MATH 145, DISCRETE MATHEMATICS FOR ENGINEERS , SECTION 1001, SPRING 2008
ANSWERS TO ASSIGNED HOMEWORK PROBLEMS

Due 4/30/08: 5.7(2,4,8)

5.7(2) Suppose a student who knows 60% of the material covered in the text is going to take a five-question objective quiz. Let X_i, $1 \le i \le 5$, be the r.v. that gives 1 if the question is answered correctly and 0 otherwise. What are $E(X_i)$ and $V(X_i)$? If X is the r.v. that counts the number of correct answers, how does $V(X)$ compare with $\sum_{i=1}^{5} V(X_i)$?

> This is a Bernoulli trials situation; success means getting the question right (no partial credit). The probability of success is then $p = .6$, so $E(X_i) = (.6)(1) + (.4)(0) = .6$. $V(X_i) = E(X_i^2) - (E(X_i))^2 = .6 - (.6)^2 = .24$. Since we're treating this process as an independent trials process, the r.v.s $X_1, \ldots, X_5$ are independent, with $X = \sum_{i=1}^{5} X_i$. Hence $V(X)$ is the sum of the variances of the X_i; i.e., $V(X) = (5)(.24) = 1.2$.

5.7(4) If the quiz in 5.7(2) has 100 questions, what are: the expected number of right answers, the variance of the expected number of right answers, and the standard deviation of the number of right answers?

> Let X be the number of right answers. Then Theorem 5.30 is applicable; namely $E(X) = np = (100)(.6) = 60$ and $\sigma(X) = \sqrt{np(1-p)} = \sqrt{(100)(.6)(.4)} = \sqrt{24} = 2\sqrt{6}$. The variance of the expected number of right answers is the variance of the constant r.v. $E(X)$, namely 0.

5.7(8) Show that if X and Y are independent r.v.s and b, c are constant, then $X - b$ and $Y - c$ are independent.

> You could try to show that $E((X-b)(Y-c)) = E(X-b)E(Y-c)$. However, this property of independent random variables is a consequence of independence; we did not prove that the property *implies* independence. You actually need to look at the independence of the events $(X-b) = x$ and $(Y-c) = y$. For any $s \in S$ (sample space), $(X-b)(s) = x$ if and only $X(s) = b + x$. Thus the events $(X-b) = x$ and $X = b + x$ are identical. Likewise, the events $(Y-c) = y$ and $Y = c + y$ are identical. By hypothesis, the events $X = b + x$ and $Y = c + y$ are independent; hence so are the events $(X-b) = x$ and $(Y-c) = y$. Thus $X - b$ and $Y - c$ are independent random variables.

Due 4/23/08: 5.4(4,6,12)

5.4(4) Assuming the process of answering the questions on a five-question quiz is an independent trials process with probability of getting an answer correct for a given student being .8, what's the probability of one particular sequence of four correct answers and one incorrect answer? What is the probability of getting exactly four questions correct?

> For the first question it's $(.8)^4(.2) = .08192$. For the second, the probability is $C(5,4)(.8)^4(.2) = .40960$.

5.4(6) What is the expected sum of the tops of n dice when you roll them?

> By linearity of expected value, it's the sum of the expected values on each individual die. Thus it is $n\frac{1+2+3+4+5+6}{6} = 21n/6 = 7n/2 = (3.5)n$.

5.4(12) Solve Problem 11 for the case of a student taking a multiple choice test with five choices per question.

If the student knows k percent of the material, then her percentage grade should be k, (assuming she gets a question right if she knows the material and is not confused by unusual wording, etc., etc.) Suppose she also answers g percent of the exam by random guessing. Since there are five answers to choose from, she's expected to get $.2g$ correct and $.8g$ incorrect; hence her raw score, inflated for guessing, is $k + .2g$. The adjusted score is obtained by subtracting a proportion y of the incorrect answers from the raw score so that it will again be k. Thus we have the equation $k = k + .2g - y(.8g)$, giving $y = .2/.8 = .25$. So the adjusted score should be calculated by subtracting a quarter of the number of incorrect answers from the number of correct ones, with questions left blank being ignored.

