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# TOPOLOGICAL REDUCED PRODUCTS VIA GOOD ULTRAFILTERS

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Good ultrafilters produce topological ultraproducts which enjoy a strong Baire category property (depending upon how good the ultrafilter is). We exploit this property to prove a "uniform boundedness" theorem as well as a theorem which says that, under the Generalized Continuum Hypothesis (GCH), many ultraproduct spaces have families consisting of closed discrete sets of high cardinality such that every nonempty open set contains one of these sets. In another section we relate the strong Baire properties to the infinite distributivity of Boolean Algebras of regular open sets. Finally, we prove that, under the GCH, a great many topological ultrapowers are homeomorphic to the corresponding ultrapower of the space of rational numbers; and we show further that the GCH is indispensible to the proof. A purely model-theoretic application of our methods solves a problem related to the Keisler–Shelah Ultrapower Theorem.

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## 0. Introduction

This report is a continuation of [3, 4, 5] and an expansion of the results announced in [6, 7]. Our notation will follow current usage as much as possible (with [8, 11, 26] as our main references); but we will assume some familiarity with the conventions established in our earlier works. The central theme is the construction of topological ultraproducts using regular and (especially) good ultrafilters. The ultraproduct construction, traditionally a part of model theory, has been investigated in a topological context and has proved to be an interesting source of uncountable zero-dimensional spaces. Here we use good ultrafilters to obtain spaces which have the Baire category property in higher cardinals as well as other combinatorial properties (including regular-open algebras which have high-cardinal distributivity).

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The broad motivation for studying topological ultraproducts is to establish a general theory of topological reduced products, of which the familar box product is a special case.

To review the basic definitions, let $\kappa$ be an infinite cardinal number and let $\langle X_\alpha : \alpha < \kappa \rangle$ be a $\kappa$-sequence of topological spaces. An *open box* is a Cartesian product $\prod_\kappa U_\alpha$ where $U_\alpha \in \tau_\alpha$, the topology on $X_\alpha$. The Cartesian product $\prod_\kappa X_\alpha$, together with the open boxes, form a space called the *box product* (again denoted $\prod_\kappa X_\alpha$). Let $D$ be a filter of subsets of $\kappa$, and define, for $x, y \in \prod_\kappa X_\alpha$, $x \underset{D}{\sim} y$ iff $\{\alpha : x_\alpha = y_\alpha\} \in D$. This relation is clearly an equivalence, and the quotient space $\prod_D X_\alpha = \prod_\kappa X_\alpha / \tilde{D}$ is the *topological reduced product via D*. The natural projection $\Gamma_D : \prod_\kappa X_\alpha \to \prod_D X_\alpha$ is an open map; and $\Gamma_D(\prod_\kappa U_\alpha) = \prod_D U_\alpha = \{[x]_D : \{\alpha : x_\alpha \in U_\alpha\} \in D\}$ is called an *open reduced box* when the $U_\alpha$ are open in the corresponding $X_\alpha$. A very elementary but important observation is that whenever $\mathcal{B}_\alpha$ is a basis for $\tau_\alpha$ then $\prod_D \mathcal{B}_\alpha = \{\prod_D U_\alpha : U_\alpha \in \mathcal{B}_\alpha$ all $\alpha < \kappa\}$ is a basis for the reduced product topology. We also note that when $D = \{\kappa\}$, the $D$-reduced product is just the box product; and when $D$ is an ultrafilter $\prod_D X_\alpha$ is the *D-ultraproduct* of the $X_\alpha$'s.

Because we will need to look at topological ultraproducts from the standpoint of model theory, we take time out here to describe reduced products in an alternate form: If $\langle X, \tau \rangle$ is a space we treat it as a relational structure $\langle X \cup \tau ; X, \tau, \underset{\sim}{\in} \rangle$ with universe $X \cup \tau$, unary relations $X, \tau$, and the binary relation $\underset{\sim}{\in}$ of membership between elements of $X$ and elements of $\tau$. The topological reduced product $\prod_D X_\alpha$ is then the structure $\langle \prod_D X_\alpha \cup \prod_D \tau_\alpha ; \prod_D X_\alpha, (\prod_D \tau_\alpha)^*, \prod_D \underset{\sim}{\in}_\alpha \rangle$, where $(\prod_D \tau_\alpha)^*$ is the topology generated by $\prod_D \tau_\alpha$ (clearly a topological base). The relation $\prod_D \underset{\sim}{\in}_\alpha$ is no longer "real" membership ($[x]_D \prod_D \underset{\sim}{\in}_\alpha \prod_D U_\alpha$ iff $\{\alpha : x_\alpha \underset{\sim}{\in}_\alpha U_\alpha\} \in D$), but it is isomorphic in the model-theoretic sense to membership, so no real difficulties arise in that regard. We will return to the model-theoretic approach to topological ultraproducts in a later section where we prove the rather surprising result that whenever $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of perfect regular spaces of cardinal + weight $\leq \exp(\kappa)$ (here "perfect" means "having no isolated points") and $D$ is a good ultrafilter on $\kappa$ then $\prod_D X_\alpha \simeq \prod_D(\mathbb{Q})$, provided $\exp(\kappa) = \kappa^+$ ($\mathbb{Q}$ denotes the space of rational numbers, and $\prod_D(Y) = \prod_D Y_\alpha$ where each $Y_\alpha$ is $Y$. Also "$\simeq$" denotes homeomorphism, not model-theoretic isomorphism which is signified by "$\cong$").

## 1. Basic concepts and combinatorial lemmas

Let $\kappa$ be a cardinal number. A space $X$ is *$\kappa$-Baire* if intersections of $<\kappa$ dense open sets are dense. $X$ is *$\kappa$-additive* (here we follow the usage of Sikorski [23]. In [3, 4, 5] we used the terminology "$\kappa$-open"; and in [12] the designation "$P_\kappa$-space" is given) if intersections of $<\kappa$ open sets of $X$ are open. $\omega_1$-additive spaces are popularly known as *P-spaces.*

Let $F$ be a family of sets, with $\kappa, \lambda \leqslant \infty$ cardinals (we abuse notation slightly). $F$ is *$\langle\kappa, \lambda\rangle$-compact* if whenever $F_0 \subseteq F$ has power $<\kappa$ and every $F_1 \subseteq F_0$ of power $<\lambda$ has nonempty intersection, then $F_0$ has nonempty intersection. A space $X$ is *$\kappa$-compact* if the collection of closed subsets of $X$ is $\langle\infty, \kappa\rangle$-compact. We borrow some terminology from model theory and say that $X$ is *weakly $\kappa$-saturated* if $X$ has an open basis which is $\langle\kappa, \omega\rangle$-compact.

**Remark.** All spaces are $\omega$-Baire, $\omega$-additive, and weakly $\omega$-saturated; the Baire spaces are precisely the $\omega_1$-Baire spaces; and Martin's Axiom is equivalent to the assertion that every compact Hausdorff space with the countable chain condition is $c$-Baire, where $c = \exp(\omega) =$ the power of the Continuum.

In the sequel we denote the set of subsets of $S$ by $\mathbb{P}(S)$, with $\mathbb{P}_\kappa(S)$ denoting $\{A \subseteq S: |A| < \kappa\}$ for $\kappa$ a cardinal. Let $D$ be an ultrafilter on $\kappa$ and let $\lambda$ be any cardinal. $D$ is *$\lambda$-regular* if there is a "regularizing" set $E \subseteq D$ of power $\lambda$ such that every $\xi < \kappa$ is contained in only finitely many members of $E$ (i.e. $E$ is point-finite). $D$ is *$\lambda$-good* if $D$ is countably incomplete (i.e. $D$ is not closed under countable intersections) and for all $\mu < \lambda$ any "monotone" $F: \mathbb{P}_\omega(\mu) \to D$ ($F$ is *monotone* if $F$ is order-reversing, i.e. $s \subseteq t \Rightarrow F(s) \supseteq F(t)$) "dominates" a "multiplicative" $G: \mathbb{P}_\omega(\mu) \to D$ (i.e. there is a function G such that $G(s) \subseteq F(s)$ for all $s \in \mathbb{P}_\omega(\mu)$ and $G(s \cup t) = G(s) \cap G(t)$ for all $s, t \in \mathbb{P}_\omega(\mu)$).

