MATH 121, SAMPLE PROBLEMS FOR EXAM 3, 04 APRIL, 2007

(Expect six problems, each worth 10 points.)

(1) Let V be the vector space $C([0,1])$ of all continuous functions $f : [0,1] \to \mathbb{R}$, with usual addition and scalar multiplication, and define the inner product $\langle f, g \rangle := \int_0^1 f(x)g(x)\,dx$.

(a) Compute $\|f\|$, where $f(x) := x$.

(b) With f defined as in (a), find all constant values c such that f is orthogonal to g, where $g(x) := x - c$.

(2) Show, by direct calculation, that the determinant of any 3×3 matrix must be zero if it has either two identical rows or two identical columns.

(3) Answer true or false:

(a) If the inner product of two vectors in an inner product space is zero, then one of the vectors must be the zero vector.

(b) The zero matrix has only one eigenvalue.

(c) The determinant of the sum of two square matrices of the same dimensions is the sum of the determinants of those matrices.

(d) A square matrix is diagonalizable iff each of its eigenvalues has the same algebraic and geometric multiplicity.

(e) An invertible matrix may have 0 as one of its eigenvalues.

(f) If two square matrices are row equivalent, then they have the same eigenvalues.

(g) Every linear transformation from a finite-dimensional vector space to itself has only a finite number of eigenvalues.

(h) If two square matrices are similar and one of them is singular, then so is the other.

(i) Every symmetric matrix has all real eigenvalues.

(4) Let A be an $n \times n$ matrix, and suppose $B = rA$, where r is a scalar. Relate $\det(B)$ to $\det(A)$, and justify your answer.

(5) Let A and B be $n \times n$ matrices, with B invertible. Show that $\det((B^{-1}AB)^k) = \det(A^k)$ for any fixed integer $k \geq 0$.

(6) What does it mean geometrically for an $n \times n$ matrix to have no real eigenvalues?

(7) Compute the eigenvalues for A, and give a basis for each eigenspace. Using this information, decide whether or not A is diagonalizable, where A is the matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(8) Let V be the vector space of all polynomials in one variable, with real coefficients; and let $T : V \to V$ be the linear transformation that takes a polynomial to its derivative. Show that 0 is the only eigenvalue for T.

(9) Suppose A is a square matrix with $A^2 = A$. What are the possible eigenvalues for A?

(10) Let V be the vector space $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, with usual addition and scalar multiplication, and define the linear transformation $T : V \to V$ by the rule $(T(f))(x) := \int_0^x f(t)\, dt$.

(a) Calculate $T(f)$, where $f(x) := \sin x$.

(b) Show that 0 is not an eigenvalue for T.