Due 4/14/08: 5.3(2,4,6)

5.3(2) In three flips of a coin, is the event that two filps in a row are heads independent from the event that there is an even number of heads?

Let E and F represent the two events in question. Then $E = \{HHH, HHT, THH\}$ and $F = \{HHT, HTH, THH, TTT\}$. So $P(E) = 3/8$ and $P(F) = 4/8$. Now $E \cap F = \{HHT, THH\}$ and $P(E \cap F) = 2/8 = 1/4$. So $P(E) \cdot P(F) = 3/16 \neq P(E \cap F)$, and so these events are not independent.

5.3(4) What is the sample space that you use for rolling two dice, a first one and then a second one? Using this sample space, explain why the event "i dots are on top of the first die" and the event "j dots are on top of the second die" are independent if you throw two dice.

The appropriate sample space is $S = \{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\}$. For any fixed i, let $E_i = \{(i, j) : 1 \leq j \leq 6\}$. Then E_i is the event "i dots are on top of the first die." Similarly, for any fixed j, let $F_j = \{(i, j) : 1 \leq i \leq 6\}$. Then F_j is the event "j dots are on top of the second die." Then $E_i \cap F_j = \{(i, j)\}$; so $P(E_i \cap F_j) = 1/36$. Now $P(E_i) = 1/6 = P(F_j)$ and so $P(E_i) \cdot P(F_j) = 1/36 = P(E_i \cap E_j)$. Therefore the two events are independent, for any fixed i and j.

5.3(6) Assume that on a true-false test, students will answer correctly any question on a subject that they know. Assume students guess at answers that they do not know. For students who know 60% of the material in a course, what is the probability that they will answer a question correctly? What is the probability that they will know the answer to a question they answer correctly?

For any given question, let K be the event "knows the answer," and let C be the event "answers correctly." Then $P(C) = P(C \cap K) + P(C \cap K')$ (where $()'$ means the complement of an event). So $P(C) = P(C|K) \cdot P(K) + P(C|K') \cdot P(K') = (1)(.6) + (.5)(.4) = .8$. The second question is to determine $P(K|C)$, which is just $P(C|K) \cdot P(K)/P(C)$ (Bayes' formula). Thus $P(C|K) = .6/.8 = .75$.

Due 4/4/08: 5.1(2,8,12)

5.1(2) When you roll two fair dice, what is the probability of getting a sum of 4 or less on the tops?

We may take as sample space the set $S = \{(m, n) : 1 \leq m \leq 6, 1 \leq n \leq 6\}$, so that $|S| = 6^2 = 36$. Making the equal-likelihood assumption, each simple outcome has a probability of 1/36 of occurrence. If E is the event "sum ≤ 4," it comprises the simple events $(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)$ and $(2, 2)$. Thus $P(E) = |E|/|S| = 6/36 = 1/6$.

5.1(8) Using five-element sets as a sample space, determine the probability that a hand of 5 cards, chosen from an ordinary deck of 52 cards, will consist of cards of the same suit.

Following the suggestion, we let S consist of all 5-element subsets of the 52-card deck. Then $|S| = C(52,5) = \frac{52!}{5!\cdot 47!} = 2,598,960$, so–under the equal-likelihood assumption–the probability of getting any particular 5-card hand is $1/2,598,960$. Let E_h be the event "all cards are hearts;" similarly define the events E_c, E_s, E_d. Then the event "all cards have the same suit" is $E = E_h \cup E_c \cup E_s \cup E_d$, a union of four mutually exclusive (i.e., pairwise disjoint) events. The number of 5-card hands that consist of all hearts is $C(13,5)$, as is the number of 5-card hands that consist of all cards of any single suit. Thus $P(E) = 4 \cdot C(13,5)/C(52,5) = 5148/2,598,960 \approx .00198$.

