MATH 121, SAMPLE PROBLEMS FOR THE FINAL EXAM (solutions), 10 MAY, 2007

(Expect nine problems, each worth 10 points. The first four of them will be taken–almost *verbatim*–from the previous exams. The following are sample questions that cover new material.)

(1) Find an orthonormal basis for the orthogonal complement of span$\{[1, -1, 3]\}$ in $\mathbb{R}^3$.

First, find a basis for the orthogonal complement, namely the solution to the system $x - y + 3z = 0$. This is a 2-parameter family, $\{[s - 3t, s, t] : s, t \in \mathbb{R}\}$; i.e., span$\{[1, 1, 0], [-3, 0, 1]\}$. For convenience, let these two vectors be $\vec{u}$ and $\vec{v}$, respectively.

Next, check whether $\vec{u}\cdot\vec{v} = 0$. It isn't. Thus we need to do a Gram-Schmidt on the pair. Set $\vec{w}_1 = \vec{u}$. Then, for $\vec{w}_2$, we take $\vec{v}-$ (the projection of $\vec{v}$ on $\vec{u}$), which is $\vec{v} - \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = [-3, 0, 1] - (-3/2)[1, 1, 0] = [-3/2, 3/2, 1]$. To obtain an ortho*normal* basis, we need to divide each of $\vec{w}_1, \vec{w}_2$ by their lengths. The result is: $\{\frac{1}{\sqrt{2}}[1, 1, 0], \frac{1}{\sqrt{22}}[-3, 3, 2]\}$.

(2) What are all possible values for x and y so that the matrix $\begin{bmatrix} .6 & x \\ .8 & y \end{bmatrix}$ is an orthogonal matrix?

First note that the left column is a indeed a unit vector. For the right column, you want $.6x + .8y = 0$ and $x^2 + y^2 = 1$. The solution set for the first equation is $\{[-4t, 3t] : t \in \mathbb{R}\}$. Plugging this into the second equation, we have $16t^2 + 9t^2 = 1$; solving for t we get $t = \pm .2$. So our possible right columns are $[-.8, .6]$ and $[.8, -.6]$.

(3) Find the projection matrix for the subspace $W = \text{span}\{[1, 1, 1], [1, -2, 1]\}$.

Let A be the matrix whose columns are the vectors in the spanning set. [Note first that these columns are linearly independent; otherwise we would need to do some row reductions to obtain a basis for the column space of A.] Then the projection matrix P is $A(A^\top A)^{-1}A^\top$. We could compute this outright, but note that the columns of A are orthogonal. So an orthonormal basis for W is $\{\frac{1}{\sqrt{3}}[1, 1, 1], \frac{1}{\sqrt{6}}[1, -2, 1]\}$. Let B now be the matrix whose columns are these. Then $P = B(B^\top B)^{-1}B^\top$. But $B^\top B$ is the identity matrix because B is an orthogonal matrix. Thus $P = BB^\top = \begin{bmatrix} .5 & 0 & .5 \\ 0 & 1 & 0 \\ .5 & 0 & .5 \end{bmatrix}$. [You could also have calculated P directly using A. Check for yourself that the result would match this one.]

(4) Find the least squares solution to the system:

$$\begin{array}{rcrcl} x & + & 2y & = & 3 \\ x & + & y & = & 1 \\ 2x & + & 3y & = & 3 \end{array}$$

Let A be the coefficient matrix, $\vec{b} = [3, 1, 3]$. Then the least squares solution is the unique (because the rank of A is the number of columns) solution to $A^\top A\vec{x} = A^\top\vec{b}$; namely the system

$$\begin{array}{rcrcl} 6x & + & 9y & = & 10 \\ 9x & + & 14y & = & 16 \end{array}$$

This solution is: $x = -\frac{4}{3}, y = 2$.

(5) Find the quadratic function that best fits the data $\{(0,0), (1,3), (2,4), (3,7)\}$, in the least squares sense.

We plug the data for x and y into the quadratic equation $y = ax^2 + bx + c$, and solve the overdetermined system of linear equations

$$\begin{array}{rcrcrcl} & & & & c & = & 0 \\ a & + & b & + & c & = & 3 \\ 4a & + & 2b & + & c & = & 4 \\ 9a & + & 3b & + & c & = & 7 \end{array}$$

After this system is premultiplied by the transpose of the coefficient matrix, we obtain

$$\begin{array}{rcrcrcl} 98a & + & 36b & + & 14c & = & 82 \\ 36a & + & 14b & + & 6c & = & 32 \\ 14a & + & 6b & + & 4c & = & 14 \end{array}$$

When you solve this system, you will have the coefficients for the desired quadratic function.

(6) Answer true or false:

(a) The image of a projection matrix is an eigenspace for that matrix.

True: Since $P^2 = P$, any vector of the form $P\vec{x}$ is fixed by P. Any nonzero such vector is an eigenvector with eigenvalue 1.

(b) Every projection matrix is an orthogonal matrix.

False: Any projection matrix for a proper subspace will have a nontrivial null space. So the columns won't even be linearly independent, let alone mutually orthogonal.

(c) The inverse of a projection matrix is a projection matrix.

False: A projection matrix is hardly ever invertible (see (b) above). However, when it is invertible, it is the identity matrix. In that special case, the inverse is a projection matrix.

(d) The inverse of an orthogonal square matrix is orthogonal.

True: The inverse of an orthogonal square matrix is the transpose of the matrix, which is also orthogonal.

(e) A nonsingular projection matrix is the identity matrix.

True: See the explanation after (c).

(f) An orthogonal linear transformation is angle preserving.

True: Orthogonal linear transformations preserve the dot product, and this ensures preservation of lengths and angles.

(g) An angle-preserving linear transformation is orthogonal.

False: The transformation on $\mathbb{R}^2$ whose standard matrix representation is $2I$ preserves angles, but not length. Hence it cannot be orthogonal.

(h) If the columns of A and the columns of B are bases for the same subspace of $\mathbb{R}^n$, then $A(A^\top A)^{-1}A^\top = B(B^\top B)^{-1}B^\top$.

True: The projection matrix depends on the subspace being projected upon, not on any particular basis for that subspace.

(i) If the least squares solution to $A\vec{x} = \vec{b}$ is an actual solution, then the coefficient matrix must be square.

False: Several of the rows of the augmented coefficient matrix could be the same, for example.

(7) Show that an invertible idempotent matrix is the identity matrix.

Let A be an invertible idempotent matrix. Then $A^2 = A$. Multiplying both sides by A^{-1} gives $A = I$.

(8) Show that the product of two orthogonal square matrices is orthogonal.

Let A and B be orthogonal matrices. To show AB is orthogonal, it suffices to show that $(AB)^{-1} = (AB)^\top$. Indeed, $(AB)^\top(AB) = B^\top(A^\top A)B = B^\top B$, because A is orthogonal. But $B^\top B = I$ because B is orthogonal. $(AB)^\top$

looks like $(AB)^{-1}$, it quacks like $(AB)^{-1}$; therefore it must *be* $(AB)^{-1}$.