(Each of the following five problems is worth 12 points. Be sure to justify all answers.)

(1) (a) You’ve determined that \( f(x) = \frac{2x}{1-x^2} \) is the generating function for a sequence \( (a_1, a_2, \ldots) \). Find this sequence.

(b) Given the initial-value recurrence \( a_n = 2a_{n-1} + 3a_{n-2}, \ n \geq 3, \ a_1 = a_2 = 1 \), and using the method for finding the golden mean from the Fibonacci sequence, find \( \lim_{n \to \infty} \frac{a_n}{a_{n-1}} \).

(2) (a) Given the initial-value recurrence \( b_n = nb_{n-1} + b_{n-2}, \ n \geq 3, \ b_1 = 2, \ b_2 = 3 \), use simple induction to prove that \( b_n > n! \) for \( n \geq 1 \).

(b) Using the fact that the derangement sequence \( d_n \) is very close to \( \frac{n!}{e} \) for large \( n \), find the approximate probability that when a 52-card deck is well shuffled, there is at least one card that occupies its original position.
(3) (a) Let $T$ be a spanning tree for the bipartite graph $K_{m,n}$. How many edges does $T$ have?

(b) Give a simple reason why $K_n$ is not bipartite for $n \geq 3$.

(c) Draw the dual graph for $K_3$.

(4) A graph $G$ has four vertices arranged to form a square; the edges of $G$ are the four sides of the square, plus one diagonal.

(a) Draw all the spanning trees for $G$.

(b) What is the smallest number of colors that you can paint the vertices of $G$ with, so that no two adjacent vertices get the same color? (This is called the chromatic number of $G$.)
(5) (a) Let $G$ be a subdivision of $K_n$ with $2n$ vertices. How many edges does $G$ have?

(b) Verify Euler’s formula for planar graphs, in the special case the graph is a tree.