We collect some combinatorial results about regular and good ultrafilters, and about the ultraproducts they form.

**1.1. Proposition.** *Every countably incomplete ultrafilter is $\omega_1$-good.*

**Proof.** This is a standard result and a straightforward application of the definition. □

**1.2. Proposition.** *Let $D$ be $\lambda$-good on $\kappa$. Then $D$ is $\mu$-regular for all $\mu < \lambda$.*

**Proof.** This result is also standard (see [8, 11]) but less straightforward, so we include a proof.

First, since $D$ is countably incomplete, we can find a sequence $\kappa = I_0 \supseteq I_1 \supseteq \cdots$ of elements of $D$ whose intersection is empty. Define $F: \mathbb{P}_\omega(\mu) \to D$ by $F(s) = I_{|s|}$. Clearly $F$ is monotone, so since $D$ is $\lambda$-good there is a multiplicative $G \leqslant F$ (i.e. $G$ is dominated by $F$). Let $\Sigma \subseteq \mathbb{P}_\omega(\mu)$ be the singleton sets and define $E = \operatorname{rng}(G | E)$. We show $E$ is a $\mu$-regularizing set for $D$. Now given $\xi < \kappa$, $\{\eta < \mu: \xi \in G(\{\eta\})\}$ can have no more elements than the number $n(\xi) = \max\{m: \xi \in I_m\}$. For suppose $\eta_1, \ldots, \eta_m$ are distinct with $\xi \in G(\{\eta_i\})$ all $1 \leqslant i \leqslant m$. Then by multiplicativity,

$$\xi \in G(\{\eta_1, \ldots, \eta_m\}) \subseteq F(\{\eta_1, \ldots, \eta_m\}) = I_m,$$

so $m \leqslant n(\xi)$. Thus $E$ is point-finite. To show $|E| = \mu$, we know that $|\Sigma| = \mu$ and for any $\xi < \kappa$, $|G^{-1}(G(\{\xi\}))| \leqslant n(\xi)$. Thus the point inverses of $G$ are finite. □

**Remark.** No ultrafilter on $\kappa$ can be $\kappa^+$-regular; and one can show quite easily that $\kappa$-regular ultrafilters abound. Thus, at best, we can hope for the existence of $\kappa^+$-good ultrafilters on $\kappa$. This was originally proved to be true by Keisler using $\kappa^+ = \exp(\kappa)$; and subsequently by Kunen using only ZFC (see [12, 17]). If $D$ is an ultrafilter on $\kappa$, we say $D$ is *regular* if $D$ is $\kappa$-regular and $D$ is *good* if $D$ is $\kappa^+$-good.

We next turn to the cardinality of ultraproducts.

**1.3. Proposition.** *Let $S$ be an infinite set, $D$ a regular ultrafilter on $\kappa$. Then* $|\prod_D (S)| = |S^\kappa|$.

**Proof.** This is a well-known result (see [8, 11]). □

**1.4. Proposition.** *Let $D$ be a countably incomplete ultrafilter on $\kappa$ and assume that $\langle n_\alpha : \alpha < \kappa\rangle$ is a sequence of natural numbers such that for all $I \in D$, $\sup\{n_\alpha : \alpha \in I\} = \omega$. Then $|\prod_D n_\alpha| \geqslant c$. Moreover if $D$ is good, then $|\prod_D n_\alpha| = \exp(\kappa)$.*

**Proof.** The first assertion is well-known and can be found in [8, 11]. The second assertion is due to Keisler and Prikry [15]. The proof proceeds as follows: Define $F : \mathbb{P}_\omega(\kappa) \to D$ by $F(s) = \{\alpha < \kappa : n_\alpha \geqslant 2^{|s|}\}$. $F$ is monotone so let $G \leqslant F$ be multiplicative. For $\alpha < \kappa$ let $s_\alpha = \{\beta < \kappa : \alpha \in G(\{\beta\})\}$, a finite set. Then $\alpha \in G(s_\alpha) \subseteq F(s_\alpha)$ so $n_\alpha \geqslant 2^{|s_\alpha|}$. Let $H_\alpha$ map $2^{s_\alpha}$ one-one into $n_\alpha$. For $f \in 2^\kappa$ define $\bar{f} \in \prod_\kappa n_\alpha$ by $\bar{f}(\alpha) = H_\alpha(f | s_\alpha)$.
If $f, g \in 2^\kappa$, $f_\beta \neq g_\beta$, then for all $\alpha \in G(\{\beta\})$ we have $\beta \in s_\alpha$, so $f | s_\alpha \neq g | s_\alpha$ and $\bar{f}_\alpha \neq \bar{g}_\alpha$; whence $f \neq g$ implies $[\bar{f}]_D \neq [\bar{g}]_D$ and $\exp(\kappa) \leqslant |\prod_D n_\alpha|$. □

The next proposition is crucial to the establishment of Baire properties for topological ultraproducts.

**1.5. Proposition.** *Let $\langle S_\alpha : \alpha < \kappa\rangle$ be a $\kappa$-sequence of sets with $D$ a $\lambda$-good ultrafilter on $\kappa$. Then any family of ultraboxes in $\prod_D S_\alpha$ is $\langle \lambda, \omega\rangle$-compact.*

**Proof.** This is proved in a manner similar to the way in which one proves that $\lambda$-good ultraproducts of relational structures are $\lambda$-saturated in the sense of Morley and Vaught (see [8, 11]).

Fix $\kappa = I_0 \supseteq I_1 \supseteq \cdots$ as in the proof of Proposition 1.2 and let $\mu < \lambda$ with $\mathcal{M} = \langle \prod_D M_{\alpha,\xi} : \xi < \mu\rangle$ a family of $\mu$ ultraboxes from $\prod_D S_\alpha$ with the finite intersection property. We show $\bigcap \mathcal{M} \neq \emptyset$. So let $F : \mathbb{P}_\omega(\mu) \to D$ be given by

$$F(s) = I_{|s|} \cap \left\{\alpha : \bigcap_{\xi \in S} M_{\alpha,\xi} \neq \emptyset\right\}.$$

$F$ is monotone so let $G \leqslant F$ be multiplicative, and define $s_\alpha = \{\xi < \mu : \alpha \in G(\{\xi\})\}$, a finite set (since $G$ is multiplicative, $|s_\alpha| \leqslant n(\alpha) = \max\{m : \alpha \in I_m\}$). Now for each $\alpha < \kappa$

define

$$x_\alpha = \begin{cases} \text{any member of } \bigcap_{\xi \in s_\alpha} M_{\alpha,\xi} & \text{if there is one,} \\ \text{arbitrary} & \text{otherwise,} \end{cases}$$

and fix $\xi < \mu$. Then $\{\alpha : x_\alpha \in M_{\alpha,\xi}\} \supseteq \{\alpha : \xi \in s_\alpha\} \supseteq G(\{\xi\}) \in D$, so $[x]_D \in \bigcap \mathcal{M}$. □

**1.6. Corollary.** *Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a $\kappa$-sequence of topological spaces with $D$ a $\lambda$-good ultrafilter on $\kappa$. Then $\prod_D X_\alpha$ is weakly $\lambda$-saturated.* □

## 2. The Baire category theorem

A central lemma of [3, 4, 5] will be often quoted in the sequel, so we state and prove it here.

