MATH 121, SAMPLE PROBLEMS (with solutions) FOR EXAM 2, 05 MARCH, 2007

(Expect six problems, each worth 10 points.)

(1) Find the rank, as well as a basis for the column space of, the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}$$

Using row reduction, we determine that

$$A \sim H = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

a matrix in row echelon form. Since the number of pivot rows is two, this gives us the rank of the matrix (= dimension of the row space = dimension of the column space). The first two columns of the reduced matrix H give a basis for its column space; so the first two columns of A give a basis for the column space of A.

(2) Enlarge $\{[2,1,1],[1,0,1]\}$ to a basis for $\mathbb{R}^3$.

Denote this set of two vectors by S. Then S is an independent set because neither of its vectors is a scalar multiple of the other. We may enlarge S to a spanning set by adding in the three standard basis vectors, and then eliminating those that fall within the span of S; alternatively we may test each of those vectors to find the first one that isn't in span(S). It's easy to check that $[1,0,0]$ works, so we know that $\{[2,1,1],[1,0,1],[1,0,0]\}$ is an independent set. Since the dimension of $\mathbb{R}^3$ is 3, this set must be a basis.

(3) Answer true or false:

(a) In any matrix, the number of independent row vectors equals the number of independent column vectors.

True: This is the rank of the matrix.

(b) The rank of an invertible square matrix equals the number of rows.

True: For a matrix to be invertible, it must be row-reducible to the identity matrix.

(c) No matrix has rank zero.

False: Any matrix of zeros has rank zero.

(d) The zero vector may fail to be in the range of a linear transformation.

False: Any linear transformation sends the zero vector in the domain to the zero vector in the range.

(e) A linear transformation is determined by where it sends the vectors in a basis.

True: Each vector in the domain has a unique representation as a linear combination of the basis vectors.

(f) Distinct vectors in a finite-dimensional vector space V have distinct coordinate vectors relative to a given ordered basis B for V.

True: Each ordered basis B for V sets up a one-one correspondence between the vectors of V and the vectors of $\mathbb{R}^n$, where $n = \dim(V)$.

(g) The vector space P_8 of polynomials of degree ≤ 8 is isomorphic to $\mathbb{R}^8$.

False: The dimension of P_n is always $n+1$.

(4) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation taking $[-1, 2]$ to $[1, 0, 0]$ and $[2, 1]$ to $[0, 1, 2]$. Find a general rule for computing $T([x, y])$, and compute $T([3, 1])$.

First determine that $[1, 0] = (-.2)[-1, 2]+(.4)[2, 1]$, and that $[0, 1] = (.4)[-1, 2]+(.2)[2, 1]$; so $T([1, 0]) = (-.2)[1, 0, 0]+(.4)[0, 1, 2] = [-.2, .4, .8]$ and $T([0, 1]) = (.4)[1, 0, 0] + (.2)[0, 1, 2] = [.4, .2, .4]$. These, respectively, give us the first and second columns of the standard matrix representation of T, so $T([x, y]) = [-.2x + .4y, .4x + .2y, .8x + .4y]$. In particular, $T([3, 1]) = [-.2, 1.4, 2.8]$.

(5) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the rule $T([x, y, z]) := [2x + y + z, x + z, y]$. Find the standard matrix representation for T.

$T([1, 0, 0]) = [2, 1, 0]$, $T([0, 1, 0]) = [1, 0, 1]$, and $T([0, 0, 1]) = [1, 1, 0]$. So the standard matrix representation is

$$A_T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(6) Decide (with justification) whether the set $\mathbb{R}^2$, with usual vector addition and scalar multiplication defined by $r[x, y] := [ry, rx]$, is a vector space.

> One of the axioms for how scalar multiplication works is that $1\vec{v} = \vec{v}$ always holds. In this instance, when $r = 1$, we have $1[x, y] = [y, x]$, which is not equal to $[x, y]$ unless $x = y$. This immediately disqualifies this notion of scalar multiplication from being legitimate for the purposes of being a vector space.

(7) Show that the set D of all diagonal 2×2 matrices–i.e., zeros off the main diagonal–is a subspace of the space M_2 of all 2×2 matrices.

> All you have to do is show that the sum of two diagonal matrices is a diagonal matrix (zeros off the main diagonal stay zero) and that multiplying a diagonal matrix by a scalar gives a diagonal matrix. Both these verifications are easy. [What's *not* acceptable is verification using examples of specific matrices.]

(8) Show that, for each natural number n, the set $\{1, x, x^2, x^3, ..., x^n\}$ is not a basis for the vector space $\mathbb{R}(x)$ of all polynomials in the indeterminate x.

> It is impossible to express x^{n+1} as a linear combination of polynomials of degree $\leq n$.

(9) Prove that $\{1, \sin x, \sin 2x\}$ is an independent set of functions in the vector space $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

> To say that the linear combination $a + b\sin x + c\sin 2x$ is zero, means that the equality $a + b\sin x + c\sin 2x = 0$ holds *identically*. So, if we plug in $x = 0$, we obtain $a = 0$ immediately. Next, if we plug in $x = \pi/2$, we obtain $a + b = 0$. But, since $a = 0$ already, we know that $b = 0$ too. Finally, if we plug in, say, $x = \pi/4$, we have (because $a = b = 0$) $c\sin\frac{\pi}{2} = c = 0$. This shows the given set of functions to be independent.

(10) Find the polynomial in P_2 whose coordinate vector relative to the ordered basis $(x^2 + 1, x^2 - 1, x)$ is $[2, 5, -1]$.

> First we verify that the given set is linearly independent by noting: (i) that neither of the first two polynomials is a scalar multiple of the other; and (ii) that the monomial x is not in the span of the first two polynomials. The polynomial we seek is $2(x^2 + 1) + 5(x^2 - 1) + (-1)x = 7x^2 - x - 3$.

(11) Let $T : P_2 \to P_2$ be differentiation. Find the matrix representation for T relative to the ordered basis in Problem 10.

First the ordered basis $(x^2 + 1, x^2 - 1, x)$ is mapped to the standard ordered basis $([1, 0, 0], [0, 1, 0], [0, 0, 1])$. Next, the differentiation operator takes the original ordered basis to the ordered triple $(2x, 2x, 1)$–of course not an ordered basis. Expressing each of these in terms of the standard ordered basis gives us the triple $([0, 0, 2], [0, 0, 2], [.5, -.5, 0])$. These give us the columns for the standard matrix representation for T:

$$\begin{bmatrix} 0 & 0 & .5 \\ 0 & 0 & -.5 \\ 2 & 2 & 0 \end{bmatrix}.$$

(12) Let C be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and let $T : C \to \mathbb{R}$ be defined by the definite integral as follows: $T(f) := \int_0^1 f(x)\, dx$. Find two distinct vectors in the kernel of T.

The kernel of a linear transformation is the set of vectors in the domain which the transformation sends to the zero vector in the codomain. So we are looking for continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $T(f) = \int_0^1 f(x)\, dx = 0$. Any continuous function that is zero in the interval $[0, 1]$ will do (there are oodles of others, too). For example, for each $n = 1, 2, \ldots$, you could let $f_n(x)$ be zero for $x \leq 1$ and set $f_n(x) = x^n - 1$ for $x \geq 1$. This gives us an infinite family of members of C that lie in the kernel of T.