondegenerate continuum containing a block point which is also extreme. Then X is a Bellamy continuum.*

Proof. Let $e \in X$ be a block point which is an extreme point as well. In view of Proposition 3.8, it suffices to show that X is indecomposable. To do this we show each proper subcontinuum of X to be nowhere dense.

Let $\mathcal{A}$ be the family of continuum components of $X \setminus \{e\}$. Since e is extreme, Lemma 3.5 and Theorem 3.6 tell us that $\mathcal{A}^- := \{A^- : A \in \mathcal{A}\}$ is a nested family of subcontinua, all containing e. Since e is block, each of these subcontinua is proper, and so $\mathcal{A}^-$ has no $\subseteq$-greatest element. Since $\bigcup \mathcal{A} = X \setminus \{e\}$ and $e \in \bigcap \mathcal{A}^-$, we have $\bigcup \mathcal{A}^- = X$.

Suppose $A, B \in \mathcal{A}$ are such that $A^- \subsetneq B^-$. Then there is some $b_0 \in B \setminus A^-$. If $a \in A$ is arbitrary, then $e \in [a, b_0]$. But A^- witnesses $b_0 \notin [a, e]$; because of e being an extreme point, it follows that $a \in [e, b_0]$.

This says that $A \subseteq [e, b_0]$ whenever $b_0 \in B \setminus A^-$. Suppose now that $b \in B$ is arbitrary and that there is some $a \in A$ with $a \notin [e, b]$. Let L be a subcontinuum containing $\{e, b\}$, but not a. Since $b_0, b \in B$, we also have a subcontinuum $M \subseteq B$ containing $\{b_0, b\}$. Hence $L \cup M$ is a subcontinuum containing $\{e, b_0\}$, but not a. This witnesses that $a \notin [e, b_0]$, a contradiction. Thus we conclude that $A^- \subseteq [e, b]$ for all $b \in B$.

So now suppose, as above, $A, B \in \mathcal{A}$ and $A^- \subsetneq B^-$. We claim that any subcontinuum intersecting both A and B must contain A^-. For if subcontinuum K contains $a \in A$ and $b \in B$, then we have $e \in [a, b] \subseteq K$, so $[e, b] \subseteq K$. But then, by the last paragraph, $A^- \subseteq [e, b]$, establishing our claim.

Finally, suppose some subcontinuum K has nonempty interior K°. Then, since $\bigcup \mathcal{A}^- = X$, there exists an $A_0 \in \mathcal{A}$ such that K° intersects A_0^-. Thus for all $B \in \mathcal{A}$ such that $A_0^- \subsetneq B^-$ (and such a B is guaranteed to exist) we have that K° intersects B^-. Since K° is open, it intersects both A_0 and B. Thus, by the claim above, we have $A_0^- \subseteq K$. Now let $A \in \mathcal{A}$ be arbitrary. If $A^- \subseteq A_0^-$, then $A^- \subseteq K$ obviously. If $A_0^- \subsetneq A^-$, then we have K° intersecting A^- and we argue as above to infer that $A^- \subseteq K$ in this case too. This immediately gives us $K = X$, and we conclude that each proper subcontinuum of X is nowhere dense. Therefore X is indecomposable. $\square$

In light of the fact [7] that Bellamy continua are indecomposable and non-irreducible, the following is now immediate.

3.10 Corollary. *Let X be a continuum which is either decomposable or irreducible. Then every extreme point of X is non-block.*

And since [20] nondegenerate indecomposable metrizable continua have many composants, we have the following.

3.11 Corollary. *In metrizable continua, all extreme points are non-block.*

If $\mathfrak{C}$ is a class of continua, we say of a continuum X that it is **hereditarily** $\mathfrak{C}$ if every nondegenerate subcontinuum of X is in $\mathfrak{C}$. So X is hereditarily unicoherent (resp. hereditarily indecomposable) if and only if any two subcontinua have connected—possibly empty—intersection (resp. are either disjoint or comparable). The following is a straightforward consequence of classic continuum theory.