**2.1. Proposition.** *Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a $\kappa$-sequence of topological spaces with $D$ a $\lambda$-regular ultrafilter on $\kappa$. Then $\prod_D X_\alpha$ is $\lambda^+$-additive.*

**Proof.** Let $E$ be a regularizing subset of $D$, say $E = \langle I_\xi : \xi < \lambda \rangle$. It suffices to show that if $\mathfrak{U} = \langle \prod_D U_{\alpha,\xi} : \xi < \lambda \rangle$ is a family of $\lambda$ open ultraboxes and if $[x]_D \in \bigcap \mathfrak{U}$ then there is an open ultrabox $\prod_D U_\alpha$ with $[x]_D \in \prod_D U_\alpha \subseteq \bigcap \mathfrak{U}$. For each $\xi < \lambda$ let $J_\xi = \{\alpha < \kappa : x_\alpha \in U_{\alpha,\xi}\} \in D$ and for $\alpha < \kappa$ define $s_\alpha = \{\xi < \lambda : \alpha \in I_\xi \cap J_\xi\}$. Then each $s_\alpha$ is finite so define $U_\alpha = \bigcap_{\xi \in s_\alpha} U_{\alpha,\xi}$.
Now $\prod_D U_\alpha$ is an open ultrabox and $\{\alpha < \kappa : x_\alpha \in U_\alpha\} = \kappa \in D$ so $[x]_D \in \prod_D U_\alpha$. And if $\xi < \lambda$ then $\{\alpha < \kappa : U_\alpha \subseteq U_{\alpha,\xi}\} \supseteq \{\alpha < \kappa : \xi \in s_\alpha\} = I_\xi \cap J_\xi \in D$. Thus $\prod_D U_\alpha \subseteq \bigcap \mathfrak{U}$. □

We can now prove our first theorem.

**2.2. Theorem.** *Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a $\kappa$-sequence of topological spaces with $D$ a $\lambda$-good ultrafilter on $\kappa$. Then $\prod_D X_\alpha$ is $\lambda$-Baire.*

**Proof.** By Proposition 1.2, Corollary 1.6 and Proposition 2.1 we have that $\prod_D X_\alpha$ is $\mu^+$-additive for all $\mu < \lambda$ (hence $\lambda$-additive), and weakly $\lambda$-saturated. We show this to be sufficient for $\prod_D X_\alpha$ to be $\lambda$-Baire. Thus we prove that for any space $X$, if $X$ is $\lambda$-saturated and $\mu$-additive for all $\mu < \lambda$, then $X$ is $\lambda$-Baire.

First pick an open basis $\mathcal{B}$ for $X$ which is $\langle \lambda, \omega \rangle$-compact, pick $\mu < \lambda$, and let $\mathfrak{U} = \langle U_\xi : \xi < \mu \rangle$ be dense open sets with $B \in \mathcal{B}$ nonempty. We show $B \cap \bigcap \mathfrak{U} \neq \emptyset$ by transfinite induction. Proceeding as in the usual proofs of the Baire Property, let $B_0 \in \mathcal{B}$ be such that $\emptyset \neq B_0 \subseteq B \cap U_0$. This is possible since $U_0$ is dense open. Now for

$0<\delta<\mu$, if $\delta=\xi+1$ then let $B_\delta \subseteq B_\xi \cap M_\delta$. If $\delta$ is a limit ordinal we wish to define $B_\delta$ in terms of $M_\delta$ and the decreasing chain $\langle B_\xi : \xi<\delta\rangle$. This can be done if $\bigcap_{\xi<\delta} B_\xi \cap M_\delta$ is nonempty and open; for we can just let $B_\delta \neq \emptyset$ be an element of $\mathscr{B}$ contained in $\bigcap_{\xi<\delta} B_\xi \cap M_\delta$. Since $M_\delta$ is dense open it suffices to show $\bigcap_{\xi<\delta} B_\xi$ to be nonempty open. But this is true since $|\delta|<\mu$; and $X$ is $\mu$- additive as well as weakly $\lambda$-saturated. We now have a decreasing chain $\langle B_\xi : \xi<\mu\rangle$ of nonempty basic open sets so again use weak $\lambda$-saturicity to conclude $\bigcap_{\xi<\mu} B_\xi \neq \emptyset$. But this set is contained in $B \cap \bigcap \mathfrak{U}$. □

Noting that the collection of dense open subsets of any topological space has the finite intersection property (it is indeed a filterbase), we have the following corollary.

**2.3. Corollary.** *Let $\langle X_\alpha : \alpha<\kappa\rangle$ be a sequence of spaces with $D$ a good ($=\kappa^+$-good) ultrafilter on $\kappa$. Then the filterbase of dense open subsets of $\prod_D X_\alpha$ is $\kappa^+$-complete (i.e. closed under $<\kappa^+$ intersections).* □

**Remarks.** (i) By Proposition 1.1 and Theorem 2.2 every countably incomplete ultraproduct is Baire ($=\omega_1$-Baire) regardless of the topologies on the factor spaces $X_\alpha$. They are also $P$-spaces (i.e. $\omega_1$-additive) which means that, unless they are discrete, they cannot be complete metric or compact Hausdorff, the usual antecedents for Baire-ness. In fact it is a fairly simple exercise to show that non-discrete $P$-spaces can never be Čech complete (We haven't checked out whether they can be co-compact in the sense of [1], but it seems doubtful that they are).

(ii) Comfort and Negrepontis have some results in [12] (vide Theorems 6.15, 15.8) about the $\kappa$-Baire property. Their proofs do not differ in spirit from ours (or indeed from the classical proofs).

(iii) The class of Baire spaces is closed under the taking of ultraproducts; for countably complete ultraproducts clearly preserve this property and countably incomplete ultraproducts create the Baire property for free (thus the class of non-Baire spaces is not closed under ultraproducts).

(iv) Since $\kappa$-good ultrafilters exist for any prescribed $\kappa$, we have that any space $X$ has ultrapowers which are $\kappa$-Baire for arbitrary $\kappa$.

(v) The converse of Proposition 2.1 is also true. That is the ultrafilter-theoretic property of $\lambda$-regularity of $D$ is characterized by the fact that topological $D$-ultraproducts are $\lambda^+$-additive (see [4]). Keisler originally proved (see [11, Problem 4.3.32]) that $D$ is $\lambda$-regular iff $D$-ultraproducts of relational structures are $\lambda^+$-universal (where the associated language has $\leqslant\lambda$ symbols).
A similar problem exists for characterizing $\lambda$-goodness of ultrafilters by what topological properties they confer on topological ultraproducts (for the model-theoretic analogue see [11, problem 6.1.17]). In particular is there an ultrafilter $D$ which is not $\lambda$-good such that $\prod_D X_\alpha$ is always $\lambda$-Baire?

## 3. Uniform boundedness

In this and in the next two sections we explore some of the consequences of the $\kappa$-Baire property, in ultraproducts and for general spaces as well.

