MATH 121, SAMPLE PROBLEMS FOR EXAM 3 (solutions), 04 APRIL, 2007

(Expect six problems, each worth 10 points.)

(1) Let V be the vector space $C([0,1])$ of all continuous functions $f : [0,1] \to \mathbb{R}$, with usual addition and scalar multiplication, and define the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\, dx$.

(a) Compute $\|f\|$, where $f(x) := x$.

$$\|f\| = \sqrt{\int_0^1 (f(x))^2\, dx} = \sqrt{\int_0^1 x^2\, dx} = \sqrt{\left[\frac{1}{3}x^3\right]_0^1} = \sqrt{\frac{1}{3}}.$$

(b) With f defined as in (a), find all constant values c such that f is orthogonal to g, where $g(x) := x - c$.

We want to solve for c in the equation $\int_0^1 x(x-c)\, dx = 0$. Evaluating the integral, we get $\frac{1}{3} - \frac{c}{2} = 0$, or $c = \frac{2}{3}$.

(2) Show, by direct calculation, that the determinant of any 3×3 matrix must be zero if it has either two identical rows or two identical columns.

There are three cases for having two rows identical and three cases for having two columns identical; six cases in all, all handled similarly. Let's do the case where the first two rows are identical; say they're both $[a, b, c]$, where the third row is $[d, e, f]$. Using the definition where we expand by minors using the top row, we get the determinant to be $a(bf - ce) - b(af - cd) + c(ae - bd) = abf - ace - abf + bcd + ace - bcd = 0$.

(3) Answer true or false:

(a) If the inner product of two vectors in an inner product space is zero, then one of the vectors must be the zero vector.

False: Two nonzero vectors can be–and often are–orthogonal.

(b) The zero matrix has only one eigenvalue.

True: The characteristic polynomial is $\lambda^n = 0$, where n is the number of rows/columns. The only solution is $\lambda = 0$.

(c) The determinant of the sum of two square matrices of the same dimensions is the sum of the determinants of those matrices.

False: Suppose the two square matrices are negatives of each other.

(d) A square matrix is diagonalizable iff each of its eigenvalues has the same algebraic and geometric multiplicity.

True: This is a major characterization of diagonalizability.

(e) An invertible matrix may have 0 as one of its eigenvalues.

False: Having 0 as an eigenvalue means that the matrix sends some nonzero vector to zero.

(f) If two square matrices are row equivalent, then they have the same eigenvalues.

False: Any invertible matrix is row equivalent to the identity matrix.

(g) Every linear transformation from a finite-dimensional vector space to itself has only a finite number of eigenvalues.

True: The eigenvalues are roots of the characteristic polynomial, which has degree at most equal to the given finite dimension.

(h) If two square matrices are similar and one of them is singular, then so is the other.

True: If $A = C^{-1}BC$ and B is invertible, then $A^{-1} = C^{-1}B^{-1}C$.

(i) Every symmetric matrix has all real eigenvalues.

True: This is a result in the book, proved using induction on the dimension of the matrix.

(4) Let A be an $n \times n$ matrix, and suppose $B = rA$, where r is a scalar. Relate $\det(B)$ to $\det(A)$, and justify your answer.

Think of obtaining B by n elementary row operations as follows: First multiply row 1 by r; then multiply the second row of *that* matrix by r, and so on. The determinant of each intermediate matrix is r times the determinant of the previous matrix; hence $\det(B) = r^n \det(A)$.

(5) Let A and B be $n \times n$ matrices, with B invertible. Show that $\det((B^{-1}AB)^k) = \det(A^k)$ for any fixed integer $k \geq 0$.

As shown in class–and in the book–$(B^{-1}AB)^k = B^{-1}A^kB$. Using the fact that the determinant of a product is the product of the determinants of the factors, we get $\det((B^{-1}AB)^k) = \det(B^{-1}A^kB) = \det(B^{-1})\det(A^k)\det(B) = \det(B^{-1})\det(B)\det(A^k) = \det(B^{-1}B)\det(A^k) = \det(I)\det(A^k) = \det(A^k)$.

(6) What does it mean geometrically for an $n \times n$ matrix to have no real eigenvalues?

Given any nonzero vector $\vec{v}$, the set $\{\vec{v}, A\vec{v}\}$ is linearly independent; and hence spans a plane.

(7) Compute the eigenvalues for A, and give a basis for each eigenspace. Using this information, decide whether or not A is diagonalizable, where A is the matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtracting λ from the main diagonal entries and computing the determinant, we obtain the characteristic equation $(1-\lambda)^3 = 0$; giving us the single eigenvalue $\lambda_1 = 1$, an eigenvalue of algebraic multiplicity 3. The corresponding eigenspace is the null space of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

namely the span of the vector $[1, 0, 0]$. Thus the geometric multiplicity is only 1. It would have to be 3 in order for the given matrix to be diagonalizable.

(8) Let V be the vector space of all polynomials in one variable, with real coefficients; and let $T : V \to V$ be the linear transformation that takes a polynomial to its derivative. Show that 0 is the only eigenvalue for T.

For any nonzero polynomial $p(x)$, the statement that λ is an eigenvalue for $p(x)$ under differentiation says that $p'(x) = \lambda p(x)$ holds identically. But, for $\lambda \neq 0$, the degree of the polynomial on the left is one less than the degree of the polynomial on the right, if that degree is positive; in the case $p(x)$ is a nonzero constant, $p'(x)$ is constantly zero. In either case equality can't hold identically. $\lambda = 0$ is an eigenvalue, however, and its eigenspace is the set of all constant polynomials.

(9) Suppose A is a square matrix with $A^2 = A$. What are the possible eigenvalues for A?

Let λ be an eigenvalue for A. Then there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Also $A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$; hence $\lambda^2\vec{v} = \lambda\vec{v}$. But then $\lambda(\lambda - 1)\vec{v} = \vec{0}$, and, because $\vec{v}$ is a nonzero vector, this can happen only when the scalar $\lambda(\lambda - 1)$ is zero. Hence $\lambda = 0$ or $\lambda = 1$.

(10) Let V be the vector space $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, with usual addition and scalar multiplication, and define the linear transformation $T : V \to V$ by the rule $(T(f))(x) := \int_0^x f(t)\, dt$.

(a) Calculate $T(f)$, where $f(x) := \sin x$.

In this case $T(f)$, evaluated at x, is just $\int_0^x \sin t\, dt = \Big[-\cos t\Big]_0^x = 1 - \cos x$.

(b) Show that 0 is not an eigenvalue for T.

For 0 to be an eigenvalue for T, we're looking for a nonzero function $f : \mathbb{R} \to \mathbb{R}$ such that $\int_0^x f(t)\, dt$ is constantly zero. Let $F(x)$ be an antiderivative for $f(x)$; i.e., $F'(x) = f(x)$ for all real x, by the fundamental theorem of calculus. Then we have $\int_0^x f(t)\, dt = F(x) - F(0)$. And for this to be identically zero, we need $F(x)$ to be identically equal to $F(0)$. But then $f(x)$ is identically zero, a contradiction. So $\lambda = 0$ is not a possible value for any eigenvalue.