3.12 Proposition. *For X a nonempty continuum, the following statements are equivalent:*

(a) *$[a, b]$ is connected for each $a, b \in X$.*
(b) *X is hereditarily unicoherent.*
(c) *$[S]$ is connected for each $S \subseteq X$.*

Proof. We need only prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Assume (a) and let K and M be overlapping subcontinua of X. For any $a, b \in K \cap M$, we have $[a, b] \subseteq K$, $[a, b] \subseteq M$, and $[a, b]$ is connected. This shows $K \cap M$ is connected, proving (b).

Assume (b) and let $S \subseteq X$ be given. If $S = \emptyset$, then either: X is degenerate and $[S]$ is a singleton; or X is nondegenerate and $[S] = \emptyset$. In either case $[S]$ is connected. So assume $S \neq \emptyset$, and denote by $\mathcal{K}_S(X)$ the family of subcontinua of X containing S. Then $\mathcal{K}_S(X)$, with the subset ordering, is downwardly directed. Hence [18, Theorem §47,II,5] $[S] = \bigcap \mathcal{K}_S(X)$ is a subcontinuum, proving (c). □

Continua consisting entirely of extreme (or even co-locally connected) points are easy to come by; one need look no further than a circle for an example. If we also insist upon hereditary unicoherence, examples become more exotic.

3.13 Theorem. *Let X be a nondegenerate continuum. The following statements are equivalent:*

(a) *X is hereditarily indecomposable.*
(b) *X is hereditarily unicoherent and every point of X is both weak cut and extreme.*
(c) *X is hereditarily unicoherent and every point of X is extreme.*

Proof. Assume (a) is true. Hereditary unicoherence trivially follows; so let $c \in X$ be fixed. By boundary bumping there is a proper nondegenerate subcontinuum K containing c. Let $a \in K \setminus \{c\}$, with $b \in X \setminus K$. If M is any subcontinuum containing $\{a, b\}$, then $M \cap K \neq \emptyset$ and $M \nsubseteq K$; hence $K \subseteq M$. Thus $c \in M$, and we have $c \in [a, b]$. This makes c a weak cut point.

We claim that no matter whether or not $c \in [a, b]$, we have $a \in [c, b]$ or $b \in [a, c]$ always. For let $a, b \in X$ be arbitrary. If $a \notin [c, b]$, we have a subcontinuum K containing $\{c, b\}$, but not a. So let M contain $\{a, c\}$. Then, as in the last paragraph, $K \subseteq M$; i.e., $b \in [a, c]$. Hence c is an extreme point.

The implication (b) ⇒ (c) is a tautology. To prove (c) ⇒ (a), assume (a) is false, but that X is hereditarily unicoherent. Then there are subcontinua $K, L \subseteq X$ with $K \setminus L$, $L \setminus K$, and $K \cap L$ all nonempty. Let $a \in K \setminus L$ and $b \in L \setminus K$. Then $[a, b] \subseteq K \cup L$. If $[a, b]$ missed $K \cap L$, then the doubleton family $\{K \cap [a, b], L \cap [a, b]\}$ would form a disconnection of $[a, b]$, contradicting hereditary unicoherence by Proposition 3.12.

So let $c \in [a, b] \cap (K \cap L)$. Then K and L respectively witnesses that $b \notin [a, c]$ and $a \notin [c, b]$, so c is not an extreme point of X. Hence (c) is false. □

It is an open problem whether a nondegenerate hereditarily indecomposable continuum can have just one composant. (M. Smith [22] has produced one with exactly two composants, and W. J. Charatonik [11] can produce one with exactly n composants, where $2 \leq n \leq \omega$.) The following result would present an obstacle to the assertion $\mathsf{E} \Rightarrow \mathsf{NB}$ if we knew that hereditarily indecomposable Bellamy continua actually exist.

3.14 Theorem. *Let X be a nondegenerate continuum. The following statements are equivalent:*

(a) *X is a hereditarily indecomposable Bellamy continuum.*
(b) *X is hereditarily unicoherent and every point of X is both block and extreme.*

Proof. Assume (a) is true, with $c \in X$ given. By Theorem 3.13, we know c is extreme. So let A be a continuum component of $X \setminus \{c\}$, with $a \in A$ fixed. Since A is continuumwise connected, there is a family $\mathcal{K}$ of subcontinua of A, all containing a, such that $A = \bigcup \mathcal{K}$. Because X has just one composant, there is a proper subcontinuum $M \subseteq X$ with $\{a, c\} \subseteq M$. By hereditary indecomposability, we have $K \subseteq M$ for each $K \in \mathcal{K}$; hence $A \subseteq M$. Since M is proper, $X \setminus M$ is a nonempty open set disjoint from A; hence A cannot be dense in X. (In fact, since the indecomposability of X guarantees that M is nowhere dense, so too is $A \subseteq M$.)

