(1) (a) Find the unique solution to the initial value problem \( \frac{dy}{dt} = \frac{1}{\sqrt{t}} + \frac{1}{t} \), \( y(e^2) = 5 \).

First antidifferentiate the right-hand side:
\[
\int t^{-1/2} + t^{-1} \, dx = 2t^{1/2} + \ln|t| + C.
\]
Next, using the initial condition, we have
\[
5 = 2\sqrt{e^2} + \ln(e^2) + C = 2e + 2 + C.
\]
Hence \( C = 3 - 2e \), so our solution is
\[
y(t) = 2\sqrt{t} + \ln|t| + 3 - 2e.
\]
(Note: Because the square root is not defined for \( t < 0 \), we could remove the absolute value sign from the \( \ln \) term and not change the function.)

(b) Find the exact area bounded by the \( x \)-axis, the lines \( x = 1 \) and \( x = 8 \), and the graph of the curve \( y = x^{\frac{1}{3}} + 1 \).

The exact area is
\[
\int_1^8 x^{\frac{1}{3}} + 1 \, dx = \left[ \frac{3}{4} x^{\frac{4}{3}} + x \right]_1^8 = (12 + 8) - (3/4 + 1) = \frac{73}{4} = 18.25.
\]

(2) A ball is tossed straight upwards from the ledge of a window, with an initial velocity of 10 feet per second. If it hits the ground 8 seconds later, then (assuming the gravitational constant for earth to be \(-32 \text{ feet/second}^2\)):

(a) What is the velocity of the ball as it hits the ground?

This is the velocity of the ball when \( t = 8 \). The velocity, as a function of time, is \( v(t) = -32t + 10 \); so \( v(8) = -246 \text{ feet/second} \).

(b) How high must the window ledge be?

A second antidifferentiation on \( a(t) = -32 \) gives us the position function \( s(t) = -16t^2 + 10t + s_0 \), where \( s_0 \) is the initial position, the (as yet) unknown height of the window ledge. We know that \( s(8) = 0 \), hence
\[
0 = -16(8^2) + 10(8) + s_0 = -1024 + 80 + s_0,
\]
so \( s_0 = 944 \text{ feet} \).
(3) Use integration by simple substitution to evaluate:

(a) \[ \int x^2 \cos(1 + 5x^3) \, dx \]

A reasonable choice for substitution is to set \( u = 1 + 5x^3 \). Let’s see if it works: Since \( du = 15x^2 \, dx \), we have \( x^2 \, dx = \frac{1}{15} \, du \); so the transformed integral is \( \int \frac{1}{15} \cos u \, du = \frac{1}{15} \sin u + C \). Reformulation in terms of \( x \) gives us the antiderivative \( \frac{1}{15} \sin(1 + 5x^3) + C \).

(b) \[ \int_{-1}^{3} x\sqrt{x + 1} \, dx \]

This would be more obvious if the term under the square root were \( x^2 + 1 \), but simple substitution still works here. As you would have set \( u = x^2 + 1 \), now set \( u = x + 1 \). Then \( du = dx \) and \( x = u - 1 \). Since the \( x \) limits of integration are \(-1 \) and \( 3 \), the corresponding \( u \) limits are \(-1 + 1 = 0 \) and \( 3 + 1 = 4 \). So the transformed definite integral is \( \int_{0}^{4} u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du = \left[ \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right]_{0}^{4} = \frac{2}{5} (32) - \frac{2}{3} (8) = \frac{112}{15} \approx 7.4667 \).

(4) Use integration by parts to evaluate:

(a) \[ \int \frac{x}{\cos^2 x} \, dx \] (given integration table entry 7: \( \int \tan x \, dx = -\ln |\cos x| + C \); as well as the fact that \( \frac{d}{dx}[\tan x] = \sec^2 x \)).

Set \( u = x \) and \( dv = \sec^2 x \, dx \). Then \( du = dx \) and \( v = \tan x \). So integration by parts gives us \( x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C \).

(b) \[ \int_{0}^{\pi/2} x \cos x \, dx \]

Set \( u = x \) and \( dv = \cos x \, dx \). Then \( du = dx \) and \( v = \sin x \). So \( \int_{0}^{\pi/2} x \cos x \, dx = \left[ x \sin x \right]_{0}^{\pi/2} - \int_{0}^{\pi/2} \sin x \, dx = \left[ x \sin x + \cos x \right]_{0}^{\pi/2} = \frac{\pi}{2} - 1 \approx .5705 \).
(5) (a) Perform the partial fraction decomposition on the rational expression, \( \frac{2x + 1}{x^4 + x^2} \).

The denominator completely factors into \( x^2(x^2+1) \), so our decomposition is of the form \( \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} \). After adding these three fractions and collecting coefficients, we have \( \frac{(A + C)x^3 + (B + D)x^2 + Ax + B}{x^4 + x^2} \).

And, since the numerator is identically equal to \( 2x + 1 \), we have \( A + C = B + D = 0, A = 2, \) and \( B = 1 \). After substitution, we solve for the remaining coefficients, \( C = -2, D = -1 \). So the completed decomposition is \( \frac{2}{x} + \frac{1}{x^2} - \frac{2x + 1}{x^2 + 1} \).

(b) Using the answer in Part (a), evaluate the integral \( \int \frac{2x + 1}{x^4 + x^2} \, dx \).

\[
\int \frac{2x + 1}{x^4 + x^2} \, dx = \int \frac{2}{x} \, dx + \int \frac{1}{x^2} \, dx - \int \frac{2x}{x^2 + 1} \, dx - \int \frac{dx}{x^2 + 1} = 2 \ln |x| - \frac{1}{x} - 2 \ln |x^2 + 1| - \arctan x + C.
\]