MATH 082, EXAM 2 (solutions), 13 JUNE, 2008

(1) Consider the vector-valued function \( \mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \).

(a) Compute the unit tangent vector \( \mathbf{T}(\pi/2) \).

\[ \mathbf{r}'(t) = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}, \]

which is constantly 2. \( \mathbf{r}(\pi/2) = -2\mathbf{i} \), so \( \mathbf{T}(\pi/2) = -\mathbf{i} \).

(b) This curve has constant curvature \( \kappa \). What is it?

You could compute \( \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} \); the numerator would be \( \|4\mathbf{k}\| = 4 \), and the denominator would be \( 2^3 = 8 \). Thus \( \kappa = 1/2 \), independently of \( t \). Alternatively, you could note that the curve is a circle of radius 2. Circles of radius \( a \) have constant curvature \( 1/a \).

(c) Compute the principal unit normal vector \( \mathbf{N}(\pi/2) \).

There are lots of ways to calculate this; but, noting that the curve is a circle, it is its own osculating circle at each point. So since \( \mathbf{T}(\pi/2) = -\mathbf{i} \), the principal unit normal vector at that time is \( -\mathbf{j} \).

(2) Consider the function \( f(x, y) = e^{2x}(\sin y + \cos y) \).

(a) Find the gradient \( \nabla f(0, 0) \).

\[ \nabla f(x, y) = (2e^{2x}(\sin y + \cos y))\mathbf{i} + (e^{2x}(\cos y - \sin y))\mathbf{j}. \]  

At \( (x, y) = (0, 0) \), we get \( \nabla f(0, 0) = 2\mathbf{i} + \mathbf{j} \).

(b) Find the largest possible value of a directional derivative \( D_{\mathbf{u}}f(0, 0) \).

As \( \mathbf{u} \) ranges over all unit vectors, \( D_{\mathbf{u}}f(a, b) \) achieves its maximum possible value in the (unit) direction of the gradient, and the maximum value attained in that direction is the magnitude of that gradient. So our maximum directional derivative is \( \|\nabla f(0, 0)\| = \sqrt{5} \).

(c) Find an equation for the plane that is tangent to the graph of \( f(x, y) \) at the point where \( (x, y) = (0, 0) \).

The plane in question has normal vector \( \nabla F(0, 0, 1) \), where \( F(x, y, z) = f(x, y) - z \). (The \( z \)-coordinate is 1 because \( f(0, 0) = 1 \).) \n
\[ \nabla F(0, 0, 1) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \]

so our equation is \( 2(x - 0) + 1(y - 0) - 1(z - 1) = 0 \), or \( z = 2x + y + 1 \). Alternatively, we could use the equation for the linear approximation to \( f(x, y) \) near \( (0, 0) \), namely \( z = f(0, 0) + \)
1  \[ f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + 2x + y. \]

2  \[(3) \text{ Consider the function } f(x, y) = x^2 + y^2, \text{ defined on the elliptical disk } R = \{(x, y) : x^2 + 4y^2 \leq 4\}. \]

(a) Find all critical points for \( f \) that lie in the interior of \( R \), and test for local max/local min/saddle point.

Here we solve the vector equation \( \nabla f(x, y) = \vec{0}; \) i.e., \( 2x = 0, 2y = 0 \). This can happen only at \( (0, 0) \); and since \( (0, 0) \) is in the interior of \( R \), this is our interior critical point. When we calculate the discriminant \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 \), we get the constant value 4, which is positive. And since \( f_{xx}(0, 0) = 2 > 0 \), we conclude \( (0, 0) \) is a local minimum for \( f \) on the interior of \( R \).

(b) Find all critical points for \( f \) that lie on the boundary ellipse of \( R \).

A boundary critical point is critical relative to the constraint condition \( g(x, y) = x^2 + 4y^2 - 4 = 0 \). So when we express that \( \nabla f(x, y) \) and \( \nabla g(x, y) \) are parallel at a point \( (x, y) \), this takes the form of the Lagrange equations \( 2x = 2x\lambda, 2y = 8y\lambda, \) and \( x^2 + 4y^2 = 4 \). We’d like to cancel the \( x \) in the first equation to obtain \( \lambda = 1 \), but that’s not possible if \( x = 0 \). So in the case \( x = 0 \), the last equation forces \( y = \pm 1 \). In that case the second equation just gives \( \lambda = 1/2 \), so we have critical points \( (0, \pm 1) \) under the assumption \( x = 0 \). Now, if we assume \( x \neq 0 \), we get \( \lambda = 1 \) from the first equation. The second equation now becomes \( 2y = 8y \), or \( 6y = 0 \) (after subtracting \( 2y \) from both sides). Hence \( y = 0 \), and the third equation kicks in to reveal that \( x = \pm 2 \). So the other critical points are \( (\pm 2, 0) \).

(c) For which points of \( R \) does \( f \) attain its absolute maximum and absolute minimum values?

We now take the five critical points obtained above and plug them into \( f(x, y) \) to compare values. At \( (0, 0) \), we get \( z = 0 \), at \( (0, \pm 1) \) we get \( z = 1 \), and at \( (\pm 2, 0) \) we get \( z = 4 \). So \( f(x, y) \) attains its absolute minimum value of 0 at \( (0, 0) \) and its absolute maximum value of 4 at the two points \( (\pm 2, 0) \).

4  \[(a) \text{ Let } f(x, y) = x^2 + 2y \text{ be defined on the region } R = \{(x, y) : x \geq 0, 0 \leq y \leq 1 - x^2\}, \text{ and let } \mathcal{P} \text{ be the inner partition of } R \text{ defined by the vertical lines } x = 0, 1/2, 1 \text{ and the horizontal lines } y = 0, 1/4, 1/2, 3/4, 1. \]

(a) Draw a picture of the region \( R \), with the grid lines from \( \mathcal{P} \), and mark the rectangles of \( \mathcal{P} \) that lie inside \( R \).
The region $R$ is bounded below by the $x$-axis, to the left by the $y$-axis, and from above-right by the downward-directed parabola $y = 1 - x^2$. The three vertical and five horizontal grid lines give us 8 rectangles, each of area $\Delta A = 1/8$. The only rectangles that lie completely in $R$, however, are the ones with upper-right corners $(1/2, 1/4), (1/2, 1/2), \text{and} (1/2, 3/4)$, the three leftmost rectangles, counting from the bottom.

(b) Approximate $\int_R f(x, y) \, dA$ using the upper-right corners of the rectangles from $\mathcal{P}$.

\[ \int_R f(x, y) \, dA \approx \left[ \left( \frac{1}{2} \right)^2 + 2 \left( \frac{1}{4} \right) \right] + \left[ \left( \frac{1}{2} \right)^2 + 2 \left( \frac{1}{2} \right) \right] + \left[ \left( \frac{1}{2} \right)^2 + 2 \left( \frac{3}{4} \right) \right] \left( \frac{1}{8} \right) = \frac{15}{32}. \]

(c) Calculate the exact value of $\int_R f(x, y) \, dA$ using an iterated double integral.

\[ \int_R f(x, y) \, dA = \int_0^1 \int_0^{1-x^2} (x^2+2y) \, dy \, dx = \int_0^1 \left[ x^2y + y^2 \right]_{y=0}^{y=1-x^2} \, dx = \int_0^1 \left( x^2(1-x^2) + (1-x^2)^2 \right) \, dx = \int_0^1 \left( 1-x^2 \right) \, dx = \left[ x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}. \]