Now assume (b) is true. We infer that X is hereditarily indecomposable from Theorem 3.13, and that X is a Belamy continuum from Theorem 3.9. □

3.15 Question. Can the assumption of hereditary unicoherence be dropped from Theorem 3.14 (b)?

3.16 Remark. Provided that a hereditarily indecomposable Belamy continuum X exists, one may obtain a Bellamy continuum with a mix of extreme and non-extreme points as follows. Choose distinct points $a, b \in X$. Since X is not irreducible and is hereditarily unicoherent, $[a, b]_X$ is a proper subcontinuum. Hence, by boundary bumping, there is a proper nondegenerate subcontinuum disjoint from $[a, b]_X$; so we may choose distinct $c, d \in X$ such that $[c, d]_X \cap [a, b]_X = \emptyset$. Now let Y be the continuum that arises when we identify b and c to the same point p. It is reasonably straightforward to show that Y is a Bellamy continuum, and that each point of Y is block. Also it can be shown that $p \in [a, d]_Y$, but the images of the subcontinua $[a, b]_X$ and $[c, d]_X$ witness how $d \notin [a, p]_Y$ and $a \notin [p, d]_Y$, respectively. It follows that p is not an extreme point of Y.

Because X is non-irreducible, there is a proper subcontinuum K containing both b and c. Since X is hereditarily indecomposable and K intersects—but is not contained within—$[a, b]_X$, we have $[a, b]_X \subseteq K$; likewise $[c, d]_X \subseteq K$. The fact that any $e \in X \setminus K$ is extreme can be used to show e is also extreme as a point of Y. Hence Y has both extreme and non-extreme points.

In the definition of *non-block point*, it suffices to have at least one continuum component of the complement of the point be dense. This raises the question of "how dense" that set can be.

Recall that a subset of a topological space is **meagre** if it is contained in a union of countably many closed nowhere dense subsets; it is **co-meagre** if it contains a countable intersection of dense open sets. A **Baire space** is a space all of whose meagre subsets have empty interior; equivalently, all of whose co-meagre subsets are dense. In a Baire space the family of co-meagre subsets forms a countably complete filter of sets; in particular, the families of meagre subsets and co-meagre subsets are disjoint. The Baire category theorem tells us that all locally compact Hausdorff spaces and all completely metrizable spaces are Baire spaces.

Define $c \in X$ to be **non-block**$^+$ (resp. **non-block**$^{++}$) if there is a continuum component of $X \setminus \{c\}$ which is a co-meagre (resp. dense open) subset of X. (Such a continuum component is necessarily unique.) Let NB^+ and NB^{++} denote these two edge point types. Then we clearly have the implications

$$\mathsf{SNC} \Rightarrow \mathsf{NB}^{++} \Rightarrow \mathsf{NB}^{+} \Rightarrow \mathsf{NB}.$$

3.17 Example. In the case where X is either the $\sin \frac{1}{x}$-continuum from Example 2.3 or the harmonic fan from Example 3.7, each non-block point is non-block^{++}. Hence $\mathsf{NB}^{++} \not\Rightarrow \mathsf{SNC}$.

The following is a classic result.

3.18 Theorem. *[18, Theorem §48, VI, 1] If X is a metrizable continuum, then each composant of X is a countable union of subcontinua of X.*

3.19 Corollary. *Let X be a nondegenerate indecomposable metrizable continuum. Then each point of X is a weak cut point which is non-block, but no point of X is non-block$^+$.*

Proof. Because of metrizability, X has a host of composants; so *weak cut* and *non-block* follow from Proposition 3.8.