Assume that $X$ is a compact Hausdorff space and that $D$ is an ultrafilter. Then (see [4] for details) there is a map $\lim_D : \prod_D(X) \to X$, the *D-limit map*, which is a left inverse for the diagonal map $\Delta_D : X \to \prod_D(X)$ taking $x \in X$ to the $D$-equivalence class of the constant map at $x$. Now although $\Delta_D$ is rarely continuous, $\lim_D$ always is; so we make the following definition: Let $f: X \to Y$ be a continuous map with $Y$ Hausdorff. $f$ is *compact* if $f[X]$ $(=\{f(x): x \in X\})$ has compact closure in $Y$. If $F$ is a family of continuous maps from $X$ to $Y$, we say $F$ is *compact at* $x \in X$ if $F(x) = \{f(x): f \in F\}$ has compact closure in $Y$. Similarly define "$F$ is compact on $A \subseteq X$". Now if $D$ is an ultrafilter on $\kappa$ and $f: X \to Y$ is compact we define $f^D : \prod_D(X) \to Y$ by the composition

$$\prod_D(X) \xrightarrow{\prod_D(f)} \prod_D(\overline{f[X]}) \xrightarrow{\lim_D} \overline{f[X]} \subseteq Y.$$

Clearly $f^D$ is a compact continuous map. If $F$ is a family of compact continuous maps we define $F^D$ in the obvious way.

**3.1. Theorem.** *Suppose $F$ is a family of compact continuous maps from $X$ to the Hausdorff space $Y$ such that there is a nonempty open $U \subseteq X$ for which $F$ is compact at each point of $U$. Then there is an ultrafilter $D$ and a nonempty open $V \subseteq \prod_D(X)$ such that $F^D$ is compact on $V$.*

**Proof.** Let $\kappa$ be the cardinality of the set $k_Y$ of compact subsets of $Y$ (the cofinality of $k_Y$ as a directed set will do), and let $D$ be a good ultrafilter on $\kappa$. Then $\prod_D(X)$ is $\kappa^+$-Baire. By hypothesis $F$ is compact at each point of $U \subseteq X$. Thus $F^D$ is compact at each point of $\prod_D(U) \subseteq \prod_D(X)$, a nonempty open set. For each $K \in k_Y$ let $V_K = \{[x]_D : F^D([x]_D) \not\subseteq K\}$. Then $V_K$ is open in $\prod_D(X)$. If all the $V_K$'s were dense it would follow that $\bigcap\{V_K : K \in k_Y\}$ is also dense. But then $F^D$ would not be compact at some point of $\prod_D(U)$, a contradiction. So let $V$ be a nonempty open set missing some $V_K$. $\square$

**Remark.** The above theorem is a topological ultraproduct analogue of the classical Banach–Steinhaus theorem. Its proof doesn't differ greatly in spirit from the classical one.

## 4. The regular open algebra of a $\kappa$-Baire space

Here we relate the $\kappa$-Baire property to a property of Boolean algebras. We were led to our result after reading Lemma C of Mansfield [18] which states essentially that if a $P$-space $X$ has an open basis such that every countable decreasing sequence of nonempty basic sets is nonempty then the algebra of regular open sets of $X$ is $\langle\omega_1, \infty\rangle$-distributive. We improve on this result (which is only stated in [18]) in the present section, after first establishing some notation.

If $X$ is a space and $A \subseteq X$ then the closure of $A$, previously denoted $\bar{A}$, will also be denoted $A^{-}$. The interior of $A$ will be denoted $A^{0}$. An open $U \subseteq X$ is *regular open* if $U = U^{-0}$. The set $\mathcal{R}X$ of regular open sets can be made into a complete Boolean algebra by defining $\bigvee_{i\in I} U_i = (\bigcup_{i\in I} U_i)^{-0}$ and $U' = (X - U)^0$. Two well-known facts about $\mathcal{R}X$ are:

(i) that if $X$ is regular then $\mathcal{R}X$ forms an open basis for $X$ ("semiregularity"); and

(ii) that for any $U, V \in \mathcal{R}X$, $U \wedge V = U \cap V$ (indeed, if $U_1, \ldots, U_n$ are open then $(U_1 \cap \cdots \cap U_n)^{-0} = U_1^{-0} \cap \cdots \cap U_n^{-0}$).

A complete Boolean algebra $B$ is $\langle\alpha, \beta\rangle$-distributive, where $\omega \leqslant \alpha$, $\beta \leqslant \infty$ are cardinals, if for any $\kappa < \alpha$, $\lambda < \beta$ and any $\kappa \times \lambda$-indexed sequence $\langle a_{\xi,\eta} : \xi < \kappa, \eta < \lambda\rangle$ of elements of $B$,

$$\bigwedge_{\xi<\kappa} \bigvee_{\eta<\lambda} a_{\xi,\eta} = \bigvee_{\sigma\in\lambda^\kappa} \bigwedge_{\xi<\kappa} a_{\xi,\sigma(\xi)}.$$

We remark that our definition differs inessentially from that given in Sikorski [22]; and that the above equation holds in $B$ iff its dual holds as well, since $B$ is complete.

**4.1. Theorem.** *Let $X$ be a regular $\kappa$-additive space, where $\kappa \geqslant \omega$ is a cardinal. The following are equivalent:*

(i) *For any $\lambda < \kappa$ and any sequence $\langle U_\xi : \xi < \lambda\rangle$ of open sets, $(\bigcap_{\xi<\lambda} U_\xi)^{-0} = \bigcap_{\xi<\lambda} U_\xi^{-0}$.*

(ii) *$X$ is $\kappa$-Baire.*

(iii) *$\mathcal{R}X$ is $\langle\kappa, \infty\rangle$-distributive.*

**Proof.** (i)$\Rightarrow$(ii). Let $\langle U_\xi : \xi < \lambda\rangle$ be a sequence of dense open sets, $\lambda < \kappa$. Then $U_\xi^{-0} = X$ for each $\xi$. By (i), $\bigcap_{\xi<\lambda} U_\xi$ is dense.

(i)$\Rightarrow$(iii). Let $\langle U_{\xi,i} : \xi < \lambda, i \in I\rangle$ be a doubly indexed sequence of regular open sets. By (i)

$$\begin{aligned}\bigwedge_{\xi<\lambda} \bigvee_{i\in I} U_{\xi,i} &= \Big(\bigcap_{\xi<\lambda} \Big(\bigcup_{i\in I} U_{\xi,i}\Big)^{-0}\Big)^{-0} = \bigcap_{\xi<\lambda} \Big(\bigcup_{i\in I} U_{\xi,i}\Big)^{-0} \\ &= \Big(\bigcap_{\xi<\lambda} \bigcup_{i\in I} U_{\xi,i}\Big)^{-0} = \Big(\bigcup_{\sigma\in I^\lambda} \bigcap_{\xi<\lambda} U_{\xi,\sigma(\xi)}\Big)^{-0} \\ &= \Big(\bigcup_{\sigma\in I^\lambda} \Big(\bigcap_{\xi<\lambda} U_{\xi,\sigma(\xi)}\Big)^{-0}\Big)^{-0} \\ &= \bigvee_{\sigma\in I^\lambda} \bigwedge_{\xi<\lambda} U_{\xi,\sigma(\xi)}.\end{aligned}$$

