Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 25, Spring 2000
Pages 45-62

CONTINUA AND THE CO-ELEMENTARY HIERARCHY OF MAPS

PAUL BANKSTON

ABSTRACT. The co-elementary hierarchy is a nested ordinal-indexed sequence of classes of mappings between compacta, with each successor level being defined inductively from the previous one using the topological ultracopower construction. The lowest level is the class of continuous surjections; and the next level up, the co-existential maps, is already a much more restricted class. Co-existential maps are weakly confluent, and monotone when their images are locally connected. These maps also preserve important topological properties, such as: being infinite, being of covering dimension $\leq n$, and being a (hereditarily decomposable, indecomposable, hereditarily indecomposable) continuum.

1. INTRODUCTION.

In this paper, a **compactum** is a compact Hausdorff space, and a **continuum** is a connected (*not* necessarily metrizable) compactum. By a **subcompactum** (resp., **subcontinuum**) we simply mean a subspace that is itself a compactum (resp., continuum).

The **co-elementary hierarchy** of maps between compacta (as introduced in [6]) is a nested ordinal-indexed sequence of mapping classes, defined in (dual) analogy with the elementary hierarchy of embeddings in model theory (i.e., embeddings classified by the

2000 *Mathematics Subject Classification.* 03C20, 54B35, 54C10, 54D30, 54D80, 54F15, 54F45.

Key words and phrases. co-elementary hierarchy, co-existential mapping, ultracopower, ultracoproduct, compactum, continuum.

quantifier complexity of the first-order formulas they preserve). The lowest level, the class $\mathrm{LEV}_{\geq 0}$, is defined to be the continuous surjections; each succeeding level is built inductively from its predecessor, using the topological ultracopower construction. What makes this hierarchy so interesting for us is that it strongly involves topological notions, especially connectedness, without mention of anything in its definition other than ultracopowers and continuous maps. We give a brief description here, the reader is referred to [1]–[6] for details.

Given a compactum X and an ultrafilter $\mathcal{D}$ on an index set I (i.e., $\mathcal{D}$ is a maximal filter in the Boolean algebra of all subsets of I), the **ultracopower of X via $\mathcal{D}$** is denoted $XI\backslash\mathcal{D}$. The most "topological" way to describe this construction is to regard I as a discrete space, letting $p : X \times I \to X$ and $q : X \times I \to I$ be the usual projection maps. Applying the Stone-Čech compactification functor $\beta(\ )$, we view $\mathcal{D}$ as a member of $\beta(I)$ and define the ultracopower to be the preimage of $\mathcal{D}$ under q^{β}. The restriction $p_{X,\mathcal{D}}$ of p^{β} to $XI\backslash\mathcal{D}$ is the **codiagonal map**, and is a continuous mapping onto X. Another way to describe the ultracopower (and, more generally, the ultraco*product*) is to start with the compactum X, let $F(X)$ be its bounded lattice of closed subsets, take the usual (model-theoretic) ultrapower $F(X)^I/\mathcal{D}$, and apply the maximal-spectrum functor to obtain $XI\backslash\mathcal{D}$. The map $p_{X,\mathcal{D}}$ is then the image under the maximal-spectrum functor of the standard embedding from $F(X)$ into the ultrapower. (This map is strongly related to the standard part map from nonstandard analysis/topology.)

With $\mathrm{LEV}_{\geq 0}$ already defined to be the continuous surjections, we can now construct the co-elementary hierarchy inductively. For each ordinal α, declare that a continuous surjection $f : X \to Y$ between compacta belongs to the class $\mathrm{LEV}_{\geq \alpha+1}$ if there is an ultracopower $YI\backslash\mathcal{D}$ and a mapping g in class $\mathrm{LEV}_{\geq \alpha}$ from $YI\backslash\mathcal{D}$ to X such that $f \circ g = p_{Y,\mathcal{D}}$. If α is a limit ordinal, then $\mathrm{LEV}_{\geq \alpha} := \bigcap_{\beta<\alpha} \mathrm{LEV}_{\geq \beta}$.

Of particular interest to us is the next-to-the-lowest class, $\mathrm{LEV}_{\geq 1}$, the class of **co-existential** maps. Co-existential maps (as well as the other classes in the co-elementary hierarchy) are introduced in [6] as topological analogues (in a category dual sense) of existential embeddings in model theory. They also arise naturally *from* existential embeddings, giving us more than just an analogue:

Suppose X and Y are compacta with lattice bases $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively. (This means they are closed-set bases that are bounded lattices under union and intersection; equivalently, that they are meet-dense sublattices of the respective closed-set lattices $F(X)$ and $F(Y)$.) If $f : \mathcal{B}_Y \to \mathcal{B}_X$ is an existential embedding, and if $f^* : X \to Y$ is the natural continuous surjection induced by f, then f^* is co-existential. (In algebra, one group (resp., field) is pure (resp., algebraically closed) in another if the inclusion embedding is existential.)

2. Central Theorems.

The theory of the co-elementary hierarchy of maps between compacta emanates from a few central results, which have proven themselves to be fundamental to the further study. The first deals with compositions, and appears as 2.5 and 2.7 in [6].