Now each proper subcontinuum is nowhere dense; hence, by Theorem 3.18, each composant of a nondegenerate indecomposable metrizable continuum is meagre. If $c \in X$ is arbitrary, the continuum components of $X \setminus \{c\}$ consist of the continuum components of $\kappa(c) \setminus \{c\}$, as well as the other composants of X. All of these sets are meagre. And since we are working in a Baire space where no subset can be simultaneously meagre and co-meagre, no continuum component of $X \setminus \{c\}$ is co-meagre. Therefore c is not non-block$^+$. $\square$

From Corollary 3.19, we immediately have $\mathsf{NB} \not\Rightarrow \mathsf{NB}^+$. When we add Theorem 3.13, we obtain the following corollary, which shows that $\mathsf{E} \not\Rightarrow \mathsf{NB}^+$.

3.20 Corollary. *Let X be a nondegenerate hereditarily indecomposable metrizable continuum. Then each point of X is a weak cut point which is also extreme and non-block, but which fails to be non-block$^+$.*

3.21 Question. Is the implication $\mathsf{NB}^{++} \Rightarrow \mathsf{NB}^+$ reversible?

4. Measuring the size of $\mathsf{E}(X)$

If $\mathfrak{T}$ is a point type and $\mathfrak{C}$ is a class of continua, having $\mathfrak{T}$ satisfy Krein-Milman (i) for $\mathfrak{C}$ is a way of saying $\mathfrak{T}(X)$ is "large" in X, for $X \in \mathfrak{C}$. In a dual way, satisfying Krein-Milman (ii) is a "smallness" condition.

For example, Corollary 2.8 tells us that all the edge point types considered here are both Krein-Milman (i) and Krein-Milman (ii) for the class of aposyndetic continua (rather a "goldilocks" state of affairs). On the other hand, the point type E is Krein-Milman (i) for the class of hereditarily indecomposable continua (Theorem 3.13), but never Krein-Milman (ii) for those hereditarily indecomposable continua that are irreducible. Since irreducibility may well be present in all nondegenerate hereditarily indecomposable continua, this state of affairs is definitely "ungoldilocks."

4.1 Proposition. *Let X be a hereditarily unicoherent continuum with $e \in X$. If e is an extreme (resp. strong non-cut, co-locally connected) point and K is a subcontinuum of X with $e \in K$, then e is an extreme (resp. strong non-cut, co-locally connected) point of K.*

Proof. Assume $e \in \mathsf{E}(X)$, with $K \in \mathcal{K}(X)$ containing e. If $a, b \in K$, we generally have $[a, b]_X \subseteq [a, b]_K$; in the case where X is hereditarily unicoherent, we have equality (see Proposition 3.12). So if $e \in [a, b]_K = [a, b]_X$, then—because $e \in \mathsf{E}(X)$—we have either $a \in [e, b]_X = [e, b]_K$ or $b \in [a, e]_X = [a, e]_K$. Hence $e \in \mathsf{E}(K)$.

In the case $e \in \mathsf{SNC}(X)$, we argue as above. But in the last sentence, we conclude either $e = a$ or $e = b$; hence $e \in \mathsf{SNC}(K)$.

Now, let $e \in \mathsf{CC}(X)$, with $K \in \mathcal{K}(X)$ and U an open set, both containing e. Then there is an open set V with $e \in V \subseteq U$ and $X \setminus V \in \mathcal{K}(X)$. Then $V \cap K$ is a relative open neighborhood of e in K, contained in $U \cap K$, such that—since X is hereditarily unicoherent—$K \setminus V = K \cap (X \setminus V)$ is connected. Thus $e \in \mathsf{CC}(K)$. $\square$

It is of interest when a continuum consists entirely of points of one type but not of a slightly more restrictive one. Here are two examples of continua consisting entirely of nested points which fail to be extreme. In particular, these show—more dramatically than Example 3.7 (1)—that $\mathsf{N} \not\Rightarrow \mathsf{E}$.