5.1(12) A die is made from a cube with a square painted on one side, a circle on two sides, and a triangle on three sides. If the die is rolled twice, what is the probability that the two shapes you see on top are the same?

There are 36 simple outcomes, just as in problem 2 above. Only one of them consists of a pair of squares, but there are $2^2 = 4$ outcomes consisting of a pair of circles. Similarly, there are $3^2 = 9$ outcomes consisting of a pair of triangles. These three events are mutually exclusive, so there are $1 + 4 + 9 = 14$ simple outcomes in which the two shapes match. Hence the probability of "both shapes match" is $14/36 = 7/18$.

Due 3/26/08: 4.3(2,4,8)

4.3(2) Draw a recursion tree diagram for $T(n) = 2T(n/2) + 2n$, for $n \geq 2$, with $T(1) = 2$ (assuming n is a power of 2). Use it to find an exact solution to the recurrence.

The recursion tree is binary, of height $\log_2 n + 1$. At level i, $0 \leq i \leq \log_2 n$, there are 2^i nodes, each representing a job of size $n/2^i$. The work per job is $2n/2^i$ units; so at each level $< \log_2 n$, the work per level is $2^i(2n/2^i) = 2n$. At the bottom level, there are n nodes, each representing a job of size 1. The work per job is $T(1) = 2$; so we have $2n$ units of work at this level too. Thus $T(n) = 2n(\log_2 n + 1)$.

4.3(4) Draw a recursion tree diagram for $T(n) = T(n/4) + n$, for $n \geq 2$, with $T(1) = 1$ (assuming n is a power of 4). Use it to find a big theta bound for the solution to the recurrence.

The recursion tree is unary, of height $\log_4 n + 1$. At level i, $0 \leq i \leq \log_4 n$, there is one node, representing a job of size $n/4^i$. The work per job is $n/4^i$ units; so at each level $< \log_4 n$, the work per level is $n/4^i$. At the bottom level, there is one node representing a job of size 1. And at this level the work is $T(1) = 1$. So $T(n) = n\sum_{i=0}^{\log_4 n}(1/4)^i = \Theta(n)$.

4.3(8) Draw a recursion tree diagram for $T(n) = T(n/3) + 1$, for $n \geq 2$, with $T(1) = 3$ (assuming n is a power of 3). Use it to find an exact solution to the recurrence.

The recursion tree is unary, of height $\log_3 n + 1$. At level i, $0 \leq i \leq \log_3 n$, there is one node, representing a job of size $n/3^i$. The work per job is one unit; so at each level $< \log_3 n$, the work per level is 1. At the bottom level, there is one node representing a job of size 1. And at this level the work is $T(1) = 3$. So $T(n) = 3 + \log_3 n$.

Due 3/05/08: 4.1(8,12), 4.2(13)

4.1(8) We can define nonnegative integer powers of a real number a by the rules: $a^0 = 1$; $a^{n+1} = a^n \cdot a$. Explain why this defines a^n for all nonnegative integers n; from this rule, prove the law of exponents:

$a^{m+n} = a^m \cdot a^n$ for nonnegative integers m, n.

If $n = 0$, then you have $a^n = 1$, by definition. If $n > 0$, then you have $a^n = a^{n-1} \cdot a$. If you already know what a^{n-1} is, then you can get a^n by multiplying that by a; otherwise you write $a^n = a^{n-2} \cdot a \cdot a$ and repeat. Eventually you get to something you know, and then you can obtain a^n.
In order to show $a^{m+n} = a^m \cdot a^n$ for arbitrary nonnegative integers m, n, first fix $m \geq 0$. Since a^m is well defined by the inductive definition of taking powers, we now turn to induction on n. At the base case, we have $a^{m+0} = a^m = a^m \cdot 1 = a^m \cdot a^0$. So assume $n \geq 0$ is fixed, and that $a^{m+n} = a^m \cdot a^n$. We need to show $a^{m+(n+1)} = a^m \cdot a^{n+1}$. Indeed, $a^{m+(n+1)} = a^{(m+n)+1} =$ (by definition) $= a^{m+n} \cdot a =$ (by induction hypothesis) $= (a^m \cdot a^n) \cdot a =$ (by associativity of multiplication) $= a^m \cdot (a^n \cdot a) =$ (by definition) $= a^m \cdot a^{n+1}$.