Theorem 2.1. (Composition). *Each class $LEV_{\geq\alpha}$ of maps is closed under composition. Moreover, if $f \circ g \in LEV_{\geq\alpha+1}$ and $g \in LEV_{\geq\alpha}$, then $f \in LEV_{\geq\alpha+1}$.*

We recall from [1] that a map $f : X \to Y$ between compacta is said to be **co-elementary** if there is a homeomorphism of ultracopowers $h : XI\backslash\mathcal{D} \to YJ\backslash\mathcal{E}$ such that $f \circ p_{X,\mathcal{D}} = p_{Y,\mathcal{E}} \circ h$. Denote by COE the class of co-elementary maps. Clearly every codiagonal map is co-elementary, and from this it is easy to show, using induction, that $\text{COE} \subseteq \bigcap_\alpha \text{LEV}_{\geq\alpha}$. This leads to the second central theorem, which appears as 2.10 in [6].

Theorem 2.2. (Hierarchy). *$\bigcap_\alpha LEV_{\geq\alpha} = LEV_{\geq\omega} = COE$.*

The next theorem along these lines deals with consequences of the collapsing of levels in the co-elementary hierarchy. For each class $\mathcal{K}$ of compacta and each ordinal α, let $\text{LEV}_{\geq\alpha}(\mathcal{K})$ consist of those maps of level $\geq \alpha$ between members of $\mathcal{K}$. The following first appears as 4.1 in [6], and is analogous to "Robinson's Test" in model theory (see [11]).

Theorem 2.3. (Collapsing). *Suppose that $\mathcal{K}$ is a class of compacta closed under ultracopowers, and that, for some $n < \omega$, $LEV_{\geq n}(\mathcal{K}) = LEV_{\geq n+1}(\mathcal{K})$. Then $LEV_{\geq n}(\mathcal{K}) = LEV_{\geq \omega}(\mathcal{K})$.*

The fourth central theorem also has its model-theoretic counterpart, the Elementary Chains Theorem (and its generalization to chains of embeddings of fixed level in the elementary hierarchy), due to A. Tarski and R. Vaught (see [11]). It appears as 3.4 in [6].

Theorem 2.4. (Limit). *Let α be a fixed ordinal, and let $\langle X_m \overset{f_m}{\leftarrow} X_{m+1} : m < \omega\rangle$ be an inverse system of compacta, with each f_m in $LEV_{\geq \alpha}$. If X_∞ is the inverse limit of this sequence, and if, for each $m < \omega$, $g_m : X_\infty \to X_m$ is the natural limit map (so $f_m \circ g_{m+1} = g_m$), then each g_m is also in $LEV_{\geq \alpha}$.*

The next theorem deals with complements of classes of compacta, and follows easily from the definition of the co-elementary hierarchy.

Theorem 2.5. (Complements). *Let $\mathcal{K} \subseteq \mathcal{M}$ be classes of compacta, both closed under ultracopowers. If $\mathcal{K}$ is closed under images of maps of level $\geq \alpha$ and $\mathcal{M}$ is closed under images of maps of level $\geq \alpha+1$, then the relative complement $\mathcal{M}\backslash\mathcal{K}$ is closed under images of maps of level $\geq \alpha + 1$.*

The last two central theorems deal with co-existential maps. If $\mathcal{K}$ is a class of compacta, we call $\mathcal{K}$ **co-inductive** if it is closed under inverse limits of directed systems, comprising members of $\mathcal{K}$ and continuous surjections. $\mathcal{K}$ is called **co-elementary** if it is closed under ultracoproducts (see any of [1]–[6] for a definition) and co-elementary images. If $\mathcal{K}$ is any class of compacta, we call $X \in \mathcal{K}$ **co-existentially closed in** $\mathcal{K}$ if whenever $Y \in \mathcal{K}$ and $f : Y \to X$ is a continuous surjection, then f is co-existential. (This notion is analogous to that of "existentially closed in a class" in model theory. Being existentially closed in the class of Boolean algebras means being atomless; being existentially closed in the class of abelian groups means being divisible and having infinitely many elements of each prime order.) The following appears as 6.1 in [5].

Theorem 2.6. (CC-Existence). *Let $\mathcal{K}$ be a class of compacta that is both co-elementary and co-inductive. Then every infinite $X \in \mathcal{K}$ is a continuous image of a member of $\mathcal{K}$ that is co-existentially closed in $\mathcal{K}$ and of the same weight as X.*

Theorem 2.7. (CC-Preservation). *Let $\mathcal{K}$ be a class of compacta that is closed under ultracopowers. If $Y \in \mathcal{K}$ is a co-existential image of some X that is co-existentially closed in $\mathcal{K}$, then Y is also co-existentially closed in $\mathcal{K}$.*

Proof: Let $f : X \to Y$ be a co-existential map, where $Y \in \mathcal{K}$ and X is co-existentially closed in $\mathcal{K}$. Let $g : Z \to Y$, be a continuous surjection, where $Z \in \mathcal{K}$. We wish to show g is co-existential. To witness the co-existentiality of f, we have an ultracopower $YI\backslash\mathcal{D}$ and a continuous surjection $h : YI\backslash\mathcal{D} \to X$ such that $f \circ h = p_{Y,\mathcal{D}}$. Take the corresponding ultracopower of Z. Ultracopowers are functorial; so there is an ultracopower map $gI\backslash\mathcal{D}$, a continuous surjection, such that $p_{Y,\mathcal{D}} \circ (gI\backslash\mathcal{D}) = g \circ p_{Z,\mathcal{D}}$. Both ultracopowers are in $\mathcal{K}$ by hypothesis; so $h \circ (gI\backslash\mathcal{D})$ is co-existential. By 2.1, then, $f \circ h \circ (gI\backslash\mathcal{D})$ is also co-existential. But this map is $g \circ p_{Z,\mathcal{D}}$. Thus g is co-existential, again, by 2.1. □

Finally we have a souped-up version of 2.4 and 2.6 in [5]. (Also note a much earlier, and weaker, version appearing as 2.8 in [3].