4.2 Examples.

(1) A *solenoid* is any nondegenerate homogeneous metrizable continuum, all of whose proper subcontinua are arcs (see, e.g., [13]). A solenoid which is not a simple closed curve is indecomposable [13, Lemma 6], hence hereditarily unicoherent. Let X be any one of these continua (well known to exist). By boundary bumping and homogeneity, every point of X is a cut point of an arc in X. Hence, by Proposition 4.1, X contains no extreme points. On the other hand, if $c \in X$ and $\mathcal{A}$ is the family of continuum components of $X \setminus \{c\}$, then $A \in \mathcal{A}$ is either a composant disjoint from $\kappa(c)$, or a continuum component of $\kappa(c) \setminus \{c\}$. In the first case A is dense in X because it is a composant of an indecomposable continuum. In the second case, if A were not dense in X, then we would have $A^- \subseteq \kappa(c)$. This would make A^- an arc, and A homeomorphic to a real interval. By Lemma 3.5, $c \in A^-$. Thus c is an end point of A^-. By homogeneity, it follows that A^- is properly contained in an arc with end point c. This contradicts the

fact that A is a continuum component of $X \setminus \{c\}$, and we conclude that each member of $\mathcal{A}$ is dense in X. Thus all points of X are nested.

(2) Let X be $\mathbb{H}^*$ (see the paragraph following Corollary 3.4; also [2,14]). X is well known to be indecomposable, as well as hereditarily unicoherent, and we first claim that each point of X is a non-extreme point of a subcontinuum of X. To see this, recall the *standard subcontinua* of X, namely those of the form

$$\bigcap_{J\in\mathcal{D}} \Big(\bigcup_{n\in J}[a_n, b_n]\Big)^-,$$

where $\langle a_0, b_0, a_1, b_1, \dots\rangle$ is an unbounded increasing sequence in $\mathbb{H}$, $\mathcal{D}$ is a free ultrafilter on ω, and closure is with respect to $\beta(\mathbb{H})$. (We refer the reader to [14], especially §2.)

If $K \subseteq X$ is a standard subcontinuum, the usual total ordering on $\mathbb{H}$ induces a preordering $\leq$ on K. And if we define $a, b \in K$ to be *equivalent* if both $a \leq b$ and $b \leq a$, then the equivalence classes, called the *layers* of K, are totally ordered by $\leq$, in such a way that the decomposition is upper semicontinuous and the associated quotient space is an arc (nonmetrizable, of course). Furthermore, each layer of K is a subcontinuum (indeed, indecomposable); any subcontinuum of K is either contained in a layer or is a union of layers.

Because the layers are arranged in a total ordering, also denoted $\leq$, it makes sense to use the interval notation $[L, R]$ to denote

$$\bigcup\{M : L \leq M \leq R\}.$$

Let $c \in X$. By [2, Lemma 3.5] a standard subcontinuum K may be chosen so that $c \in K \setminus (L_0 \cup L_1)$, where L_0 and L_1 respectively are the leftmost and the rightmost layer of K (both degenerate, by the way).

For any $x \in K$, let L_x be the layer of K containing x. If $L_c = \{c\}$, then c is a cut point of K, and is hence a non-extreme point of K. If L_c is nondegenerate, pick $a \in [L_0, L_c] \setminus \{c\}$ and $b \in [L_c, L_1] \setminus \{c\}$. Then the subcontinuum interval $[a, b]$, relative to K, is a subcontinuum of K, by hereditary unicoherence (and is also the subcontinuum interval relative to X). Since $[a, b]$ intersects both L_a and L_b, it follows that $[a, b] = [L_a, L_b]$; hence $c \in [a, b]$. But $[L_0, L_c]$ (resp. $[L_c, L_1]$) witnesses that $b \notin [a, c]$ (resp. $a \notin [c, b]$). Hence c is a non-extreme point of K in this case too. By Proposition 4.1, X is devoid of extreme points.

