MATH 1451, FINAL EXAM, 13 DECEMBER, 2010

(Each part of the following seven problems is worth 6 points. You’re much more likely to get partial credit if you show your work. And, when in doubt, DRAW A PICTURE.)

(1) (a) Use integration by simple substitution to evaluate \( \int x^2 \sqrt{1 + 5x^3} \, dx \).

Set \( u = 1 + 5x^3 \). Then \( \frac{du}{dx} = 15x^2 \), so \( x^2 \, dx = \frac{1}{15} \, du \). Thus our integral transforms into \( \int \frac{1}{15} u^{\frac{1}{2}} \, du = \frac{2}{45} (1 + 5x^3)^{\frac{3}{2}} + C \).

(b) Use integration by parts to evaluate \( \int x \cos x \, dx \).

Set \( u = x \), \( dv = \cos x \, dx \). Then \( du = dx \) and \( v = \sin x \). Following the formula \( \int u \, dv = uv - \int v \, du \), we obtain \( \int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C \).

(2) (a) Approximate \( \pi = \int_0^1 \frac{dx}{1 + x^2} \) using TRAP(2).

We have \( \Delta x = 1/2 \), so TRAP(2) = \( \frac{1}{2} \left( \frac{1}{1 + 0} + \frac{1}{1 + (\frac{1}{2})^2} + \frac{1}{2 \left( 1 + \frac{1}{2} \right)} \right) \frac{1}{2} = \frac{31}{40} \).

(b) The region in the first quadrant, bounded on the left by the y-axis, above by the line \( y = 6 - x \), and below by the parabola \( y = x^2 \), is rotated about the x-axis. Set up—but do not evaluate—a definite integral that represents the resulting volume of revolution.

The curves \( y = 6 - x \) and \( y = x^2 \) cross—in the first quadrant—at the point \( (2, 4) \). For each \( 0 \leq x \leq 2 \) the washer resulting from rotating the vertical segment from \( y = x^2 \) to \( y = 6 - x \) about the x-axis has infinitesimal volume \( \pi [(6 - x)^2 - (x^2)^2] \, dx \), so our integral representing the volume of rotation is \( \int_0^2 \pi [(6 - x)^2 - x^4] \, dx \).

(3) (a) A cylindrical tank has height and diameter both twelve feet. Its axis is perpendicular to the ground, and it is full of oil that weighs 50 pounds per cubic foot. Set up—but do not evaluate—a definite integral that expresses the amount of work required to pump all the oil to the level of the top of the tank. (Be sure to use the correct units.)

The tank is full, so \( 0 \leq h \leq 12 \). At each level \( h \), the wafer of “frozen” oil is lifted a distance of \( 12 - h \) feet, and it weighs \( (50)(\pi)(6^2) \, dh \) pounds. Thus the total amount of work is \( \int_0^{12} 1800\pi \, dh \) foot-pounds.

(b) Find the median of the p.d.f. \( p(x) = \frac{2/\pi}{1 + x^2} \), defined on \( [0, \infty) \).

The median is where the cumulative probability equals 1/2. If this median is \( m \), then we know \( \frac{1}{2} = \int_0^m \frac{2}{\pi} \frac{1}{1 + x^2} = \frac{2}{\pi} \arctan m \). So \( m = \tan(\pi/4) = 1 \).

(4) (a) If the geometric series \( \sum_{n=1}^{\infty} r^n \) converges to 5, what must \( r \) be?

We know \( |r| < 1 \) and that \( 5 = \sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \ldots \). Thus \( 5r = r^2 + r^3 + \ldots \); so \( 5 - 5r = r \). This gives \( r = 5/6 \).
(b) Find the interval of convergence for the power series \( \sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)2^n} \).

Using the ratio test, we want the successive ratios \( \frac{|x-1|^{n+1}}{2(n+1)|x-1|} = \frac{1}{2} \left( \frac{n+1}{n+2} \right) |x-1| \)
to converge to something less than 1. But this happens just in case \(|x-1| < 2\); hence the radius of convergence is 2 and the series converges absolutely on the open interval \((1-2, 1+2) = (-1, 3)\). When we test the end point \(-1\), we get the series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \), which converges, by the alternating series test. When we test the end point 3, we get the series \( \sum_{n=0}^{\infty} \frac{1}{n+1} \), which diverges, by the p-test \((p=1)\). Hence the interval of convergence for this power series is \([-1, 3]\).

(5) (a) Find the Taylor series for \( f(x) = 1/x \), centered at \( c = 2 \). Be sure to include the general term.