4.1(12) Prove by induction that the number of subsets of an n-element set is 2^n.

Let's be very specific and let $P(n)$ be the statement, "All n-element sets have exactly 2^n subsets." Then $P(0)$ is the statement that the empty set has exactly $2^0 = 1$ subset; a true statement since the only subset of the empty set is the empty set itself. Now let $n \geq 0$ be fixed, and assume that $P(n)$ holds. We wish to show that every $(n+1)$-element set has exactly 2^{n+1} subsets. So let S be any $(n+1)$-element set. Since $n+1 > 0$, we may fix an element $x \in S$. Let S' be the complement of x in S; i.e., $S' = S \setminus \{x\}$. Then the subsets of S consist of two disjoint families $\mathcal{S}_1$–the subsets that contain x as an element–and $\mathcal{S}_2$–the subsets that don't contain x as an element. The members of $\mathcal{S}_1$ are defined by deciding what elements of S' to add to x to make a subset. Hence $\mathcal{S}_1$ has the same number of elements as the number of subsets of S'–which is 2^n, by the induction hypothesis. Similarly $\mathcal{S}_2$ has exactly 2^n elements. Since the collection of all subsets of S is $\mathcal{S}_1 \cup \mathcal{S}_2$, and these two families are disjoint, there are exactly $2^n + 2^n = 2^n \cdot 2 = 2^{n+1}$ subsets of S.

4.2(13) Solve the recurrence $T(n) = 2T(n-1) + 3^n$; $T(0) = 1$.

By iterating, we get: $T(1) = 2 \cdot 1 + 3^1$, $T(2) = 2(2+3) + 3^2 = 2^2 + 2 \cdot 3 + 3^2$, $T(3) = 2^3 + 2^2 \cdot 3 + 2 \cdot 3^2 + 3^3$, etc. This leads to the conjecture that the solution to this recurrence is the function $S(n) = 2^n + \sum_{i=1}^{n} 2^{n-i} \cdot 3^i$. Before showing this is right, let's use the sum formula $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$, for $r \neq 1$, to obtain a tidier formulation of $S(n)$. We quickly have $S(n) = 2^n + 2^n \sum_{i=1}^{n} (3/2)^i = 2^n(1 + \sum_{i=1}^{n}(3/2)^i) = 2^n(\sum_{i=0}^{n}(3/2)^i) = 2^n \frac{1-(3/2)^{n+1}}{1-(3/2)} = 2^n \frac{(3/2)^{n+1}-1}{1/2} = 2^{n+1}((3/2)^{n+1} - 1) = 3^{n+1} - 2^{n+1}$.
Now let's show this works. First, we have $S(0) = 3 - 2 = 1 = T(0)$; so the base case is taken care of. Next, assume $n \geq 0$ is fixed and that $S(n) = T(n)$. Show $S(n+1) = T(n+1)$. So we have $S(n+1) = 3^{n+2} - 2^{n+2} = (2+1)3^{n+1} - 2 \cdot 2^{n+1} = 2(3^{n+1} - 2^{n+1}) + 3^{n+1} = 2S(n) + 3^{n+1} =$ (by induction hypothesis) $= 2T(n) + 3^{n+1} = T(n+1)$.

Due 2/27/08: 3.2(6,12,14)

3.2(6) Using $s(x, y, z)$ to be the statement $x = yz$ and $t(x, y)$ to be the statement $x \leq y$, what is the form of the definition of the greatest common divisor d of m and n? (Use the set $\mathbb{Z}$ of integers for your universe.)