(iii)⇒(i). Let $\langle U_\xi: \xi<\lambda\rangle$ be given. Now $\bigcap_{\xi<\lambda} U_\xi \subseteq U_\eta$ for each $\eta<\lambda$ so $(\bigcap_{\xi<\lambda} U_\xi)^{-0} \subseteq \bigcap_{\xi<\lambda} U_\xi^{-0}$ always. For the reverse inclusion, assume first that the $U_\xi$'s are regular open. By $\kappa$-additivity $\bigcap_{\xi<\lambda} U_\xi^{-0} = \bigcap_{\xi<\lambda} U_\xi$ is open, hence it is contained in $(\bigcap_{\xi<\lambda} U_\xi)^{-0}$. Back to the general case, we have that $X$ is regular. Thus we can write $U_\xi = \bigcup_{i\in I} U_{\xi,i}$ where $U_{\xi,i}$ is regular open. Now

$$\Big(\bigcap_{\xi<\lambda} U_\xi\Big)^{-0} = \Big(\bigcap_{\xi<\lambda}\bigcup_{i\in I} U_{\xi,i}\Big)^{-0} = \Big(\bigcup_{\sigma\in I^\lambda}\bigcap_{\xi<\lambda} U_{\xi,\sigma(\xi)}\Big)^{-0} = \Big(\bigcup_{\sigma\in I^\lambda}\Big(\bigcap_{\xi<\lambda} U_{\xi,\sigma(\xi)}\Big)^{-0}\Big)^{-0}$$

since each $U_{\xi,\sigma(\xi)}$ is regular open. This last expression is

$$\bigvee_{\sigma\in I^\lambda}\bigwedge_{\xi<\lambda} U_{\xi,\sigma(\xi)} = \bigwedge_{\xi<\lambda}\bigvee_{i\in I} U_{\xi,i}$$

by (iii). This is now

$$\Big(\bigcap_{\xi<\lambda}\Big(\bigcup_{i\in I} U_{\xi,i}\Big)^{-0}\Big)^{-0} = \Big(\bigcap_{\xi<\lambda} U_\xi^{-0}\Big)^{-0} \supseteq \bigcap_{\xi<\lambda} U_\xi^{-0}$$

since, by $\kappa$-additivity, the intersection is open. The desired equality thus holds.

(ii)⇒(i). To show $(\bigcap_{\xi<\lambda} U_\xi)^{-0} \supseteq \bigcap_{\xi<\lambda} U_\xi^{-0}$ we just show $(\bigcap_{\xi<\lambda} U_\xi)^{-} \supseteq \bigcap_{\xi<\lambda} U_\xi^{-0}$. This will do since $\bigcap_{\xi<\lambda} U_\xi^{-0}$ is open. To obtain a contradiction, suppose $x\in\bigcap_{\xi<\lambda} U_\xi^{-0} - (\bigcap_{\xi<\lambda} U_\xi)^- = V$, an open neighborhood of $x$ by $\kappa$-additivity. Then for each $\eta<\lambda$, $V\subseteq U_\eta^{-0}$ so $V\subseteq U_\eta^-$; whence $V^-\subseteq U_\eta^-$. Also we have $V\cap(\bigcap_{\xi<\lambda} U_\xi)^- = \emptyset$. Now when, $M, N$ are disjoint open sets then $(M^-\cap N^-)^0 = M^{-0}\cap N^{-0} = (M\cap N)^{-0} = \emptyset$, so $M^-\cap N^-$ is closed nowhere dense (c.n.d.). Thus $V^-\cap(\bigcap_{\xi<\lambda} U_\xi)^-$ is c.n.d. But also each $V^- - U_\eta \subseteq U_\eta^- - U_\eta$ is c.n.d. By (ii),

$$\bigcup_{\xi<\lambda}(V^- - U_\xi) = V^- - \Big(\bigcap_{\xi<\lambda} U_\xi\Big)$$

is also c.n.d.; whence

$$A = \Big(V^-\cap\Big(\bigcap_{\xi<\lambda} U_\xi\Big)^-\Big)\cup\Big(V^- - \Big(\bigcap_{\xi<\lambda} U_\xi\Big)\Big)$$

is c.n.d. But $x\in V\subseteq A$, a contradiction. □

**Remark.** If we drop the regularity hypothesis (we actually use only semiregularity) in Theorem 4.1 all we lose is the (iii)⇒(i)-direction. In particular it is always true that (i)⇔(ii), and either implies (iii).

**4.2. Corollary.** *Let $D$ be a $\lambda$-good ultrafilter on $\kappa$ with $\langle X_\alpha: \alpha<\kappa\rangle$ a family of spaces. Then $\prod_D X_\alpha$ is $(\lambda,\infty)$-distributive.* □

## 5. Resolvability

Following the terminology of Hewitt [13], we say a space $X$ is *resolvable* if there are two disjoint subsets of $X$ which are each dense in $X$. In [13] various conditions are found which ensure resolvability or unresolvability; and we show in the present section that topological ultraproducts tend to be resolvable in a very strong sense.

Let $\lambda$ be a cardinal number (possibly finite). $X$ is $\lambda$-resolvable if there are $\lambda$ disjoint subsets of $X$ which are each dense in $X$. Clearly $\lambda$-resolvability gets increasingly restrictive as $\lambda$ increases and in particular 2-resolvable implies perfect (= no isolated points). Hewitt [13] describes a machine for producing perfect Hausdorff spaces which are unresolvable (the "submaximal" spaces (Bourbaki terminology [10]) where every dense set is open). We go in the other direction and produce spaces which can support as many pairwise disjoint subsets as there are points in the space.

We strengthen $\lambda$-resolvability in the following way: A space $X$ is *strongly $\lambda$-resolvable* if there is a family $\mathcal{M}$ of pairwise disjoint subsets of $X$ such that each $M \in \mathcal{M}$ is closed discrete, of power $\lambda$, such that each nonempty open $U \subseteq X$ contains a member of $\mathcal{M}$. Clearly if $X$ is strongly $\lambda$-resolvable then $X$ is $\lambda$-resolvable, even if we omit the requirement that each $M$ be closed discrete. Again, strong $\lambda$-resolvability is a chain of properties; increasing in strength as $\lambda$ increases.

The first theorem in this section is about how (strongly) resolvable topological ultraproducts can be. We first quote a time-honored combinatorial lemma (the original discoverer of which is unknown to us).

**5.1. Lemma.** *Let $\kappa$ be an infinite cardinal and let $\langle M_\alpha : \alpha < \kappa \rangle$ be a collection of subsets of $M$, each of power $\kappa$. Then for each $\alpha < \kappa$ there is a set $N_\alpha \subseteq M_\alpha$ such that the $N_\alpha$'s are pairwise disjoint and each $N_\alpha$ has power $\kappa$.*

**Proof.** First well order $\kappa \times \kappa$ in type $\kappa$ as the sequence $\langle\langle \alpha_\xi, \beta_\xi\rangle : \xi < \kappa\rangle$ in such a way that if $\beta_\xi = \beta_\eta = \beta$, then

$$M_{\langle \alpha_\xi, \beta_\xi\rangle} = M_{\langle \alpha_\eta, \beta_\eta\rangle} = M_\beta.$$

Use induction. Suppose for each $\eta < \xi$ we've chosen a point $x_{\langle \alpha_\eta, \beta_\eta\rangle} \in M_{\langle \alpha_\eta, \beta_\eta\rangle}$, all distinct. Then choose a new $x_{\langle \alpha_\xi, \beta_\xi\rangle} \in M_{\langle \alpha_\xi, \beta_\xi\rangle}$ since $|\xi| < \kappa$. For all $\beta < \kappa$, $|\{\xi : \beta_\xi = \beta\}| = \kappa$. So let $N_\beta = \{x_{\langle \alpha_\xi, \beta_\xi\rangle} : \beta_\xi = \beta\}$. □

We define a *$\pi$-basis* for a space $X$ to be a collection $\mathfrak{U}$ of nonempty open subsets of $X$ such that every nonempty open set contains a member of $\mathfrak{U}$. The $\pi$-weight, denoted $\pi(X)$, is the least cardinal of a $\pi$-basis for $X$. Plainly $\pi(X) \leqslant w(X)$, the weight of $X$.