Theorem 2.8. (Mapping Structure). *Let $f : X \to Y$ be a co-existential map between compacta. Then there is a function $f^* : F(Y) \to F(X)$ (between the closed set lattices) that satisfies the following conditions:*

(i) *f^* is a bounded-$\cup$-semilattice embedding that takes atoms to atoms and preserves disjointness.*
(ii) *For any $K \in F(Y)$, $f|f^*(K)$ is a co-existential map from $f^*(K)$ onto K.*
(iii) *For any $K \in F(Y)$, $f^{-1}[int(K)] \subseteq f^*(K)$ ($int(\ )$ is the topological interior operator).*
(iv) *Suppose $\langle K_1, \dots, K_n\rangle$ is an n-tuple of members of $F(Y)$ such that $\bigcap_{i=1}^n K_i = \emptyset$, and suppose that there exist $A_1, \dots, A_n$ in $F(X)$ such that $f^*(K_i) \subseteq A_i$, $1 \leq i \leq n$, $\bigcap_{i=1}^n A_i = \emptyset$,*

and $\bigcup_{i=1}^{n} A_i = X$. *Then there exist* $F_1, \ldots, F_n$ *in* $F(Y)$ *such that* $K_i \subseteq F_i$, $1 \leq i \leq n$, $\bigcap_{i=1}^{n} F_i = \emptyset$, *and* $\bigcup_{i=1}^{n} F_i = Y$.

(v) *If* $\mathcal{K}$ *is any class of compacta that is closed under ultracopowers and continuous images, and if* $K \in F(Y)$, *as a compactum, is in* $\mathcal{K}$, *then* $f^*(K)$ *is also in* $\mathcal{K}$.

Remark 2.9. **(i)** 2.8(i,ii,iii,v) is used to prove (see 2.5 in [5]) that co-existential maps preserve being infinite, being disconnected, being totally disconnected, being an indecomposable continuum, and being a hereditarily indecomposable continuum.
(ii) 2.8(iv) is used to prove (see 2.6 in [5]) that co-existential maps cannot raise dimension, and is an assertion that would follow easily if f^* were an existential embedding of lattices. We do not know whether this is true in general; although in the case that X is an ultracopower of Y and f is the codiagonal map, f^* may be taken to be elementary from $F(Y)$ into the canonical lattice base of closed set ultraproducts. (This lattice base is a far cry from $F(X)$, though.)

3. Co-existentially Closed Compacta.

A compactum (resp., continuum) is a **co-existentially closed compactum** (resp., **co-existentially closed continuum**) if it is co-existentially closed in the class of compacta (resp., continua). It should not be inferred that a co-existentially closed continuum is a co-existentially closed compactum that happens to be a continuum, any more than it should be inferred that a free abelian group is a free group that happens to be abelian (or that a jumbo shrimp is a jumbo-sized thing that happens to be a shrimp). In fact, as we see below, there is no such thing as a connected, co-existentially closed compactum; while there are plenty of co-existentially closed continua.

The following appears as 6.2 in [5]; we give a slightly different proof here.

Theorem 3.1. *The co-existentially closed compacta are precisely the Boolean spaces (i.e., the totally disconnected compacta) without isolated points.*

Proof: Suppose X is a co-existentially closed compactum, and let I be discrete, of the same cardinality as X. Let W be any

Boolean space with no isolated points. Then there is an obvious continuous surjection $f : \beta(I) \times W \to X$, which, by hypothesis, is co-existential. Now co-existential maps cannot raise dimension; hence X must be Boolean. Also, by 2.8(iii,v), they must preserve the property of having no isolated points.

Conversely, suppose X is Boolean, with no isolated points. By 2.6, there is a co-existentially closed compactum Y and a continuous $f : Y \to X$. Now, by the previous paragraph, Y is also Boolean, with no isolated points; hence the induced homomorphism $f^B : B(X) \to B(Y)$ of clopen algebras is an embedding between atomless Boolean algebras. By classic model theory (see [11]), the class of atomless Boolean algebras is model complete; hence every embedding in that class is elementary. This tells us that f is co-elementary (hence co-existential); and, by 2.7, that X is a co-existentially closed compactum. $\square$

4. Co-existentially Closed Continua.

The class of continua satisfies the conditions of 2.6; hence every infinite (equivalently, nondegenerate) continuum is a continuous image of a co-existentially closed continuum of the same weight. By 4.5 in [6], every co-existentially closed continuum is indecomposable and of covering dimension one. We can say a little more here, but we still have nothing like a characterization of the kind that 3.1 affords.

