Refer to the function \( f(x, y) = x^3 - 3x + y^3 - 3y \) in answering the following.

(1) Find all four critical points of \( f \), and classify each as local maximum, local minimum, or saddle point.

First solve \( \nabla f(x, y) = (3x^2 - 3)i + (3y^2 - 3)j = \vec{0} \). We get \( x = \pm 1 \) and \( y = \pm 1 \), giving the four critical points \((1, 1), (1, -1), (-1, 1), \) and \((-1, -1)\). To bring in the second derivative test, we have \( f_{xx} = 6x, f_{yy} = 6y, \) and \( f_{xy} = f_{yx} = 0 \), so \( D = D(x, y) \) will be positive just in case \( x \) and \( y \) have the same sign and negative just in case \( x \) and \( y \) have opposite signs. Thus we have saddle points at \((1, -1)\) and \((-1, 1)\) because \( f_{xx}(1, 1) = 6 > 0 \), and a local maximum at \((-1, -1)\) because \( f_{xx}(-1, -1) = -6 < 0 \).

(2) Set up—but do not solve—the three Lagrange equations that determine the constrained critical points for \( f \) on the curve \( x^2 + y^2 = 4 \).

The constraint curve is the level curve \( g(x, y) = 4 \), where \( g(x, y) = x^2 + y^2 \). Then \( \nabla g(x, y) = (2x)i + (2y)j \), so the vector equation \( \nabla f = \lambda \nabla g \), plus the constraint equation \( g(x, y) = 4 \), give the three equations,

\[
\begin{align*}
3x^2 - 3 &= 2x\lambda \\
3y^2 - 3 &= 2y\lambda \\
x^2 + y^2 &= 4 \\
\end{align*}
\]

[Now solving this system is a little tricky: First note, by symmetry, we could have \( x = y \). In that case we have \( 2x^2 = 4 \), giving us the two critical points \((\sqrt{2}, \sqrt{2})\) and \((-\sqrt{2}, -\sqrt{2})\). \( \lambda = 3/(2\sqrt{2}) \) in the first case, \(-3/(2\sqrt{2}) \) in the second. So let’s now assume \( x \neq y \). If you add the first two equations, you get \( 3(x^2 + y^2) - 6 = 2\lambda(x + y) \). But \( x^2 + y^2 = 4 \), so this becomes the equation \( 3 = \lambda(x + y) \). (This will prove useful to know.) Using the quadratic formula, we can solve for \( x \) in terms of \( \lambda \) in the first equation and for \( y \) in terms of \( \lambda \) in the second. We get the same result, though: \( x \) (or \( y \)) is \( \frac{2\lambda \pm \sqrt{4\lambda^2 + 36}}{6} \). But since \( x \neq y \), one of them must have the plus sign and the other must have the minus sign. Regardless of which is the case, \( x + y \) must be \( \frac{4\lambda}{3} = \frac{2\lambda}{3} \). But we showed above that \( x + y = 3/\lambda \). Hence \( 2\lambda^2 = 9 \) and \( \lambda = \pm 3/\sqrt{2} \). Now we can solve for \( x \) and \( y \) in each case, giving two new points \((\frac{3\sqrt{2} + \sqrt{54}}{6}, \frac{-3\sqrt{2} + \sqrt{54}}{6})\) and \((\frac{-3\sqrt{2} + \sqrt{54}}{6}, \frac{3\sqrt{2} + \sqrt{54}}{6})\). So we have four critical points. To see which give the global maximum/minimum for \( f \) on the constraint curve, simply plug them into the original equation \( z = f(x, y) \) and compare the \( z \)-values.]