**5.2. Theorem.** *Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of perfect $T_1$ spaces such that for $\alpha < \kappa$, $\pi(X_\alpha) \leqslant \exp(\kappa)$; and let $D$ be an ultrafilter on $\kappa$.*

(i) *If $D$ is regular, then $\prod_D X_\alpha$ is strongly $\kappa$-resolvable and* $\exp(\kappa)$*-resolvable.*
(ii) *If $D$ is good and if* $\exp(\kappa)=\kappa^+$, *then $\prod_D X_\alpha$ is strongly* $\exp(\kappa)$*-resolvable.*

**Proof.** (i) For each $\alpha<\kappa$ let $\mathfrak{U}_\alpha$ be a $\pi$-basis for $X_\alpha$ of power $\leqslant\exp(\kappa)$. Then

$$\prod_D \mathfrak{U}_\alpha=\left\{\prod_D U_\alpha\colon U_\alpha\in\mathfrak{U}_\alpha \text{ for all } \alpha<\kappa\right\}$$

is a $\pi$-basis for $\prod_D X_\alpha$ of power $\leqslant|\exp(\kappa)^\kappa|=\exp(\kappa)$. Moreover since each $X_\alpha$ is perfect, each $U_\alpha$ is then infinite; so by Proposition 1.3 each $\prod_D U_\alpha$ has power $\geqslant\exp(\kappa)$.

Well order $\prod_D \mathfrak{U}_\alpha=\langle\prod_D U_{\alpha,\xi}\colon \xi<\exp(\kappa)\rangle$. By Lemma 5.1 we can shrink each $\prod_D U_{\alpha,\xi}$ to a set $N_\xi$ where $N_\xi\cap N_\eta=\emptyset$ for $\xi<\eta<\exp(\kappa)$ and each $N_\xi$ has power $\exp(\kappa)$. This shows that $\prod_D X_\alpha$ is $\exp(\kappa)$-resolvable. To see that $\prod_D X_\alpha$ is also strongly $\kappa$-resolvable, note that ultraproducts preserve the $T_1$ axiom and $\kappa$-regular ultraproducts are $\kappa^+$-additive. Thus all sets of power $\leqslant\kappa$ are closed discrete. So let $M_\xi\subseteq N_\xi$ have power $\kappa$ for each $\xi<\exp(\kappa)$.

(ii) Let $D$ be $\kappa^+$-good and assume $\kappa^+=\exp(\kappa)$. Now each $X_\alpha$ is perfect $T_1$, so if $U_\alpha\subseteq X_\alpha$ is nonempty open there are closed discrete subsets of $U_\alpha$ of arbitrary finite cardinality. Fix nonempty $U_\alpha\subseteq X_\alpha$ open, $\alpha<\kappa$, and let $\langle n_\alpha\colon \alpha<\kappa\rangle$ be a sequence of natural numbers such that for each $I\in D$, $\sup\{n_\alpha:\alpha\in I\}=\omega$ (if such a sequence did not exist then every member of $\omega^\kappa$ would be $D$-bounded, hence $D$-constant. Thus $\prod_D(\omega)$ would be countable, contradicting the first clause in Proposition 1.4), and let $F_\alpha\subseteq U_\alpha$ be finite of power $n_\alpha$. Then $\prod_D F_\alpha\subseteq\prod_D U_\alpha$ is closed discrete of power $\exp(\kappa)$, by Proposition 1.4.

Let $\prod_D \mathfrak{U}_\alpha$ now be the $\pi$-basis from (i), $\prod_D \mathfrak{U}_\alpha=\langle\prod_D U_{\alpha,\xi}\colon\xi<\exp(\kappa)\rangle$; and assume as an induction hypothesis that for fixed $\xi<\exp(\kappa)$, there is a closed discrete $M_\eta\subseteq\prod_D U_{\alpha,\eta}$, each $\eta<\xi$, such that the $M_\eta$'s are all pairwise disjoint, of power $\exp(\kappa)$. Since $\prod_D X_\alpha$ is perfect, each $M_\eta$ is c.n.d. Also since $\prod_D X_\alpha$ is $\kappa^+$-Baire and $\kappa^+=\exp(\kappa)$, we have $\bigcup_{\eta<\xi}M_\eta$ is nowhere dense. Thus there is an open ultrabox $\prod_D U_\alpha\subseteq\prod_D U_{\alpha,\xi}$ which misses $\bigcup_{\eta<\xi}M_\eta$. Let $M_\xi=\prod_D F_\alpha$ as described above. □

In the rest of this section we explore a little more deeply the relationships among the various resolvability notions which we've introduced. Our main theme is that the only implications among these properties are the obvious ones. Since many implications fail trivially when we allow indiscrete counterexamples, we make some minimal separation assumptions, such as the $T_1$ axiom. The three obvious implications are:

(i) If $X$ is strongly $\kappa$-resolvable, then $X$ is $\kappa$-resolvable;
(ii) If $X$ is $\kappa$-resolvable and $\lambda<\kappa$, then $X$ is $\lambda$-resolvable; and
(iii) If $X$ is strongly $\kappa$-resolvable and $\lambda<\kappa$, then $X$ is strongly $\lambda$-resolvable.

We first show how badly the converse of (i) can fail. It is easy to find counter examples for $\kappa$ infinite and more difficult for $\kappa$ finite.

**5.3. Proposition.** *No locally compact space is strongly $\omega$-resolvable.*

**Proof.** Let $X$ be locally compact. Then no relatively compact open set can contain an infinite closed discrete subset. □

**5.4. Proposition.** *Let $X$ be infinite discrete of power $\kappa$. Then $X^* = \beta X - X$ (where $\beta X$ is the Stone-Čech compactification of $X$) is* $\exp(\exp(\kappa))$*-resolvable but not strongly $\omega$-resolvable. $X^*$ is, however, strongly $n$-resolvable for every $n < \omega$.*

**Proof.** (see [26] for details). $X^*$ has an open basis $\mathscr{B}$ of power $\exp(\exp(\kappa))$ and each nonempty open set has power $\exp(\exp(\kappa))$. By Lemma 5.1. $X^*$ is $\exp(\exp(\kappa))$-resolvable as well as strongly $n$-resolvable for each $n < \omega$. $X^*$ is not strongly $\omega$-resolvable by Proposition 5.3. □

Before we state the next result, we define two cardinal invariants on a space $X$: the *dispersion character*, $\delta(X)$, is the least cardinal of a nonempty open set in $X$; the *character*, $\chi(X)$, is the least cardinal $\chi$ such that every point of $X$ has a local basis of power $\chi$.

**5.5. Theorem.** *Let $X$ be perfect. If $X$ is first countable $T_0$, then $X$ is $\omega$-resolvable. If $X$ is locally compact Hausdorff, then $X$ is* $\exp(\omega)$*-resolvable. In either case, $X$ is strongly $n$-resolvable for each $n < \omega$.*

**Proof.** We draw heavily from Hewitt's paper [13].

**Lemma a.** *Let $X$ be an infinite $T_1$ space where $\pi(X) \leq \delta(X)$. Then $X$ is $\delta(X)$-resolvable, as well as strongly $n$-resolvable for each $n < \omega$.*

**Proof.** Use Lemma 5.1.

**Lemma b.** *$X$ is $\kappa$-resolvable iff every nonempty open subset of $X$ contains a nonempty set which is $\kappa$-resolvable in its relative topology.*

**Proof.** Just mimic the proof of Theorem 20 of [13], which is stated for $\kappa = 2$.

**Lemma c.** *Let $X$ be a perfect $T_0$ space such that every nonempty open $U \subseteq X$ contains an open $V \neq \emptyset$ such that for each $x \in V$, $\chi(X, x) \leq |V|$ (where $\chi(X, x)$ = the least $\chi$ such that $x$ has a local basis of power $\chi$). Then $X$ is $\delta(X)$-resolvable.*

**Proof.** Use Lemmas a and b together as in the proof of Theorem 46 of [13].

Now to prove Theorem 5.5, first let X be perfect $T_0$, and first countable. Use Lemma c directly to get the conclusion. If X is locally compact Hausdorff then, by a well-known theorem of Aleksandrov and Urysohn (see [46]), $\chi(X) \leq \delta(X)$, so by Lemma c, $X$ is $\delta(X)$-resolvable. But $\delta(X) \geq \exp(\omega)$, since $X$ is locally compact Hausdorff and perfect.