Theorem 4.1. *Every co-existentially closed continuum is indecomposable and of covering dimension one; every metrizable co-existentially closed continuum is hereditarily indecomposable. There exist at least two topologically distinct metrizable co-existentially closed continua; hence there is a metrizable co-existentially closed continuum that is not arc-like.*

Proof: Only the part after the first clause is new; so suppose X is a metrizable co-existentially closed continuum. By a result of D. Bellamy [9], every metrizable continuum is a continuous image of a hereditarily indecomposable metrizable continuum. Let $f : Y \to X$ witness this fact. Then f is co-existential. Now, by 2.5

of [5] (see 2.9(i) above), co-existential maps preserve (hereditary) indecomposability; hence X is hereditarily indecomposable.

Suppose there were only one (up to homeomorphism) metrizable co-existentially closed continuum X. By 2.6, then, every metrizable continuum would be a continuous image of X. This contradicts another result of D. Bellamy [7], however, that no metrizable continuum continuously surjects onto every metrizable continuum. Now only one metrizable hereditarily indecomposable continuum is arc-like, and that is the celebrated **pseudo-arc** (see R. H. Bing's early work; e.g., [10]). Thus there must be a metrizable co-existentially closed continuum that is not arc-like. □

Remark 4.2. (i) Re 4.1, we do not know whether the pseudo-arc is indeed a co-existentially closed continuum, but we do know several familiar examples that are not (e.g., solenoids, buckethandle continua, lakes of Wada, etc.) because they are not hereditarily indecomposable.
(ii) We do not know whether the Stone-Čech remainder of the half-line (well known ([8]) to be a non-metrizable indecomposable continuum that is not hereditarily indecomposable, and that continuously surjects onto every metrizable continuum) is a co-existentially closed continuum. If so, then there are co-existentially closed continua that are not hereditarily indecomposable.
(iii) A significant step towards an understanding of how the pseudo-arc relates to the co-elementary hierarchy would be deciding whether the classes of hereditarily indecomposable continua and arc-like continua are closed under ultracopowers and co-existential images. (We use a non-metric definition of "arc-like" here: For each open cover of X there is a continuous map from X to $[0, 1]$ whose point-preimages refine the open cover.) The only one of these questions that we can answer is the preservation of hereditary indecomposability by co-existential maps (2.9).
(iv) We note that the question of whether the class of arc-like continua is closed under *confluent* images is a problem first posed by A. Lelek ([14]).

5. A Characterization of Indecomposability.

As an application of the Limit Theorem (2.4 above), we can characterize indecomposable continua in terms of the kinds of self-maps they admit.

Recall that a map $f : X \to Y$ between continua is **indecomposable** if whenever A and B are subcontinua of X, whose union is X, then either $f[A] = Y$ or $f[B] = Y$. Every mapping with indecomposable range is indecomposable, of course; but there are lots of indecomposable maps between decomposable continua; e.g., the squaring, cubing, etc., maps on the unit circle in the complex plane, the tent maps on the unit interval.

The following well known result appears as 2.7 in [16].

Theorem 5.1. *Let $\langle X_m \overset{f_m}{\leftarrow} X_{m+1} : m < \omega \rangle$ be an inverse system of compacta, where each connecting map f_m is indecomposable. Then the inverse limit space is an indecomposable continuum.*

Our characterization theorem is now easy, given the results so far.

Theorem 5.2. *Let X be a continuum. Then X is indecomposable if and only if X admits a self-map that is both indecomposable and co-existential.*

Proof: Assume first that X is indecomposable. Then the identity map on X is both indecomposable and co-existential. For the converse, let $f : X \to X$ be indecomposable and co-existential, and let $\langle X_m \overset{f_m}{\leftarrow} X_{m+1} : m < \omega \rangle$ be the inverse system, where each X_m is X and each f_m is f. Then, by 5.1, the inverse limit space X_∞ is indecomposable. By the Limit Theorem (2.4), there is a co-existential map $g : X_\infty \to X$. Now co-existential maps preserve indecomposability, as mentioned in the proof of 4.1; hence X is an indecomposable continuum. $\square$

6. Other Special Types of Maps.

By far the most familiar class of specal maps in the study of continua (and compacta in general) is the class of **monotone** maps, those continuous surjections with connected point-preimages (the topological embodiment of the (order-)monotonic functions from analysis). Because we are in the compact setting, this is equivalent to saying that preimages of subcompacta are subcompacta. Hence there are natural generalizations of monotonicity, namely: A mapping $f : X \to Y$ is called **confluent** (resp., **weakly confluent**) (see, say, [16] for the provenance of these notions, as well as related ones) if whenever K is a subcontinuum of Y, then every (resp., some) component of $f^{-1}[K]$ maps onto K via f. In this section we are interested in relating the co-elementary hierarchy of maps between continua to the classes of monotone, confluent, and weakly confluent maps.