Quantification is taken over $U = \mathbb{Z}$.

$$d = \gcd(m, n) \equiv \exists x \; s(m, d, x) \wedge \exists y \; s(n, d, y) \wedge \forall z \; [(\exists x \; s(m, z, x) \wedge \exists y \; s(n, z, y)) \Rightarrow t(z, d)]$$

3.2(12) Let $p(x)$ stand for "x is prime," $q(x)$ for "x is even," and $r(x,y)$ stand for "$x = y$." Use these three symbolic statements and appropriate logical notation to express "There is exactly one even prime." (Use the set $\mathbb{Z}^+$ of positive integers for your universe.)

Quantification is taken over $U = \mathbb{Z}^+$.

$$\exists x\ [p(x) \wedge q(x) \wedge \forall y\ [(p(y) \wedge q(y)) \Rightarrow r(x,y)]]$$

3.2(14) Why is $(\exists x \in U(p(x))) \wedge (\exists y \in U(q(y)))$ not equivalent to $\exists z \in U(p(z) \wedge q(z))$? Are the statements $(\exists x \in U(p(x))) \vee (\exists y \in U(q(y)))$ and $\exists z \in U(p(z) \vee q(z))$ equivalent? (Explain.)

Intuitively, you could have something that satisfies $p(x)$ and, perhaps, something entirely different that satisfies $q(y)$; but nothing that satisfies both $p(x)$ and $q(y)$ *simultaneously*. For example–and this was suggested by a student–you could work in the universe of integers, and have $p(x)$ be $x = 0$, $q(y)$ be $y = 1$. Then $(\exists x \in U(p(x))) \wedge (\exists y \in U(q(y)))$ is true; however $\exists z \in U(p(z) \wedge q(z))$ is false.
However, when we turn $\wedge$ to $\vee$, the two corresponding statements are equivalent. To see one direction, suppose $(\exists x \in U(p(x))) \vee (\exists y \in U(q(y)))$ holds in universe U. Then one of the disjuncts holds; say it's $\exists x \in U(p(x))$. Then there is some $a \in U$ such that $p(a)$ is true in U. Then, since statements of the form $A \Rightarrow (A \vee B)$ are tautologies, we know that $p(a) \vee q(a)$ is also true in U. Thus $\exists z \in U(p(z) \vee q(z))$ is true in U. Similarly we have $\exists y \in U(q(y))$ implying $\exists z \in U(p(z) \vee q(z))$, so the disjunction implies $\exists z \in U(p(z) \vee q(z))$. The opposite direction is just as easy.

Due 2/18/08:

(1) What's $\varphi(p^3)$ when p is a prime? How about $\varphi(p^k)$ for any positive integer k?

If the number of non-units of $\mathbb{Z}_n$ is m, then $\varphi(n)$, the number of units of $\mathbb{Z}_n$, is $n - m$. The non-units of $\mathbb{Z}_n$ when $n = p^k$ are the multiples of p that are $< p^k$; i.e., $0 \cdot p, 1 \cdot p, \ldots, (p^{k-1} - 1) \cdot p$. This gives p^{k-1} non-units in all, hence $\varphi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$.

(2) Show that $\varphi(n)$ is not a one-to-one function.

All this needs is two distinct integers m and n, such that $\varphi(m) = \varphi(n)$. If you start computing $\varphi(k)$ for small values of k, you soon get a repetition: the numbers relatively prime to 3 are 1 and 2; the numbers relatively prime to 4 are 1 and 3. Hence $\varphi(3) = \varphi(4) = 2$.

(3) If n is composite (i.e., not prime), show that there are at least $\sqrt{n}$ non-units in $\mathbb{Z}_n$. Then show $\varphi(n) \leq n - \sqrt{n}$.

Since n is composite, we may write $n = km$, where both k and m are positive and less than n; so in particular, both k and m are members of $\mathbb{Z}_n$. Suppose $k \leq m$. Then $m \geq \sqrt{n}$, and the numbers $0 \cdot k, 1 \cdot k, \ldots, (m-1) \cdot k$ give us m distinct non-units of $\mathbb{Z}_n$. Since there are at least $\sqrt{n}$ non-units of $\mathbb{Z}_n$ (and exactly $\sqrt{n}$ non-units when n is the square of a prime), there are at most $n - \sqrt{n}$ units of $\mathbb{Z}_n$. Hence $\varphi(n) \leq n - \sqrt{n}$.