The strong $n$-resolvability ($n < \omega$) in each of the above cases is immediate from the construction. □

Thus any perfect locally compact Hausdorff space witnesses the fact that even $\exp(\omega)$-resolvability needn't imply strong $\omega$-resolvability; and moreover that strong $n$-resolvability for each $n < \omega$ needn't imply strong $\omega$-resolvability (whether or not $n$-resolvability for each $n < \omega$ implies $\omega$-resolvability is still an open question).

It is clear that one can get the converses of (i)–(iii) above to fail decisively when the cardinals $\kappa$, $\lambda$ are infinite. We confine ourselves in the rest of this section to the case where $\kappa$, $\lambda$ are finite.

**5.7. Theorem.** *For every positive natural number n there is a (countable) $T_1$ space which is n-resolvable but not $(n+1)$-resolvable (nor even strongly n-resolvable, when $n \geq 2$).*

**Proof.** The case $n = 1$ has been explored in [13]. In this case the space may be taken to be Hausdorff.

We prove the case $n = 2$. The higher cases are treated similarly (the added complexity being insubstantial).

Our space $Z$ is defined as follows: the points of $Z$ are taken from two disjoint countable sets $X$, $Y$; the nonempty open sets are of the form $A \cup B$ where $A \in D$, $B \in E$, and $D$, $E$ are nonisomorphic free ultrafilters on $X$, $Y$ respectively.

It is easy to see that $Z$ is a resolvable $T_1$ space, since both $X$, $Y$ are dense in $Z$. $Z$ is not 3-resolvable, for suppose $U_i \cup V_i$ ($1 \leq i \leq 3$) are pairwise disjoint. Then for some $1 \leq i \leq 3$ $U_i \notin D$ and $V_i \notin E$. Thus $U_i \cup V_i$ cannot be dense. $Z$ is not strongly 2-resolvable either. To see this, suppose $\mathcal{M} = \mathcal{M}_X \cup \mathcal{M}_Y \cup \mathcal{N}$ is a collection of pairs from $Z$, where elements of $\mathcal{N}_X$ are subsets of $X$, elements of $\mathcal{M}_Y$ are subsets of $Y$, and elements of $\mathcal{N}$ have one point taken from each of $X$, $Y$. Since the collection of nonempty open subsets of $Z$ forms a filter, we need only find nonempty open sets $A_X \cup B_X$, $A_Y \cup B_Y$, $A \cup B$ such that no member of $\mathcal{M}_X$ lies in $A_X \cup B_X$, etc. We can then take the set $(A_X \cap A_Y \cap A) \cup (B_X \cap B_Y \cap B)$. Let

$$\mathcal{M}_X = \{\{x_0, y_0\}, \{x_1, y_1\}, \ldots\}.$$

If $\bigcup \mathcal{M}_X \notin D$, set $A_X = X - \bigcup \mathcal{M}_X$, $B_X = Y$. If $\bigcup \mathcal{M}_X \in D$, assume, say, that $\{x_0, x_1, \ldots\} \in D$. Then $\{y_0, y_1, \ldots\} \notin D$. Let $A_X = \{x_0, x_1, \ldots\}$, $B_X = Y$. We treat $\mathcal{M}_Y$ similarly. So suppose $\mathcal{N} = \{\{x_0, y_0\}, \{x_1, y_1\}, \ldots\}$, and let $f: X \to Y$ be a bijection with $y_n = f(x_n)$, $x < \omega$. Since $D$, $E$ are nonisomorphic there is a set $I \in D$ such that $f[I] \notin E$. If either $\{x_0, x_1, \ldots\} \notin D$ or $\{y_0, y_1, \ldots,\} \notin E$ we can easily find a nonempty open $A \cup B$ containing none of the pairs $\{x_n, y_n\}$. If $\{x_0, x_1, \ldots\} \in D$ and $\{y_0, y_1, \ldots\} \in E$ then set $A = I \cap \{x_0, x_1, \ldots\}$, $B = \{y_0, y_1, \ldots\} - f[A]$. □

**Remarks.** (i) The space in Theorem 5.7 is connected $T_1$ but not Hausdorff. It would be nice to have a Hausdorff counterexample.

(ii) In the space $Z$ of Theorem 5.7, assume that $D$ and $E$ are isomorphic, say via the bijection $f: X \to Y$. Then the collection $\{\{x, f(x)\}: x \in X\}$ shows that this new space is a strongly 2-resolvable $T_1$ space which isn't 3-resolvable, let alone strongly 3-resolvable. Again, it would be nice to have a Hausdorff counterexample.

## 6. Topological ultrapowers of the rational line

In this section we prove that, under the GCH, a large number of topological ultraproducts look alike. We first repeat the main result of [5].

**6.1. Theorem.** *Let $\kappa$ be an infinite cardinal. Then* $\exp(\kappa) = \kappa^+$ *iff for every $\kappa$-sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of regular spaces of weight* $\leqslant \exp(\kappa)$, *and every regular ultrafilter $D$ on $\kappa$, $\prod_D X_\alpha$ is paracompact.* $\square$

We prove here a similarly phrased theorem about good ultraproducts. The conclusion will of course be quite a bit stronger, but the Theorem will not be as sharp since we will be unable to deduce the negation of the conclusion merely by assuming $\kappa^+ < \exp(\kappa)$. Rather it will be apparently necessary to use the equiconsistent "$\exp(\kappa) = \exp(\kappa^{++})$".

**6.2. Theorem.** *Let $\kappa$ be an infinite cardinal.*

(i) *If* $\exp(\kappa) = \kappa^+$, *then for every $\kappa$-sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of regular perfect spaces of cardinality + weight* $\leqslant \exp(\kappa)$, *and every good ultrafilter $D$ on $\kappa$, $\prod_D X_\alpha \simeq \prod_D (\mathbb{Q})$ (where $\mathbb{Q}$ denotes the space of rational numbers, and "$\simeq$" denotes homeomorphism).*

(ii) *If* $\exp(\kappa) = \exp(\kappa^{++})$, *then there is a perfect compact Hausdorff space $X$ of cardinality + weight* $\leqslant \exp(\kappa)$ *such that for any regular ultrafilter $D$ on $\kappa$, $\prod_D (X)$ fails to be normal, so in particular $\prod_D (X) \not\simeq \prod_D (\mathbb{Q})$.*

Before proving Theorem 6.2, a few comments are in order. First, in order to avoid a lot of repetition, we assume familiarity with [4, 5], only stating the results we use. Second, so that the proof of (i) be more intelligible, we treat spaces as relational structures as outlined in the Introduction. In particular we define a *basoid* to be a structure of the form $\langle X \cup \mathscr{B}; X, \mathscr{B}, \in \rangle$ where $\mathscr{B}$ is a basis for a topology on $X$ ($\mathscr{B}^*$ is the associated topology). Two basoids $\mathfrak{A}$, $\mathfrak{B}$ are *isomorphic* if they are isomorphic in the model-theoretic sense and we write $\mathfrak{A} \cong \mathfrak{B}$. If $\mathfrak{A}^* \cong \mathfrak{B}^*$ (i.e. the generated topological structures are isomorphic) then $\mathfrak{A}$ and $\mathfrak{B}$ are *homeomorphic*, and we write $\mathfrak{A} \simeq \mathfrak{B}$.