Remark 6.1. (i) Open maps between metric compacta are confluent, by an old result of G. T. Whyburn (13.14 of [16]).
(ii) 13.27 in [16] makes the same statement for confluent maps as the Composition Theorem (2.1) does for co-existential maps.
(iii) It is interesting to comtemplate what form the Mapping Structure Theorem (2.8) would take if "co-existential" were replaced by "monotone surjective," and the map f^* were f^{-1}. Clause (i) would have to be reformulated to read, "f^* is a bounded *lattice* (not just $\cup$-semilattice) embedding." Then preservation of disjointness would be automatic, but atoms would no longer necessarily go to atoms. Clauses (ii) and (iii) would go through intact, and Clause (v) would certainly go through for $\mathcal{K} = \{\text{continua}\}$; but Clause (iv) would break down because monotone maps can raise dimension (see [15], [17]).

Given a class $\mathcal{M}$ of maps between continua, there is a natural class of continua associated with $\mathcal{M}$, which, to be consistent with current usage in the continuum theory literature, we denote by Class($\mathcal{M}$). This class consists of those continua X such that if Y is any continuum and $f : Y \to X$ is any continuous surjection, then f is in $\mathcal{M}$. Clearly the class of co-existentially closed continua is precisely Class($\mathcal{M}$), where $\mathcal{M}$ is the class of co-existential maps between continua. Let us now refer to this class as Class(1). The general idea of studying Class($\mathcal{M}$) for various classes $\mathcal{M}$ is not new, and probably goes back to A. Lelek's topology seminar at the University of Houston in the 1970s. Lelek's Class(C) (resp., Class(W)) is just the metrizable members of Class($\mathcal{M}$), where $\mathcal{M}$ is the class of confluent (resp., weakly confluent) maps between continua. To avoid confusion in the sequel, we indicate the metrizable members of Class($\mathcal{M}$) with a subscript 0. In keeping with this, we redenote Lelek's classes as Class$(C)_0$ and Class$(W)_0$, respectively. Now, both of these classes have been characterized (see [16], [12] for details); what is most worthy of mention here is that Class$(C)_0$ consists precisely of the hereditarily indecomposable metrizable continua; hence, by 4.1, Class$(1)_0 \subseteq$ Class$(C)_0$. It is easy to show that every member of Class(C), metrizable or not, is hereditarily indecomposable; however the converse appears strongly to rely on metric considerations (see Exercise 13.72 in [16]). It would be nice to have a "weightless" characterization of this interesting class of continua.

Remark 6.2. **(i)** It is not always easy to show that Class($\mathcal{M}$) is nonempty; indeed the class is often trivial (in some sense). For example, let $\mathcal{M}$ be the class of monotone maps between continua. Now clearly, if X is degenerate, then $X \in$ Class($\mathcal{M}$). But if X is nondegenerate, we can simply "spot-weld" two copies of X together; i.e., form the quotient space $Y := (X \times \{0,1\})/(\{x_0\} \times \{0,1\})$, where x_0 is a fixed point of X. Then the natural projection $f : Y \to X$ fails to be monotone. On a similar note, if we define Class(2) := Class($\mathcal{M}$), where $\mathcal{M}$ is the class of level ≥ 2 maps between continua, then we can easily see that this class is empty. For if $X \in$ Class(2), then X is co-existentially closed; hence, by 4.1, it is of covering dimension one. Now, by taking products and using projection maps, it is always possible to find a continuous surjection $f : Y \to X$, where Y is a continuum of dimension ≥ 2. The property of being of covering dimension $\leq n$, for any finite n, is closed under ultracopowers and co-existential images. Hence, by the Complements Theorem (2.5), level ≥ 2 maps actually preserve covering dimension. So our f cannot be of level ≥ 2, a contradiction.
(ii) By the CC-Preservation Theorem (2.7), Class(1) is closed under co-existential images. Similar questions concerning analogous classes immediately suggest themselves; e.g, whether Class(C) (resp., Class(W)) is closed under confluent (resp., weakly confluent) maps. In the case of Class$(C)_0$, the answer is yes; for this class coincides with the class of metrizable hereditarily indecomposable continua,

a class closed under confluent images (see 7.2(ii) below). However, Class(W) is *not* closed under weakly confluent images. Indeed, {arc-like continua} $\subseteq$ Class(W) $\subseteq$ {unicoherent continua}, so the unit interval is in Class(W) and the unit circle is not. But the map that wraps the first twice around the second is weakly confluent.
(iii) We do not know whether Class(1) is closed under ultracopowers. If it is, then, by the Collapsing Theorem (2.3), every mapping between co-existentially closed continua is co-elementary. This situation is analogous to the model-theoretic notion that a particular first-order theory has a model companion. Also if so, then Class(1) $\subseteq$ {hereditarily indecomposable continua}, and we have a negative answer to the question in 4.2(ii). [Start with $X \in$ Class(1). Then (by a Löwenheim-Skolem-style argument) there is a metrizable continuum X_0 and a co-elementary map $f : X \to X_0$. But then, because Class(1) is closed under co-existential maps (see 2.7), X_0 is co-existentially closed, hence, hereditarily indecomposable (see 4.1). Because, by assumption, Class(1) is closed under ultracopowers, we have X as a co-elementary image of an ultracopower of X_0.])

The following appears in [5]. The first assertion is an immediate consequence of the Mapping Structure Theorem (2.8); the second, a somewhat less immediate consequence of 2.8, appears as 2.7 in [5].