Due 2/06/08: 2.1(14); 2.2(4,14)

2.1(14) Write the multiplication table for $\mathbb{Z}_7$

$\cdot_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

2.2(4) How many elements a are there such that $a \cdot_{31} 22 = 1$? How many elements a are there such that $a \cdot_{10} 2 = 1$?

$a \cdot_{31} 22 = 1$ holds for some $a \in \mathbb{Z}_{31}$ because 22 and 31 are relatively prime–hence 22 is a unit of $\mathbb{Z}_{31}$–and multiplicative inverses are unique, when they exist. (Note that $a \cdot 22 = 1 \mathrm{mod}\ 31$ holds for infinitely many integers a: just find the unique a with $1 \leq a \leq 30$ and add any multiple of 31 to it.)

$a \cdot_{10} 2 = 1$ has no solution in $\mathbb{Z}_{10}$ because $\gcd(10, 2) = 2 \neq 1$.

2.2(14) Which elements of $\mathbb{Z}_{35}$ do not have multiplicative inverses in $\mathbb{Z}_{35}$?

The numbers $0 \leq a \leq 34$ that have no multiplicative inverses in $\mathbb{Z}_{35}$ are the ones that are not relatively prime to 35. These are just the multiples of either 5 or 7; namely 0,5,7,10,14,15, 20,21,25,28, and 30. There are eleven of these.

Due 1/30/08: 1.2 (17,18); 1.3 (10)

1.2(17) Explain why a function from an n-element set to an n-element set is one-to-one iff it is onto.

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be n-element sets, with $f : A \to B$ a function. There are two things to prove: one-to-one implies onto; and onto implies one-to-one. So assume f is one-to-one. Then the set $\{f(a_1), \ldots, f(a_n)\}$ is an n-element subset of B, and so all the elements of B are used up. Thus f is onto. Another way to say this is this: if there were some $b \in B$ that wasn't of the form $f(a)$ for some $a \in A$, then$\{f(a_1), \ldots, f(a_n)\}$ could have no more than $n - 1$ elements. But this would force $f(a_i) = f(a_j)$ for some $1 \leq i < j \leq n$.
Conversely, if f were not one-to-one, then there would be some $1 \leq i < j \leq n$ such that $f(a_i) = f(a_j)$. This would force the image set $\{f(a_1), \ldots, f(a_n)\}$ to have at most $n - 1$ elements; so f couldn't be onto. (Note: The fact that n is finite allows us to conclude that $n - 1$ is strictly less than n. This is no longer true if our sets are infinite.)

1.2(18) The function g is called an *inverse* of the function f if the domain of g is the range of f, if $g(f(x)) = x$ for every x in the domain of f, and $f(g(y)) = y$ for every y in the range of f. Explain why: (a) a function is a bijection iff it has an inverse; and (b) an inverse for a function is unique, if it exists at all.

Re (a): Let $f : A \to B$ be a function. If f is a bijection, then f is both one-to-one and onto–by definition. So if $b \in B$ is given, then there is a unique $a \in A$ so that $b = f(a)$. Define $g(b)$ to be this element a. Thus, by definition, $g(f(a)) = a$ for every $a \in A$. If $b \in B$ is given, then $g(b)$ is the unique $a \in A$ such that $b = f(a)$. So $f(g(b)) = f(a) = b$. The

proof that f is a bijection if there's an inverse g is just as easy: for each $b \in B$, $g(b)$ is the unique $a \in A$ such that $b = f(a)$. This says f is a bijection.
Re (b): Suppose $f : A \to B$, with $g, h : B \to A$ satisfying the conditions for being an inverse for A. For each $b \in B$, both $g(b)$ and $h(b)$ are the unique $a \in A$ such that $f(a) = b$. Thus $g(b) = h(b)$, so g and h are equal as functions.