To see that any $c \in X$ is a nested point, let $\mathcal{A}$ be the collection of continuum components of $X \setminus \{c\}$. By Lemma 3.5, $c \in A^-$ for all $A \in \mathcal{A}$; hence all members of $\mathcal{A}^-$ overlap. By [14, Theorem 5.9], if two subcontinua of X overlap and one of them is indecomposable, then they are comparable. Thus, it suffices to show that every subcontinuum in $\mathcal{A}^-$ is indecomposable. Suppose this is not the case for some A. Then, by [14, Theorem 5.8], there is a standard subcontinuum $K = [L_0, L_1]$ of which A^- is a nondegenerate subinterval. By the proof of [2, Lemma 3.5], K may be chosen so that neither end point layer intersects A^-. Let L_c be the layer of K containing c. If A were to lie in L_c, then so would A^-; and this would contradict the fact that A^- is a nondegenerate subinterval of K. Hence A intersects some layer $L \neq L_c$. Suppose $L_0 \leq L < L_c$. Then $[L_0, L]$ is a subcontinuum of $X \setminus \{c\}$ which intersects A, itself a continuum component of $X \setminus \{c\}$. Thus $[L_0, L] \subseteq A$, contradicting the fact that $L_0 \cap A^- = \emptyset$. If $L_c < L \leq L_1$, we argue the same way, showing that $[L, L_1] \subseteq A$. This completes the argument that c is a nested point of X.

4.3 Remarks.

(1) Theorem 3.6 tells us that $\mathsf{N} \Rightarrow \mathsf{S}$, and—especially in light of Corollary 3.10—prompts the question of whether $\mathsf{N} \Rightarrow \mathsf{NB}$. This cannot have a general positive answer, however, owing to the facts that: (1)

every point of $\mathbb{H}^*$ is nested, by the previous example; and (2) under NCF, every point of $\mathbb{H}^*$ is block [2, Theorem 4.8].

(2) As a side issue, note that in the process of showing above that all points of $\mathbb{H}^*$ are nested, we also showed that if c is any point of $\mathbb{H}^*$ and A is any continuum component of $\mathbb{H}^* \setminus \{c\}$, then A^- is an indecomposable subcontinuum of $\mathbb{H}^*$. This appears to be a new fact about $\mathbb{H}^*$.

4.4 Questions.

(1) Are the nested points of a metrizable continuum always non-block?
(2) Is there a ZFC counterexample to the general assertion $\mathsf{N} \Rightarrow \mathsf{NB}$?

Since all nested points are shore, every indecomposable continuum consists entirely of shore points ([3, Theorem 3]), and all points of a solenoid or $\mathbb{H}^*$ are nested (Example 4.2), it makes sense to ask whether all points of *any* indecomposable continuum are nested. The answer is *no*.

4.5 Example. Start with $X = \langle X, d\rangle$ any nondegenerate indecomposable metric continuum, with $a \in X$ arbitrary. Letting Y be the interval $[-1, 1]$ in the real line, let $f : X \setminus \{a\} \to Y$ be defined by $f(x) = \sin(\frac{1}{d(x,a)})$. Let Z be the result of *resolving* $a \in X$ *to* Y *via* f [23]; in other words,

$$Z = (\{a\} \times Y) \cup \{\langle x, f(x)\rangle : x \in X \setminus \{a\}\} \subseteq X \times Y.$$

It is easy to show that Z is a continuum. Let $\pi : Z \to X$ be the restriction of the first coordinate map from $X \times Y$ to X. Then π is [17, Theorem 3.1] both *irreducible* (i.e., taking proper subcontinua to proper subcontinua) and *atomic* (i.e., fibers are *rungs*, subcontinua which cannot intersect other subcontinua without being comparable with them). In particular, the subcontinuum $\{a\} \times Y$ is a rung of Z. The irreducibility of π implies immediately that Z is an indecomposable continuum.

Now, let $c = \langle a, 0\rangle$, with $A = \{a\} \times [-1, 0)$ and $B = \{a\} \times (0, 1]$. If K is any subcontinuum of $Z \setminus \{c\}$ and K intersects A then, because $\{a\} \times Y$ is a rung which both intersects K and is not contained in K, we have $K \subseteq A \cup B$. But then $A \cup B$ is disconnected; so $K \subseteq A$. Similarly, if K intersects B then $K \subseteq B$. Hence both A and B are continuum components of $Z \setminus \{c\}$. Since $A^- = \{a\} \times [-1, 0]$ and $B^- = \{a\} \times [0, 1]$ are incomparable, we conclude that c is not a nested point of Z.

4.6 Question. Can an indecomposable continuum consist entirely of non-nested points? (As noted in the discussion following Corollary 3.4, when two disjoint copies of $\mathbb{H}^*$ are joined at a point, the result is a decomposable continuum which is devoid of shore points—and hence of nested points—under NCF.)