Theorem 6.3. *Co-existential maps between compacta are always weakly confluent; they are monotone whenever the range space is locally connected.*

Remark 6.4. **(i)** We do not know whether co-existential maps between compacta are always confluent. If so, then Class(1) $\subseteq$ Class(C) $\subseteq$ {hereditarily indecomposable continua}, giving another negative answer to the question in 4.2(ii).
(ii) Co-elementary (in particular, co-existential) maps need not be monotone: Start with a co-existentially closed continuum X, fix $x_0 \in X$, and let Y be the quotient space $(X \times \{0,1\})/(\{x_0\} \times \{0,1\})$. Let $f : Y \to X$ be induced by projection onto the first factor. Then f is clearly not monotone, but it is co-existential. Let $g : XI \backslash \mathcal{D} \to Y$ witness this; i.e., $f \circ g = p_{X,\mathcal{D}}$. If $K \subseteq Y$ is arbitrary, then $f^{-1}[K] = g[p_{X,\mathcal{D}}^{-1}[K]]$. So if K is a subcontinuum of

Y that witnesses the nonmonotonicity of f, then it witnesses the nonmonotonicity of the co-elementary map $p_{X,\mathcal{D}}$ as well.

To facilitate the discussion, we identify an important strengthening of weak confluence. Define a map $f : X \to Y$ between continua to be **affluent** if whenever K is a subcontinuum of Y, then there is a subcontinuum C of X such that $f[C] = K$ and $f^{-1}[\mathrm{int}(K)] \subseteq C$. The Mapping Structure Theorem clearly shows that co-existential maps are not just weakly confluent; they're affluent. Letting Class(A) denote the continua that are images of affluent maps only, it is obvious that this class interpolates between Class(1) and Class(W).

Theorem 6.5. *Class(A) = Class(W)$\cap${indecomposable continua}.*

Proof: Let $X \in$ Class(A). We show X is indecomposable. If not, then there is a proper subcontinuum K of X, whose interior is nonempty. Let $x_0 \in X \setminus K$; and take Y to be the quotient space $(X \times \{0,1\})/(\{x_0\} \times \{0,1\})$, with $f : Y \to X$ the map induced by projection onto the first factor. Then $f^{-1}[K]$ consists of two components, and $f^{-1}[U]$ must intersect both of them whenever $U \subseteq K$ is nonempty and open in Y. Thus f is not affluent, a contradiction.

Conversely, if $X \in$ Class(W) is indecomposable and $f : Y \to X$ is a continuous surjection between compacta, then f is automatically affluent because no proper subcontinuum of X has nonempty interior. $\square$

Remark 6.6. **(i)** The second clause of 6.3 easily generalizes to the assertion that if $f : X \to Y$ is an affluent mapping between compacta, and if $y \in Y$ is a point at which Y is connected *im kleinen* (i.e., there is a neighborhood base at y of connected, not necessarily open, sets), then $f^{-1}[\{y\}]$ is a subcontinuum of X. The local behavior of y does not always influence the topological nature of $f^{-1}[\{y\}]$, however: Just start with a co-existentially closed continuum Y, let Z be any compactum whatsoever, and let X be the quotient space $(Y \times Z)/(\{y_0\} \times Z)$, where y_0 is a fixed point in Y. Then X is a continuum, and the natural map $f : X \to Y$ induced by projection onto the first factor is co-existential. So if $y \in Y$ is different from y_0 (chosen at random), then $f^{-1}[\{y\}]$ is homeomorphic to Z.

(ii) Monotone (even monotone-and-open) maps need not be co-existential, because they can raise dimension (see [15], [17]) and co-existential maps cannot (see 2.9 above).

(iii) Pursuant to the comments in 6.2(ii), we do not know whether Class(A) is closed under affluent maps.

7. Preservation of Decomposability and its Variants.

An important part of any inquiry into special kinds of continuous maps is their preservation of topological properties not preserved by continuous maps in general. In this section we consider the four properties, "(hereditarily) (in)decomposable."

Theorem 7.1. *Level ≥ 1 maps between continua preserve indecomposability, hereditary indecomposability, and hereditary decomposability. They do not preserve decomposability, however. Level ≥ 2 maps between continua preserve all four properties.*

Proof: The preservation of (hereditary) indecomposability is an easy application of the Mapping Structure Theorem (2.8), and appears as part of 2.5 in [5]. (Indecomposability is preserved by affluent maps in general.) The preservation of hereditary decomposability proceeds as follows: Co-existential maps are weakly confluent (6.3); weakly confluent maps preserve hereditary decomposability (not at all a trivial result, see Exercise 13.66 in [16]). Here is an easy example that co-existential maps do not preserve decomposability: Let $X \in$ Class(1), with $Y := X \times [0,1]$ and $f : Y \to X$ the canonical projection. Then X is indecomposable (4.1), Y is decomposable, and f is co-existential.