[Useful formula: \( f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \).]

Plugging in \( c = 2 \) to the formula for the \( n \)th derivative, we get the \( n \)th Taylor coefficient \( \frac{(-1)^n}{2^{n+1}} \). So the Taylor series is \( T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n \).

(b) Find the Taylor polynomial of order 2 for \( f(x) = x^3 \), centered at \( c = -1 \).

\( f(-1) = -1 \). Since \( f'(x) = 3x^2 \), we have \( f'(-1) = 3 \). Since \( f''(x) = 6x \), we have \( f''(-1) = -3 \). So \( T_2(x) = -1 + 3(x+1) - 3(x+1)^2 \).

(6) (a) Given the IVP \( \frac{dy}{dx} = 1 + y^2 \), \( y(0) = 0 \), use Euler’s method to approximate \( y(0.2) \) in two steps.

If \( y = y(x) \) is the solution, then we have \( \Delta x = .1 \) because we’re moving right a distance of .2 in two equal increments. Euler’s recurrence relation is \( y_{n+1} = y_n + y_n' \Delta x \), where \( y_n \) approximates \( y(x_n) \). In this case \( x_0 = 0 \), \( x_1 = .1 \), and \( x_2 = .2 \). Correspondingly, \( y_0 = y(x_0) = 0 \), \( y_1 = y_0 + y_0' \Delta x = 0 + (1 + 0^2)(.1) = .1 \), and \( y_2 = .1 + (1 + (.1)^2)(.1) = .201 \).

(b) Use the separation of variables technique to find the exact solution to the IVP in Part (a).

Separation of variables gives us \( \frac{dy}{1+y^2} = dx \). Then integration of both sides gives us \( \arctan y = x + C \), and application of the initial condition \( y(0) = 0 \) shows \( C \) to be zero. Hence the solution to this IVT is \( y = \tan x \), \( -\frac{\pi}{2} < x < \frac{\pi}{2} \). [Check: \( \tan 0 = 0 \), so the initial condition checks out. The derivative of \( y = \tan x \) is \( y' = \sec^2 x \), which equals \( 1 + \tan^2 x \), by the well-known trigonometric identity obtained from dividing both sides of the fundamental identity \( \sin^2 x + \cos^2 x = 1 \) by \( \cos^2 x \).]

(7) (a) Decide whether \( y = xe^x \) is a solution to the ODE \( \frac{d^2y}{dx^2} = y + \frac{2y}{x} \).

If \( y = xe^x \), then \( y' = (1 + x)e^x \) and \( y'' = (2 + x)e^x \). The right-hand side of the ODE above is \( xe^x + \frac{2xe^x}{x} = e^x(x + 2) \), which checks with the second derivative. Hence the given function is a solution, as long as \( x \neq 0 \). But we could overcome this annoyance, simply by multiplying both sides of the given equation by \( x \), to obtain
(b) A given population takes ten years to increase by 5%. How long does it take to double?

The basic ODE modeling simple population growth is \( \frac{dP}{dt} = kP \), whose general solution is \( P = P_0e^{kt} \), where \( P_0 \) is the initial population and \( t \) is measured in years.

The information given is that \( 1.05P_0 = P_0e^{10k} \), so that \( k = \frac{\ln 1.05}{10} \). The population growth function is then \( P = P_0e^{kt} \), with this value of \( k \), and the next step is to substitute \( 2P_0 \) for \( P \) and solve for \( t \). Thus the doubling time is \( \frac{10 \ln 2}{\ln 1.05} \approx 14.21 \) years.

(c) What are the equilibrium solutions to the ODE \( \frac{dy}{dx} = y^3 - y \), and which of them are stable?

Because \( \frac{dy}{dx} \) depends on \( y \) alone, the slopes are zero along the horizontal lines corresponding to the solutions of the (algebraic) equation \( y^3 - y = 0 \). So the three constant functions \( y = 0, \pm 1 \) are the equilibrium solutions. For nonequilibrium solutions above \( y = 1 \) or between \( y = -1 \) and \( y = 0 \), the slopes are all positive; and for the rest of the nonequilibrium solutions, they’re all negative. So when initial conditions are slightly above or below \( y = \pm 1 \), the corresponding solutions will stray further and further from the corresponding equilibrium with increasing values of \( x \). And when the initial conditions are close to \( y = 0 \), the solutions will have that equilibrium as horizontal asymptote as \( x \to \infty \). Hence the only stable equilibrium solution is \( y = 0 \).