**Proof of Theorem 6.2(i).** Let $\langle X_\alpha : \alpha < \kappa \rangle$ be regular perfect and of power + weight $\leq$ exp($\kappa$), with $D$ a $\kappa^+$-good ultrafilter on $\kappa$. Then the basoid

$$\prod_D X_\alpha = \langle \prod_D X_\alpha \cup \prod_D \tau_\alpha ; \prod_D X_\alpha ; \prod_D \tau_\alpha, \underset{\sim}{\in} \rangle$$

is $\kappa^+$-saturated as a relational structure (see [8, 11] for details). For each $\alpha < \kappa$ let $Y_\alpha$ be an elementary substructure of $X_\alpha$ of countable cardinality. Then $Y_\alpha$ is a basoid which has countably many points, countably many basic sets; and $Y_\alpha^*$ is perfect regular (the details can be found in [4]). Now all perfect regular countable and second countable spaces are homeomorphic to $\mathbb{Q}$, so let $f_\alpha : Y_\alpha \to \mathbb{Q}$ be a homeomorphism for each $\alpha < \kappa$. Then $\prod_D f_\alpha : \prod_D Y_\alpha \to \prod_D (\mathbb{Q})$ is also a homeomorphism. Now $\prod_D X_\alpha$ is $\kappa^+$-saturated of power exp($\kappa$) $= \kappa^+$. So also is $\prod_D Y_\alpha$. In addition $\prod_D X_\alpha$ and $\prod_D Y_\alpha$ are elementarily equivalent relational structures. Thus $\prod_D X_\alpha \cong \prod_D Y_\alpha$ (see [8, 11]). Consequently $\prod_D X_\alpha \simeq \prod_D (\mathbb{Q})$. □

**Remarks.** (i) $\prod_D (\mathbb{Q})$ as a topological space is perfect, linearly orderable (the ultraproduct of the natural ordering on $\mathbb{Q}$ will do), and $\lambda$-metrizable for some $\kappa^+ \leq \lambda \leq$ exp($\kappa$) (where a space $X$ is *$\lambda$-metrizable* (see [19, 20, 23]) if $X$ has a uniformity which, as a filter of binary relations on $X$, has a basis linearly ordered by inclusion in confinality $\lambda$). Consequently $\prod_D (\mathbb{Q})$ is hereditarily paracompact regardless of the combinatorial nature of $D$.

(ii) In Theorem 6.1 we could replace the statement, "$\prod_D X_\alpha$ is paracompact" with the stronger assertion, "$\prod_D X_\alpha$ is exp($\kappa$)-metrizable" (this is not done in [5]). One simply uses the fact that $\prod_D X_\alpha$ is regular, $\kappa^+$-additive ($\kappa^+ =$ exp($\kappa$)), and of weight $\leq$ exp($\kappa$). Then, in a manner analogous to the way in which one embeds regular second countable zero-dimensional spaces in the Cantor discontinuum $2^\omega$, we embed $\prod_D X_\alpha$ within the space $(2^{(\kappa^+)})_{\kappa^+}$ (where $(X)_\lambda$ is the expansion of $\tau_X$ formed by closing $\tau_X$ under intersections of length $< \kappa$. These spaces are studied in various places (see [4, 5, 12, 19, 20, 23, 25]). For $\lambda$ a regular cardinal, the space $(2^\lambda)_\lambda$ is a $\lambda$-additive analogue to the Cantor discontinuum and is $\lambda$-metrizable [23].

**Proof of Theorem 6.2(ii).** We use the same counterexample we used in [5], namely $X = 2^{(\kappa^{++})}$. In that proof we showed that

(a) $(X)_{\kappa^+}$ is not normal; and

(b) $(X)_{\kappa^+}$ embeds as a retract of $\prod_D (X)$ for any regular $D$ (on $\kappa$), whence $\prod_D (X)$ is not normal.

So if, in the proof of Theorem 6.1, we assume exp($\kappa$) $\geq \kappa^{++}$ then $X$ is a regular space of weight $\leq$ exp($\kappa$) such that no regular ultrapower $\prod_D (X)$ is normal, let alone paracompact.

In the present proof we must also force exp($\kappa^{++}$) to be small; and it is consistent to have exp($\kappa^{++}$) $=$ exp($\kappa$). Under that assumption $X$ is a regular perfect space of power + weight $\leq$ exp($\kappa$) such that no regular ultrapower $\prod_D (X)$ is normal (so $\prod_D (X) \not\simeq \prod_D (\mathbb{Q})$). □

## 7. Appendix: On a problem in model theory

The following is a purely model-theoretic application of the techniques of Section 6.

**7.1. Conjecture** (Chang and Keisler [11], p. 514]). Let $L$ be a first order language of power $\leqslant \kappa$ with $\mathfrak{A}$, $\mathfrak{B}$ elementarily equivalent $L$-structures of power $\leqslant \kappa$. Then $\prod_D(\mathfrak{A}) \cong \prod_D(\mathfrak{B})$ for any regular ultrafilter $D$ on $\kappa$.

We show that a related conjecture, one which Keisler has shown to be true assuming segments of the GCH (see [11]), is independent of ZFC. In particular we show that the statement, "Let $L$ be a first order language of power $\leqslant \kappa$ with $\mathfrak{A}$, $\mathfrak{B}$ elementarily equivalent $L$-structures of power $\leqslant \exp(\kappa)$. Then $\prod_D(\mathfrak{A}) \cong \prod_D(\mathfrak{B})$ for any good ultrafilter $D$ on $\kappa$.", a consequence of $\exp(\kappa) = \kappa^+$, is false if we work in the relatively consistent set theory, ZFC + $(\exp(\kappa) = \exp(\kappa^{++}))$. As our counterexample we choose $L$ to be the language of basoids, and assume $\exp(\kappa) = \exp(\kappa^{++})$. Pick

$$\mathfrak{A} = \langle 2^{(\kappa^{++})} \cup \tau; 2^{(\kappa^{++})}, \tau, \underset{\sim}{\in} \rangle,$$

where $\tau$ is the Tichonov topology on the set of maps from $\kappa^{++}$ to 2. Let $\mathfrak{B}$ be any countable structure which is elementarily equivalent to $\mathfrak{A}$. It is easy to check that $\mathfrak{B}$ is a basoid and that $\mathfrak{B}^*$ is a countable second countable regular perfect topological space. Thus the space $\mathbb{Q}$ of rationals has an open basis $\mathcal{B}$ such that

$$\mathfrak{B} \cong \langle \mathbb{Q} \cup \mathcal{B}, \mathbb{Q}, \mathcal{B}, \underset{\sim}{\in} \rangle.$$

If $D$ is any regular ultrafilter on $\kappa$ and if $\prod_D(\mathfrak{A}) \cong \prod_D(\mathfrak{B})$ then in particular the spaces $(\prod_D(\mathfrak{A}))^*$, $(\prod_D(\mathfrak{B}))^*$ are homeomorphic. But we saw earlier (proved in [5]) that $(\prod_D(\mathfrak{A}))^*$ is not normal. However $(\prod_D(\mathfrak{B}))^*$ is an ultrapower of the rationals and is quite normal. □

Noting that $\kappa^+$-good ultraproducts are $\kappa$-saturated, we have also proved the independence from ZFC of the statement, "Let $L$ be a first order language of power $\leqslant \kappa$, with $\mathfrak{A}$, $\mathfrak{B}$ $\kappa^+$-saturated elementarily equivalent $L$-structures of power $\exp(\kappa)$. Then $\mathfrak{A} \cong \mathfrak{B}$".

We mention in parting that Shelah [21] has proved in ZFC that a weak version of Conjecture 7.1 holds with the new conclusion, "Then $\prod_D(\mathfrak{A}) \cong \prod_D(\mathfrak{B})$ for some good ultrafilter $D$ on $\exp(\kappa)$.". His proof constructs $D$ using induction with an "independent sets" argument (à la Hausdorff).

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