Finally, to prove that level ≥ 2 maps preserve decomposability, we use the Complements Theorem (2.5), where $\mathcal{K}$ is the class of indecomposable continua, and $\mathcal{M}$ is the class of continua. $\mathcal{K}$ is closed under level ≥ 1 images, as mentioned above, and is also closed under ultracopowers, by a theorem of R. Gurevič ([13]). $\square$

Remark 7.2. **(i)** Monotone mappings preserve hereditary decomposability because weakly confluent mappings do. They preserve indecomposability because they are affluent; they preserve hereditary indecomposability because restrictions of monotone maps to preimages of subcompacta are monotone onto those subcompacta (see also 6.1). They do not preserve decomposability, however, for almost exactly the same reason as why co-existential maps do not. (See the proof of 7.1 above.)
(ii) Confluent mappings preserve hereditary decomposability and hereditary indecomposability, but neither of the other two variants. The following simple example and argument come courtesy of Wayne Lewis (private communication): First, the n-adic solenoid ($n \geq 2$) is an inverse limit of circles, and the projection maps are open, hence confluent. So confluent maps can take indecomposable continua to (hereditarily) decomposable ones. Next, suppose $f : X \to Y$ is a confluent map between continua, and assume Y is not hereditarily indecomposable. Then there are subcontinua $A, B \subseteq Y$ such that $A \cup B$ is a subcontinuum containing both A and B properly. Let $y \in A \cap B$, with $y' \in f^{-1}[\{y\}]$. Let A' (resp., B') be the component of $f^{-1}[A]$ (resp., $f^{-1}[B]$) containing y'. Then A' and B' are overlapping subcontinua of X. Suppose $A' \subseteq B'$. Then $A = f[A'] \subseteq f[B'] = B$, contradicting our assumption about A and B. Likewise, we cannot have $B' \subseteq A'$; hence $A' \cup B'$ is a decomposable subcontinuum of X.
(iii) Weakly confluent mappings preserve hereditary decomposability only, among the four variants. Indeed, by 12.46 in [16], every continuous mapping onto an arc-like continuum (in particular, an arc) is weakly confluent. So if $K \subseteq [0,1]^2$ is hereditarily indecomposable, then either projection map onto $[0,1]$ takes K onto a nondegenerate subcontinuum of $[0,1]$; i.e., onto an arc. So weakly confluent images of hereditarily indecomposable continua can be hereditarily decomposable.

8. Preservation of Decomposability Degree.

One way of measuring how "decomposable" a continuum is is to count how many pairwise disjoint subcontinua-with-interior it can contain. For conciseness, let us call a subcontinuum **plump** if it has nonempty interior. The **decomposability degree** $d(X)$ of a compactum X is the least cardinal δ such that there is no family of pairwise disjoint plump *proper* subcontinua of X, of cardinality δ. In this section we consider the preservation of decomposability degree by the kinds of mappings under consideration.

Remark 8.1. **(i)** Note that $d(X) \geq 1$, by definition; and that, for a continuum, $d(X) = 1$ just in case that continuum is indecomposable. ($d(X) = 1$ also when X is a Boolean space without isolated points.) If X is locally connected and infinite, then $d(X) \geq \aleph_1$. If X is an indecomposable continuum and Y is a "spot weld" of two copies of X, then $d(Y) = 2$. By repeating this process any finite number of times (being careful to use different points of X for the welding), we obtain continua of any given finite decomposability degree.
(ii) The notion of "decomposability degree δ" is a renaming of the property of being of "width δ" in [5], [6]. We discovered that the word "width" had been introduced earlier as a cardinal invariant.

Theorem 8.2. *Level ≥ 1 maps between compacta cannot raise decomposability degree; they can actually lower it by an arbitrary amount. Level ≥ 2 maps preserve finite decomposability degree; also any level ≥ 2 image of a compactum of infinite decomposability degree must also have infinite decomposability degree.*

Proof: Indeed, affluent maps cannot raise decomposability degree. For if $f : X \to Y$ is an affluent map between compacta, and $\langle K_\alpha : \alpha < \kappa \rangle$ is a cardinal-indexed family of pairwise disjoint plump proper subcontinua of Y, then, by affluence, there is a likewise-indexed family $\langle C_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint plump proper subcontinua of X (where $f[C_\alpha] = K_\alpha$ and $f^{-1}[\mathrm{int}(K_\alpha)] \subseteq C_\alpha$. for each $\alpha < \kappa$).

To see why co-existential maps can properly lower decomposability degree, just review the proof of 7.1; i.e., the example showing that co-existential images of decomposable continua can be indecomposable. (In fact it is possible for $f : X \to Y$ to be a co-existential map between continua, where $d(Y) = 1$ and $d(X)$ is as large as we like.)

In order to prove that level ≥ 2 maps preserve finite decomposability degree (as well as the infinitude of decomposability degree), it suffices to show that this invariant is preserved by the taking of ultracopowers. Then we can invoke the Complements Theorem (2.5, $\mathcal{M}$ being the class of continua). It is clear that the decomposability of an ultracopower of X is at least as high as that of X because the co-diagonal map is co-existential. So start with a collection of n pairwise disjoint plump proper subcontinua of the ultracopower $XI\backslash\mathcal{D}$; and, for simplicity, consider the case $n = 2$ (easily seen to be an inessential loss of generality). The following argument assumes a reasonable familiarity with the ultracoproduct construction (see, e.g., [1]–[6] for details). Given the disjoint plump subcontinua K and M, we first note that, since they have nonempty interior, and the ultracoproducts of singletons in X form a set dense in the ultracopower, we can find points $\sum_{\mathcal{D}} x_i \in K$ and $\sum_{\mathcal{D}} y_i \in M$. (Also, in the special case $n = 1$, we need a point in the complement, witnessing that the subcontinuum is proper.)

