MATH 121, SAMPLE PROBLEMS (with solutions) FOR EXAM 1, 09 FEB, 2007

(1) Find a unit vector parallel to $[5, 0, -12]$.

Our vector has length $\sqrt{5^2 + 0^2 + (-12)^2} = 13$, so there are two unit vectors that meet the requirements: $\pm\frac{1}{13}[5, 0, -12]$.

(2) Find a vector in $\mathbb{R}^3$ that is perpendicular to both $[1, 1, 1]$ and $[1, 0, 1]$.

We're looking for a vector $[x, y, z]$ whose dot product with each of the given vectors is 0. This amounts to solving the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ x & & & + & z & = & 0 \end{array},$$

whose solution set may be expressed parametrically as

$$\begin{array}{ccc} x & = & -t \\ y & = & 0 \\ z & = & t \end{array}.$$

Any nonzero choice of t will give a vector that satisfies the given conditions.

(3) Answer the following statements true or false, where all matrices are square, of the same dimensions:

(a) If A is invertible, then so is its transpose, A^T.

True: Multiply A^T by $(A^{-1})^T$ to obtain I. Conclude that $(A^T)^{-1} = (A^{-1})^T$.

(b) If $AC = BC$ for some matrix C, then $A = B$.

False: This is not guaranteed to work unless C is invertible.

(c) If $AB = BA$, then $A = B$.

False: B could be A^2, which is not necessarily equal to A.

(d) If A is symmetric, then so is A^T.

True: A symmetric means $A = A^T$.

(e) If A and B are symmetric, then so is AB.

True: Use the theorem about taking transposes of products.

(4) Compute the *reduced* row echelon form H for the matrix A below, and write H in the form BA, where B is a product of elementary matrices.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You can get from A to H by applying $R_1 \leftrightarrow R_2$, then $-R_2 + R_1 \to R_1$, then $-R_3 + R_2 \to R_2$. The rref is

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) A$$

(5) Express as a parameterized family of vectors, the solution set for the system:

$$\begin{array}{ccccccccc} x_1 & & & + & 2x_3 & & & = & 1 \\ & & x_2 & & & + & 3x_4 & = & 2 \end{array}$$

The augmented coefficient matrix for this system is already in rref, so the lead variables are x_1 and x_2, with x_3 and x_4 free. The required parameterized family is:

$$\begin{array}{rcl} x_1 & = & 1 - 2s \\ x_2 & = & 2 - 3t \\ x_3 & = & s \\ x_4 & = & t \end{array}.$$

This is a two-parameter family; any independent real values of s and t will give rise to a particular solution for the system.

(6) Show that $[1, 2]$ is in the span of the set $\{[1, 3], [2, -6]\}$.

We need to find scalars x and y so that $x[1, 3] + y[2, -6] = [1, 2]$. This translates to solving the system

$$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 3x & - & 6y & = & 2 \end{array}.$$

The unique solution is $x = \frac{5}{6}$, $y = \frac{1}{12}$, and we're done.

(7) Find a 2×2 non-invertible matrix A with no zero entries.

All you need is to be able to get the rref to fail to be the identity matrix. One possibility (out of oodles) is the 2×2 matrix with entries all equal to, say, 1.

(8) Determine (with justification) whether the set of column vectors of A is a basis for the column space of A, where A is the matrix:

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix}$$

We try to find a nontrivial linear combination of the column vectors that adds up to the zero vector. If this is impossible, the given set is linearly independent, a basis for its span; otherwise we don't have a basis. The latter is true because $\frac{12}{7}[2,1] + (-\frac{3}{7})[1,4] + (-1)[3,0] = [0,0]$.

(9) Find a basis for the null space of the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

We're looking for all vectors $\vec{x} \in \mathbb{R}^4$ such that $A\vec{x} = \vec{0}$, where A is the given matrix. The coefficient matrix is already in rref, and (see Problem 5 above) a parameterization for the solution set is

$$\begin{array}{rcr} x_1 & = & -2s \\ x_2 & = & -3t \\ x_3 & = & s \\ x_4 & = & t \end{array}.$$

This may also be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

So the vectors $[-2,0,1,0]$ and $[0,-3,0,1]$ span the null space for the matrix; they form a basis because they are linearly independent. (This being due to the reduction to ref.)