1.3(10) Explain the difference between choosing four pairwise disjoint 3-element sets from a 12-element set, and labeling a 12-element set with 3 labels each of types 1–4. In how many ways can you choose 4 pairwise disjoint 3-element subsets (respectively, 3 pairwise disjoint 4-element subsets) from a 12-element set?

The difference is that in the first instance order doesn't matter, but in the second instance order does matter. It's the same difference as that between choosing three delegates from a 12-element club, and choosing three officers from that club, with three distinct officer designations.
Let x (resp., y) be the number of ways of choosing 4 pairwise disjoint 3-element subsets (resp., 3 pairwise disjoint 4-element subsets) from a 12-element set. Then $4!x$ (resp., $3!y$) is the number of ways of choosing: 3 out of 12, then 3 out of 9, then 3 out of 6, then 3 out of 3 $= C_{12,3}C_{9,3}C_{6,3}C_{3,3} = \frac{12!}{(3!)^4}$ (resp., 4 out of 12, then 4 out of 8, then 4 out of 4 $= C_{12,4}C_{8,4}C_{4,4} = \frac{12!}{(4!)^3}$). Finally, we have $x = \frac{12!}{(4!)(3!)^4} = 15,400$ and $y = \frac{12!}{(3!)(4!)^3} = 5,775$.

Due 1/23/08: 1.2 (11,13,15)

1.2(11) Panel discussion; choose 4 administrators from among 10; choose 4 students from among 20. How many ways can this be done?

The unmentioned assumption is that the set of administrators is disjoint from the set of students. So assembling a panel may be decomposed into a two-step process: (1) choose the 4 administrators; (2) choose the 4 students. For each of the $C_{10,4} = \frac{10!}{(4!)(6!)} = \frac{(10)(9)(8)(7)}{(4)(3)(2)} = 210$ ways to complete step 1, there are $C_{20,4} = \frac{20!}{(4!)(16!)} = \frac{(20)(19)(18)(17)}{(4)(3)(2)} = 4,845$ ways to complete step 2. By the product principle, the total number of ways to assemble the panel is $(210)(4,845) = 1,017,450$.

1.2(13) A sundae consists of two scoops of ice cream–from among 10 flavors–plus any one of three flavors of topping. On top of this comes a combination of whipped cream/nuts/cherry. How many sundaes are possible?

The total number–using the product principle–is the product: (number of ways of choosing one or two flavors of ice cream out of 10) $\times$ (number of ways of choosing one flavor of topping out of 3)$\times$ (number of ways of picking a subset of {whipped cream,nuts,cherry}) $= (C_{10,1} + C_{10,2})C_{3,1}2^3 = (10 + 45)(3)(8) = 1,320$.

1.2(15) A tennis club has $2n$ members. How many ways to pair them up for n singles matches? How many ways to pair them up if we also specify who has first serve?

Let x be the total number of pairings. If we were to run all n matches consecutively, this could be accomplished in two separate procedures. The first procedure involves two steps: (1) pick the n pairs; (2) give each pair an order of appearance. There are $xn!$ ways to accomplish this procedure. The second procedure involves n steps: (1) pick the first pair from $2n$ players; (2) pick the second pair from the remaining $2n - 2$ players; $\ldots$; (n) pick the nth pair from the remaining 2 players. This gives $C_{2n,2}C_{2n-2,2} \ldots C_{2,2} = \frac{(2n)!}{2^n}$ ways in all. Hence we obtain $x = \frac{C_{2n,2}C_{2n-2,2}\ldots C_{2,2}}{n!} = \frac{(2n)!}{n!2^n}$.

To answer the second part, where the order counts for each pairing, simply replace each $C_{2n-i,2}$ in the formula above with $P_{2n-i,2}$. Thus the answer here is $x2^n = \frac{(2n)!}{n!}$.