(1) Describe precisely the domain of the function \( f(x, y) = \frac{x + y}{\sqrt{4 - (x^2 + y^2)}} \). What does this subset of the plane look like?

You need the term under the square root to be positive. (Can’t be zero because of the Prime Directive: never divide by zero.) So this means \( x^2 + y^2 < 4 \); the graph of this set is the open disk of radius 2, centered at the origin (bounding circle not included).

(2) Can two different level curves for a function \( z = f(x, y) \) share a point in common? (Explain why/why not.)

No they can’t: each level curve corresponds to a specific output value of the function. If \( \langle a, b \rangle \) were to lie on two different level curves, this would amount to saying that \( f(a, b) \) takes on two different values. Functions don’t work this way.

(3) Find \( \lim_{\langle x, y \rangle \to \langle 1, 2 \rangle} \frac{xy^2}{x^2 + y} \), or give reasons why such a limit does not exist.

As \( \langle x, y \rangle \) gets closer and closer to \( \langle 1, 2 \rangle \), the fraction gets closer and closer to \( \frac{(1)(2)^2}{1^2 + 2} = \frac{4}{3} \). This is the limit.

(4) Find the cosine of the angle formed by \( \vec{u} = \vec{i} - 2\vec{j} + 2\vec{k} \) and \( \vec{v} = 3\vec{i} + 4\vec{k} \).

\[ \cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} = \frac{3 + 8}{11} = \frac{11}{15} \]

(5) Find an equation for the plane that goes through the points \( \langle 1, 0, 0 \rangle \), \( \langle 0, 2, 0 \rangle \), and \( \langle 0, 0, 3 \rangle \).

Let the given points be \( P, Q, R \), respectively. We get a normal vector by taking the cross product \( \vec{PQ} \times \vec{PR} = 6\vec{i} + 3\vec{j} + 2\vec{k} \). Using \( P \) as our base point on the plane, we have the equation \( 6(x - 1) + 3(y - 0) + 2(z - 0) = 0 \), or \( 6x + 3y + 2z = 6 \).

(6) Find the vector projection of \( \vec{u} = 4\vec{i} - 3\vec{j} \) onto \( \vec{v} = \vec{i} + 2\vec{j} \).

The vector projection is \( \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{-2}{5}(\vec{i} + 2\vec{j}) \).

(7) If \( f(x, y) = x^2 + y^2 \), find all unit vectors \( \vec{u} \) such that \( f_\vec{u}(1, 2) = 2 \).

The gradient at \( (1, 2) \) is \( \nabla f(1, 2) = 2\vec{i} + 4\vec{j} \). If \( \vec{u} = a\vec{i} + b\vec{j} \) is the unknown unit vector, we know that \( \nabla f(1, 2) \cdot \vec{u} = 2a + 4b = 2 \) and \( a^2 + b^2 = 1 \). From the first equation we have \( a = 1 - 2b \). Substituting for \( a \) in the second equation gives \( (1 - 2b)^2 + b^2 = 1 - 4b + 5b^2 = 1 \), or \( b(5b - 4) = 0 \). The two solutions are \( b = 0 \) (giving \( a = 1 \), or \( \vec{u} = \vec{i} \)), and \( b = \frac{4}{5} \) (giving \( a = -\frac{3}{5} \), or \( \vec{u} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} \)). These are the only two possibilities for \( \vec{u} \) so that \( f_\vec{u}(1, 2) = 2 \).
(8) Let \( f(x, y, z) = x^3e^{2y} + z^2 \), and find the unit vector \( \vec{u} \) for which \( f_\vec{u}(1, 0, 2) \) is as large as possible. What is this largest possible value?

The maximum possible value for the directional derivative is \( \| \nabla f(1, 0, 2) \| = \| 3\hat{i} + 2\hat{j} + 4\hat{k} \| = \sqrt{29} \). Our unit vector \( \vec{u} \) is in the direction of the gradient; hence \( \vec{u} = \frac{3}{\sqrt{29}} \hat{i} + \frac{2}{\sqrt{29}} \hat{j} + \frac{4}{\sqrt{29}} \hat{k} \).

(9) Find an equation for the plane that is tangent to the oblate spheroid \( 2x^2 + y^2 + z^2 = 4 \), at the point \( (1, 1, 1) \).

The spheroid is the level surface corresponding to \( f(x, y, z) = 4 \) for \( f(x, y, z) = 2x^2 + y^2 + z^2 \). Hence a vector normal to the tangent plane is \( \nabla f(1, 1, 1) = 4\hat{i} + 2\hat{j} + 2\hat{k} \), and an equation for this plane is \( 4(x - 1) + 2(y - 1) + 2(z - 1) = 0 \).

(10) Find the local linearization \( L(x, y) \) of \( f(x, y) = x^2(\cos y - \sin y) \), at the point \( (3, 0) \).

\[
L(x, y) = f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)(y - 0) = 9 + 6(x - 3) - 9y = 6x - 9y - 9.
\]