Now we find disjoint ultracoproducts of closed subsets of X (such sets form a closed-set base for $XI\backslash\mathcal{D}$), $\sum_{\mathcal{D}} F_i \supseteq K$ and $\sum_{\mathcal{D}} H_i \supseteq M$. For each $i \in I$, let C_i be the component of x_i in F_i. Let C be the component of $\sum_{\mathcal{D}} x_i$ in $\sum_{\mathcal{D}} F_i$. Then clearly $\sum_{\mathcal{D}} C_i \subseteq C$. Suppose $P \in \sum_{\mathcal{D}} F_i \setminus \sum_{\mathcal{D}} C_i$. Then there is a closed-set ultraproduct $\prod_{\mathcal{D}} G_i \in P$ such that $J := \{i \in I : C_i \cap G_i = \emptyset\} \in \mathcal{D}$. Pick arbitrary $i \in J$. Since C_i is a component of F_i, and we are working with compacta, there is a set B_i, clopen in F_i, such that $C_i \subseteq B_i$ and $B_i \cap G_i = \emptyset$. Thus $\sum_{\mathcal{D}} B_i$ is clopen in $\sum_{\mathcal{D}} F_i$, contains $\sum_{\mathcal{D}} C_i$, and does not contain P. This shows that $P \notin C$; hence that $\sum_{\mathcal{D}} C_i$ is the component of $\sum_{\mathcal{D}} x_i$ in $\sum_{\mathcal{D}} F_i$. Thus $K \subseteq \sum_{\mathcal{D}} C_i$. Moreover, since $\sum_{\mathcal{D}} x_i$ is in the interior of $\sum_{\mathcal{D}} C_i$, $\{i \in I : x_i \text{ is in the interior of } C_i\} \in \mathcal{D}$.

Repeat the immediately preceding argument for the components E_i of y_i in H_i, $i \in I$. Then $\{i \in I : \{C_i, E_i\}$ is a family of two pairwise disjoint plump subcontinua of $X\} \in \mathcal{D}$. So if $d(XI\backslash\mathcal{D})$ is finite, then it is no larger than $d(X)$; furthermore, if $d(XI\backslash\mathcal{D})$ is infinite, then one can find families of pairwise disjoint plump subcontinua of X, of arbitrarily large finite cardinality. □

Remark 8.3. Of the mapping classes considered here, only the level ≥ 2 maps actually preserve decomposability degree to any extent. As seen in the proof of 8.2 (the example in 7.1 being monotone as well as co-existential), monotone maps may lower this invariant by an arbitrary amount; but affluent maps may not raise it. Confluent maps, on the other hand (see 7.2(ii)), may raise decomposability degree by in infinite amount.

References

[1] P. Bankston, "Reduced coproducts of compact Hausdorff spaces," *J. Symbolic Logic* **52**(1987), 404–424.

[2] ———, "Model-theoretic characterizations of arcs and simple closed curves," *Proc. A. M. S.* **104**(1988), 898–904.

[3] ———, "Taxonomies of model-theoretically defined topological properties," *J. Symbolic Logic* **55** (1990), 589–603.

[4] ———, "Co-elementary equivalence, co-elementary maps, and generalized arcs," *Proc. A. M. S.* **125** (1997), 3715–3720.

[5] ———, "Some applications of the ultrapower theorem to the theory of compacta," *Applied Categorical Structures* **8** (2000), 45–66.

[6] ———, "A hierarchy of maps between compacta," *J. Symbolic Logic* **64** (1999), 1628–1644.

[7] D. Bellamy, "Mappings of indecomposable continua," *Proc. A. M. S.* **30** (1971), 179–180.

[8] ———, "A non-metric indecomposable continuum," *Duke Math. J.* **38** (1971), 15–20.

[9] ———, "Continuous mappings between continua," (Topology Conference, 1979 (Greensboro, N. C., 1979), pp. 101–111, Guilford College, Greensboro, N. C., 1980.

[10] R. H. Bing, "Concerning hereditarily indecomposable continua," *Pac. J. Math* **1** (1951), 43–51.

[11] C. C. Chang and H. J. Keisler, "Model Theory (Third Edition)," North Holland, Amsterdam, 1989.

[12] J. Grispolakis and E. D. Tymchatyn, "Weakly confluent mappings and the covering property of hyperspaces," *Proc. A. M. S.* **74**(1979), 177–182.

[13] R. Gurevič, "On ultracoproducts of compact Hausdorff spaces," *J. Symbolic Logic* **53**(1988), 294–300.

[14] A. Lelek, "Some problems concerning curves," *Colloq. Math.* **23** (1971), 93–98.

[15] W. Lewis, "Monotone maps of hereditarily indecomposable continua," *Proc. A. M. S.* **75** (1979), 361–364.

[16] S. B. Nadler, Jr., "Continuum Theory, An Introduction," Marcel Dekker, New York, 1992.

[17] D. C. Wilson, "Open mappings of the universal curve onto continuous curves," *Trans. A. M. S.* **168** (1972), 497–515.

Marquette University, Milwaukee, WI 53201-1881
E-mail address: `paulb@mscs.mu.edu`