One way of saying a subset of a continuum is “small” is to say that it is somehow “granular;” another is to say it has empty interior. We next show how $\mathsf{E}(X)$ may be regarded as “small” in both senses.

Recall that a space X is **totally (continuumwise) disconnected** if no connected subset (subcontinuum) of X has more than one point. The two disconnectedness notions are clearly equivalent if the space is a compactum.

4.7 Proposition. *If a subset of a continuum X is totally continuumwise disconnected, then it has empty interior in X.*

Proof. Suppose A is a totally continuumwise disconnected subset of a continuum X. If U is a nonempty open subset of X, then boundary bumping tells us there is a nondegenerate subcontinuum K of X contained in U. By hypothesis $K \not\subseteq A$, and we conclude that A has empty interior in X. □

4.8 Theorem. *Let X be a hereditarily unicoherent continuum that contains no hereditarily indecomposable subcontinua. Then $\mathsf{E}(X)$ is totally continuumwise disconnected, and hence has empty interior in X.*

Proof. Suppose there is a nondegenerate subcontinuum $K \subseteq \mathsf{E}(X)$. Then, by Proposition 4.1, K—as a continuum—consists entirely of extreme points. Because K is hereditarily unicoherent, it is hereditarily indecomposable, by Theorem 3.13. This contradicts our assumption; hence $\mathsf{E}(X)$ is totally continuumwise disconnected. The second assertion now follows from Proposition 4.7. □

Hence for hereditarily unicoherent continua that are hereditarily decomposable, the set of non-extreme points is dense. The following example shows that cut points—even block points—are not guaranteed to *span* a hereditarily unicoherent, hereditarily decomposable continuum, let alone be dense in one.

4.9 Example. Let X be a *Cantor fan*; i.e., a cone over a Cantor discontinuum. X is both hereditarily unicoherent and hereditarily decomposable, but the vertex of X is the only cut point—indeed, the only block point. Also observe that the extreme points, which coincide with the co-locally connected points, consist of the points of the original Cantor discontinuum. Hence, as predicted by Theorem 4.8, the set of extreme points of X is totally (continuumwise) disconnected and the set of non-extreme points is dense.

4.10 Questions.

(1) In Theorem 4.8, when can we infer that $\mathsf{E}(X)$ is totally disconnected? When can we infer that $\mathsf{E}(X)$ is nowhere dense (or perhaps meagre) in X? (If X is the *shrinking harmonic fan*

$$\bigcup_{n=1}^{\infty} \left[\langle 0,0\rangle, \langle \frac{1}{n}, \frac{1}{n^2}\rangle\right]_{\mathsf{L}}$$

in the euclidean plane then $\mathsf{E}(X)^{-} = \mathsf{E}(X) \cup \{\langle 0,0\rangle\}$ still has empty interior.)

(2) Being dense in a continuum is a lot stronger than spanning. Is there a reasonable weakening of the hypothesis of Theorem 4.8 so that $X \setminus \mathsf{E}(X)$ spans X without necessarily being dense?

Recall that a space X is **(uniquely) arcwise connected** if each two points of X are the end points of a (unique) arc in X. The space satisfies the **arc-nestedness property** if the union of any nested family of arcs in X is contained in an arc in X. Following [21], a continuum is an **arboroid** if it is hereditarily unicoherent and arcwise connected. Metrizable arboroids are known as **dendroids**; the harmonic fan is one of the better-known dendroids which fail to be aposyndetic. Here is a list of some of the salient features of arboroids we will be using.

4.11 Lemma. *Let X be an arboroid. The following are true for X.*

(1) [21] *Each subcontinuum of X is uniquely arcwise connected.*
(2) [21] *X satisfies the arc-nestedness property.*
(3) [6] *X is antisymmetric.*
(4) [6] *X is hereditarily decomposable.*

We end with a study of how E satisfies Krein-Milman conditions for the class of arboroids.

4.12 Theorem. *Let X be a nondegenerate arboroid. Then the following are true for X.*

(1) *$\mathsf{E}(X)$ is totally continuumwise disconnected.*

(2) *$X \setminus \mathsf{E}(X)$ is dense in X.*
(3) *$\mathsf{E}(X)$ spans X.*
(4) *If $S \subseteq X$ spans X and $\mathsf{E}(X) = \mathsf{CC}(X)$, then $\mathsf{E}(X) \subseteq S^-$.*

Proof. Because of Lemma 4.11 (4), Theorem 4.8 applies to arboroids. This establishes (1) and (2).

By Lemma 4.11 (3), X is antisymmetric; so, by Proposition 2.9, $\mathsf{E}(X) = \mathsf{SNC}(X)$. Furthermore, by Lemma 4.11 (1), each subcontinuum interval $[a, b]$ is the unique arc in X with end points a and b.

Let $\mathcal{A}(X)$ be the collection of arcs (as well as singletons) in X, partially ordered by inclusion. By Lemma 4.11 (2), X satisfies the arc-nestedness property. Hence a simple Zorn's Lemma argument allows us to conclude that every arc (or singleton) in X is contained in a *maximal arc*; i.e., an arc not properly contained in any arc of X. Let $\mathsf{M}(X)$ consist of all points $e \in X$ such that e is an end point of a maximal arc of X.

Suppose $e \in \mathsf{E}(X)$. By Proposition 4.1, e is an end point of any containing arc. Since there is a maximal arc containing e, we infer $e \in \mathsf{M}(X)$.

So $\mathsf{E}(X) \subseteq \mathsf{M}(X)$. To obtain the reverse inclusion, note first that if A and B are any two overlapping arcs, hereditary unicoherence dictates that $C = A \cap B$ is either a singleton or an arc. Moreover, if both $A \setminus C$ and $B \setminus C$ are connected, then $A \cup B$ is an arc as well.

Suppose $e \in \mathsf{M}(X)$, say $[d, e]$ is a maximal arc with end point e. We show $e \in \mathsf{SNC}(X)$. Given $e \in [a, b]$, we know that $[d, e] \cap [a, b]$ is of the form $[c, e]$ for some $c \in [a, b]$. Either $e \in [c, b]$ or $e \in [a, c]$. In the first case it follows that $[d, e] \cup [e, b]$ is an arc; and, by the maximality of $[d, e]$, we infer that $b \in [d, e]$. But then $b \in [c, e]$; hence $e = b$ (since we assumed $e \in [c, b]$). In the case $e \in [a, c]$, we infer $e = a$. This establishes (3); condition (4) is just a special case of Proposition 2.6. □

The hypothesis in Theorem 4.12 (4) that $\mathsf{E}(X) = \mathsf{CC}(X)$ cannot be dropped.

4.13 Example. Let X be the harmonic fan from Example 3.7 (1), and for each $n \in \mathbb{N}$, let $m_n = \langle 1, \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})\rangle$ be the midpoint of $[b_n, b_{n+1}]_{\mathsf{L}}$. Now let $c_n = \langle \frac{1}{2}, \frac{1}{4}(\frac{1}{n} + \frac{1}{n+1})\rangle$ be the midpoint of $[a, m_n]_{\mathsf{L}}$, with Y the result of attaching to X the line segments $[b_n, c_n]_{\mathsf{L}}$. The points b_n are now cut points of Y (forming infinitely many "knuckles"). The points c_n are the co-locally connected points of Y. The point b is co-locally connected in X, but is merely a strong non-cut point of Y. (Our continuum Y is similar to an example in [10], used to show $\mathsf{SNC} \not\Rightarrow \mathsf{CC}$, and we call it a *harmonic rake*.) Note that $\mathsf{CC}(Y)$ spans Y, but that $\mathsf{E}(Y) \not\subseteq \mathsf{CC}(Y)^-$.

4.14 Remarks.

(1) [16, Corollary 3.8] says that if X is any metrizable arcwise connected continuum, then $\mathsf{CC}(X)$ spans X, thus (along with Proposition 2.6) presenting us with another "goldilocks" situation, in the Krein-Milman sense. The machinery used to prove this result relies heavily on the metrizability assumption, but nevertheless prompts the question of whether metrizability is essential.
(2) Hereditary unicoherence without arcwise connectedness is not enough for a metrizable continuum to have a suitably large set of co-locally connected points. Indeed, the solenoid of Example 4.2 (1) has no extreme points, let alone co-locally connected ones.

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